

Pacific Journal of Mathematics

**LEVEL CROSSING PROBABILITIES FOR A
MULTI-PARAMETER BROWNIAN PROCESS**

PEGGY STRAIT

LEVEL CROSSING PROBABILITIES FOR A MULTI-PARAMETER BROWNIAN PROCESS

PEGGY TANG STRAIT

Let $\{\xi_{s_1, s_2}^{\hat{\cdot}}$: $-\infty < s_1, < \infty, -\infty < s_2 < \infty\}$ be a Gaussian process with $\xi_{s_1, s_2}^{\hat{\cdot}} = 0$ if $s_1 = 0$ or $s_2 = 0$, mean values $E(\xi_{s_1, s_2}^{\hat{\cdot}}) = 0$, and covariances $E(\xi_{s_1, s_2}^{\hat{\cdot}} \xi_{s_1', s_2'}^{\hat{\cdot}}) = 1/2 \min(s_1, s_1') \min(s_2, s_2')$. This is the two parameter Brownian process studied by J. D. Kuelbs, W. J. Park, P. T. Strait, and J. Yeh. In this paper, upper and lower bounds for level crossing probabilities of this process are derived.

More specifically, let (t_1, t_2) and (τ_1, τ_2) be two pairs of constants chosen so that $0 < t_1 < \infty, 0 < t_2 < \infty, 0 < \tau_1 < \infty$, and $0 < \tau_2 < \infty$. Let $\delta_1 = \tau_1/m, \delta_2 = \tau_2/n$ where m and n are integers, and define random variables $X_{h,k}$ for $h = 0, 1, 2, \dots, m$ and $k = 0, 1, 2, \dots, n$ as follows.

$$(1) \quad X_{h,k} = \begin{cases} 0 & \text{for } h = 0 \text{ or } k = 0 \\ \xi_{t_1+(h-1)\delta_1, t_2+(k-1)\delta_2}^{\hat{\cdot}} & \text{for } h = 1, 2, \dots, m; k = 1, 2, \dots, n. \end{cases}$$

For any given number a , define

$$(2) \quad \begin{aligned} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \\ = P(X_{i,j} > a \text{ for } i = 1, 2, \dots, m; j = 1, 2, \dots, n). \end{aligned}$$

In this paper, upper and lower bounds (Theorems 1 and 2) for $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2)$ are derived.

2. Preliminary lemmas.

LEMMA 1. *Let $\zeta_{h,k} = X_{h,k} - X_{h,k-1} - X_{h-1,k} + X_{h-1,k-1}$ for $h = 1, \dots, m$ and $k = 1, \dots, n$. Then,*

(i) $\zeta_{h,k}, h = 1, \dots, m, k = 1, \dots, n$ are independent Gaussian random variables with means 0 and variances $\sigma_{h,k}^2$ given by

$$(3) \quad \begin{aligned} \sigma_{1,1}^2 &= \frac{1}{2} t_1 t_2 \\ \sigma_{1,k}^2 &= \frac{1}{2} \delta_1 t_2 \quad \text{for } k = 2, \dots, n \\ \sigma_{h,1}^2 &= \frac{1}{2} \delta_2 t_1 \quad \text{for } h = 2, \dots, m \\ \sigma_{h,k}^2 &= \frac{1}{2} \delta_1 \delta_2 \quad \text{for } h = 2, \dots, m; k = 2, \dots, n \end{aligned}$$

and

$$(4) \quad (ii) \quad X_{i,j} = \sum_{h=1}^i \sum_{k=1}^j \zeta_{h,k} \quad \text{for } i = 1, \dots, m; j = 1, \dots, n.$$

Proof of Lemma 1. To prove part (i) of the lemma, observe that

$$(5) \quad E(\zeta_{h,k}) = E(X_{h,k} - X_{h,k-1} - X_{h-1,k} + X_{h-1,k-1}) = 0$$

for $h = 1, \dots, m; k = 1, \dots, n$.

$$(6) \quad E(\zeta_{h,k}^2) = E[(X_{h,k} - X_{h-1,k} - X_{h,k-1} + X_{h-1,k-1})^2]$$

$$= \begin{cases} \frac{1}{2}t_1t_2 & \text{for } h = 1, k = 1 \\ \frac{1}{2}\delta_1t_2 & \text{for } h = 1; k = 2, \dots, n \\ \frac{1}{2}\delta_2t_1 & \text{for } h = 2, \dots, m; k = 1 \\ \frac{1}{2}\delta_1\delta_2 & \text{for } h = 2, \dots, m; k = 2, \dots, n. \end{cases}$$

$$(7) \quad E(\zeta_{h,k}\zeta_{p,q})$$

$$= E[X_{h,k} - X_{h-1,k} - X_{h,k-1} + X_{h-1,k-1}](X_{p,q} - X_{p-1,q} - X_{p,q-1} + X_{p-1,q-1})$$

$= 0$ when $(h, k) \neq (p, q)$.

(In each of the equations (5), (6), and (7), the last term is derived by direct computation of the expression of the preceding term.) Thus, the random variables $\zeta_{h,k}$, $h = 1, \dots, m, k = 1, \dots, n$ are independent, Gaussian random variables with mean values 0 and variances given by equation (3).

To prove part (ii) of the lemma, observe that

$$(8) \quad \sum_{h=1}^i \sum_{k=1}^j \zeta_{h,k} = \sum_{h=1}^i \sum_{k=1}^j (X_{h,k} - X_{h-1,k} - X_{h,k-1} + X_{h-1,k-1}) = X_{i,j}$$

For the remaining lemmas and theorems we add the following notations and definitions. Let $C = \{(i, j): i = 1, \dots, m; j = 1, \dots, n\}$, $C^* = C - \{(1, 1)\}$. Furthermore, let $P_{m,n}^-$ and $P_{m,n}^*$ be probabilities defined as follows.

$$(9) \quad P_{m,n}^-(a, t_1, t_2, \tau_1, \tau_2) = P(X_{i,j} > a \text{ for all } (i, j) \in C^* \text{ and } X_{1,1} \leq a)$$

$$(10) \quad P_{m,n}^*(a, t_1, t_2, \tau_1, \tau_2) = P(X_{i,j} > a \text{ for all } (i, j) \in C^*).$$

Then clearly,

$$(11) \quad P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) = P_{m,n}^*(a, t_1, t_2, \tau_1, \tau_2) - P_{m,n}^-(a, t_1, t_2, \tau_1, \tau_2).$$

LEMMA 2. Let η_1, η_2, η_3 be normal random variables with $E(\eta_i) = 0$ and $\text{Var}(\eta_i) = (1/6)t_1 t_2$ for $i = 1, 2, 3$. Assume also that $\eta_1, \eta_2, \eta_3, \zeta_{1,1}, \zeta_{1,2}, \dots, \zeta_{m,n}$ form a set of independent random variables. Then

$$\begin{aligned}
 & P_{m,n}^*(a, t_1, t_2, \tau_1, \tau_2) \\
 & \geq P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > \frac{a}{3} \quad \text{for all } (i, j) \in C^*\right) \\
 (12) \quad & \cdot P\left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > \frac{a}{3} \quad \text{for all } j = 1, 2, \dots, n\right) \\
 & \cdot P\left(\eta_3 + \sum_{h=2}^i \zeta_{h,1} > \frac{a}{3} \quad \text{for all } i = 1, 2, \dots, m\right).
 \end{aligned}$$

Proof of Lemma 2. For $i < 2$ or $j < 2$ define

$$\sum_{k=2}^i \sum_{k=2}^j \zeta_{h,k} = 0, \quad \sum_{k=2}^j \zeta_{1,k} = 0, \quad \text{and} \quad \sum_{h=2}^i \zeta_{h,1} = 0.$$

The proof of Lemma 2 follows.

$$\begin{aligned}
 & P_{m,n}^*(a, t_1, t_2, \tau_1, \tau_2) = P(X_{i,j} > a \quad \text{for } (i, j) \in C^*) \\
 & = P\left(\sum_{h=1}^i \sum_{k=1}^j \zeta_{h,k} > a \quad \text{for } (i, j) \in C^*\right) \\
 & = P\left(\sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} + \sum_{k=2}^j \zeta_{1,k} + \sum_{h=2}^i \zeta_{h,1} + \zeta_{1,1} > a \quad \text{for } (i, j) \in C^*\right) \\
 & = P\left(\sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} + \sum_{k=2}^j \zeta_{1,k} + \sum_{h=2}^i \zeta_{h,1} + \eta_1 + \eta_2 + \eta_3 > a \right. \\
 (13) \quad & \left. \text{for } (i, j) \in C^*\right) \\
 & \geq P\left[\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > \frac{a}{3}\right) \quad \text{and} \quad \left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > \frac{a}{3}\right) \right. \\
 & \quad \left. \text{and} \quad \left(\eta_3 + \sum_{h=2}^i \zeta_{h,1} > \frac{a}{3}\right) \quad \text{for } (i, j) \in C^*\right] \\
 & = P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > \frac{a}{3} \quad \text{for } (i, j) \in C^*\right) \\
 & \cdot P\left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > \frac{a}{3} \quad \text{for } j = 1, 2, \dots, n\right) \\
 & \cdot P\left(\eta_3 + \sum_{h=2}^i \zeta_{h,1} < \frac{a}{3} \quad \text{for } i = 1, 2, \dots, m\right).
 \end{aligned}$$

LEMMA 3.

$$\begin{aligned}
 (14) \quad & \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > a \quad \text{for } (i, j) \in C^*\right) \\
 & \geq \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - 4 \sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}, \quad \alpha = \frac{a}{\sqrt{3t_1 t_2}}.
 \end{aligned}$$

Proof of Lemma 3. Let $Y_{h,k}$, $h = 1, \dots, m$, $k = 1, \dots, n$ be independent standard normal random variables. Let

$$\begin{aligned}
 (15) \quad L_{m,n} &= \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left(\sum_{h=1}^i \sum_{k=1}^j Y_{h,k} \right) \\
 L_{m,n}^- &= \min(0, L_{m,n}) \\
 U_{m,n} &= \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left(\sum_{h=1}^i \sum_{k=1}^j Y_{h,k} \right), \quad U_{m,n}^+ = \max(0, U_{m,n})
 \end{aligned}$$

It is shown in [6] that

$$(16) \quad E(U_{m,n}^+) < 4\sqrt{mn}.$$

To prove Lemma 3, it is first shown that

$$\begin{aligned}
 (17) \quad P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > a \text{ for } (i, j) \in C^*\right) \\
 = \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - E^-\left(\int_0^{-\sqrt{3\delta_1\delta_2}/t_1t_2} L_{m-1,n-1} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right)
 \end{aligned}$$

where E^- denotes that portion of the expectation obtained by integration over the negative range of $L_{m-1,n-1}$. (Later, E^+ shall also be used to denote expectation obtained by integration over the positive range of values.)

To prove equation (17), observe that

$$\begin{aligned}
 &P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > a \text{ for } (i, j) \in C^*\right) \\
 &= P\left(\eta_1 - a > -\sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} \text{ for } (i, j) \in C^*\right) \\
 &= P\left(\eta_1 - a > \sqrt{\frac{\delta_1\delta_2}{2}} \sum_{h=2}^i \sum_{k=2}^j Y_{h,k} \text{ for } (i, j) \in C^*\right) \\
 &= P\left(\eta_1 - a > -\sqrt{\frac{\delta_1\delta_2}{2}} L_{m-1,n-1}^- \right) \\
 &= E\left(\int_{-\sqrt{\delta_1\delta_2}/2}^{\infty} \frac{1}{\sqrt{\pi} \frac{t_1t_2}{3}} e^{-(u+\alpha)^2/(1/3)t_1t_2} du\right) \\
 &= E\left(\int_{-\sqrt{3\delta_1\delta_2}/t_1t_2}^{\infty} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
 (18) \quad &= E^+\left(\int_{-\sqrt{3\delta_1\delta_2}/t_1t_2}^{\infty} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
 &+ E^-\left(\int_{-\sqrt{3\delta_1\delta_2}/t_1t_2}^{\infty} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right)
 \end{aligned}$$

$$\begin{aligned}
 &= E^+\left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
 &\quad + E^-\left(\int_{-\sqrt{3\delta_1\delta_2}/t_1 t_2}^{L_{m-1}, n-1} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
 &= E^+\left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) + E^-\left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
 &\quad - E^-\left(\int_0^{-\sqrt{3\delta_1\delta_2}/t_1 t_2}^{L_{m-1}, n-1} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
 &= \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - E^-\left(\int_0^{-\sqrt{3\delta_1\delta_2}/t_1 t_2}^{L_{m-1}, n-1} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right).
 \end{aligned}$$

Next, apply the inequality

$$(19) \quad \int_0^x e^{-1/2(u+\alpha)^2} du < x \quad \text{for } x > 0$$

to obtain

$$(20) \quad E^-\left(\int_0^{-\sqrt{3\delta_1\delta_2}/t_1 t_2}^{L_{m-1}, n-1} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) < -\sqrt{\frac{2}{\pi}} \sqrt{\frac{3\delta_1\delta_2}{t_1 t_2}} E^-(L_{m-1}, n-1).$$

Now, observe that

$$(21) \quad E^-(L_{m-1}, n-1) = E(L_{m-1}^-, n-1) = -E(U_{m-1}^+, n-1)$$

then combine this with equation (16), to obtain

$$(22) \quad E^-(L_{m-1}, n-1) > -4\sqrt{(m-1)(n-1)}.$$

Finally, apply equations (17), (20), and (22) to obtain

$$\begin{aligned}
 &\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > a \quad \text{for } (i, j) \in C^*\right) \\
 (23) \quad &\geq \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \left\{ \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du + \sqrt{\frac{2}{\pi}} \sqrt{\frac{3\delta_1\delta_2}{t_1 t_2}} E^-(L_{m-1}, n-1) \right\} \\
 &\geq \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \left\{ \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du + \sqrt{\frac{2}{\pi}} \sqrt{\frac{3\delta_1\delta_2}{t_1 t_2}} (-4\sqrt{(m-1)(n-1)}) \right\} \\
 &= \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - 4\sqrt{\frac{6\delta_1\delta_2}{\pi t_1 t_2}}.
 \end{aligned}$$

LEMMA 4.

$$\begin{aligned}
 (a) \quad &\lim_{n \rightarrow \infty} P\left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > 0 \quad \text{for } j = 1, 2, \dots, n\right) \\
 &= \frac{1}{\pi} \sin^{-1}\left[\left(1 + \frac{3\tau_1}{t_1}\right)^{-1/2}\right]
 \end{aligned}$$

$$(b) \lim_{m \rightarrow \infty} P\left(\eta_3 + \sum_{k=2}^i \zeta_{k,1} > 0 \text{ for } i = 1, 2, \dots, m\right) \\ = \frac{1}{\pi} \sin^{-1}\left[\left(1 + \frac{3\tau_2}{t_2}\right)^{-1/2}\right]$$

$$(24) \quad (c) \text{ For small } \frac{\tau_1}{t_1}, \\ \lim_{n \rightarrow \infty} P\left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > a \text{ for } j = 1, 2, \dots, n\right) \\ = \Phi\left(-\frac{6a}{t_1 t_2}\right) - \frac{1}{\pi} \left(\frac{3\tau_1}{t_1}\right)^{1/2} e^{-3a^2/t_1 t_2} + O\left(\frac{3\tau_1}{t_1}\right)$$

$$(d) \text{ For small } \frac{\tau_2}{t_2}, \\ \lim_{m \rightarrow \infty} P\left(\eta_3 + \sum_{k=2}^i \zeta_{k,1} a \text{ for } i = 1, 2, \dots, m\right) \\ = \Phi\left(-\frac{6a}{t_1 t_2}\right) - \frac{1}{\pi} \left(\frac{3\tau_2}{t_2}\right)^{1/2} e^{-\frac{3a^2}{t_1 t_2}} + O\left(\frac{3\tau_2}{t_2}\right)$$

where $\Phi(\cdot)$ is the standardized normal distribution function

$$\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du.$$

Proof of Lemma 4. Let

$$(25) \quad Y_k = \sqrt{\frac{2}{\delta_1 t_2}} \zeta_{1,k} \text{ for } k = 2, \dots, n$$

so that Y_2, Y_3, \dots, Y_n are independent standard normal random variables. Let

$$(26) \quad \delta = \frac{\delta_1 t_2}{2}$$

$$(27) \quad X_1 = \eta_2$$

$$X_2 = \eta_2 + \zeta_{1,2} = \eta_2 + \sqrt{\frac{\delta_1 t_2}{2}} Y_2 = X_1 + \delta^{1/2} Y_2 \\ \vdots \\ X_n = \eta_2 + \sum_{k=2}^n \zeta_{1,k} = \eta_2 + \sqrt{\frac{\delta_1 t_2}{2}} \sum_{k=2}^n Y_k = X_1 + \delta^{1/2} \sum_{k=2}^n Y_k$$

where X_1 is normal, $E(X_1) = 0$, $\text{Var}(X_1) = (1/6)t_1 t_2$. Let

$$(28) \quad t = \frac{1}{6} t_1 t_2, \quad \tau = \frac{n \delta_1 t_2}{2} = \frac{\tau_1 t_2}{2}.$$

For random variables X_1, X_2, \dots, X_n satisfying the conditions given

above, J. A. McFadden and J. L. Lewis proved in [3] (equation (34) on page 310) that

$$(29) \quad \lim_{n \rightarrow \infty} P(X_i > 0 \text{ for all } i = 1, \dots, n) = \lim_{n \rightarrow \infty} P_{n+1}(0, t) \\ = \frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{\tau}{t} \right)^{-1/2} \right].$$

Hence in this case

$$(30) \quad \lim_{n \rightarrow \infty} P \left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > 0 \text{ for } j = 1, 2, \dots, n \right) \\ = \lim_{n \rightarrow \infty} P(X_i > 0 \text{ for all } i = 1, \dots, n) \\ = \frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{\tau}{t} \right)^{-1/2} \right] = \frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_1}{t_1} \right)^{-1/2} \right].$$

Similarly,

$$(31) \quad \lim_{m \rightarrow \infty} P \left(\eta_3 + \sum_{h=2}^i \zeta_{h,1} > 0 \text{ for } i = 1, \dots, m \right) \\ = \frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_2}{t_2} \right)^{-1/2} \right].$$

For the case $a \neq 0$ and $\tau_1/t_1, \tau_2/t_2$ small, use equation (37) of McFadden and Lewis [3] which states that

$$(32) \quad \lim_{n \rightarrow \infty} P(X_i > a \text{ for all } i = 1, \dots, n) \\ = \lim_{n \rightarrow \infty} P_{n+1}(a, t) = \Phi \left(\frac{-a}{t^{1/2}} \right) - \frac{1}{\pi} \left(\frac{\tau}{t} \right)^{1/2} e^{-a^2/2t} + O(\tau/t)$$

to obtain

$$(33) \quad \lim_{n \rightarrow \infty} P \left(\eta_2 + \sum_{k=2}^j \zeta_{i,k} > a \text{ for } i = 1, 2, \dots, n \right) \\ = \Phi \left(-\frac{a}{\sqrt{\frac{1}{6} t_1 t_2}} \right) - \frac{1}{\pi} \left(\frac{3\tau_1}{t_1} \right)^{1/2} e^{-3a^2/t_1 t_2} + O \left(\frac{3\tau_1}{t_1} \right).$$

Similarly,

$$(34) \quad \lim_{m \rightarrow \infty} P \left(\eta_3 + \sum_{h=2}^i \zeta_{h,1} > 0 \text{ for } i = 1, 2, \dots, m \right) \\ = \Phi \left(-\frac{\sqrt{6a}}{\sqrt{t_1 t_2}} \right) - \frac{1}{\pi} \left(\frac{3t_2}{t_2} \right)^{1/2} e^{-3a^2/t_1 t_2} + O \left(\frac{3\tau_2}{t_1} \right).$$

LEMMA 5.

$$(35) \quad \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) = \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}^*(a, t_1, t_2, \tau_1, \tau_2).$$

Proof of Lemma 5. Consider the term $P_{m,n}^-(a, t_1, t_2, \tau_1, \tau_2)$ of equation (11).

$$(36) \quad \begin{aligned} P_{m,n}^-(a, t_1, t_2, \tau_1, \tau_2) &= P(X_{i,j} > a \text{ for } (i, j) \in C^* \text{ and } X_{1,1} \leq a) \\ &= P(X_{i,j} > a \text{ for } (i, j) \in C^* \text{ and } X_{1,1} < a). \end{aligned}$$

Therefore, by continuity of the sample paths (see [4] for proof of this fact)

$$(37) \quad \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}^-(a, t_1, t_2, \tau_1, \tau_2) = 0 \text{ for } a \neq 0.$$

But this implies that equation (11) is equivalent to

$$(38) \quad P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) = P_{m,n}^*(a, t_1, t_2, \tau_1, \tau_2) - P_{m,n}^-(a, t_1, t_2, \tau_1, \tau_2)$$

Upon taking limits on both sides of (38), equation (35) of Lemma 5 is obtained.

3. The main theorems.

THEOREM 1. *A lower bound.*

$$(39) \quad \begin{aligned} &\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(0, t_1, t_2, \tau_1, \tau_2) \\ &\geq \left(1 - 4\sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}\right) \cdot \left(\frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_1}{t_1}\right)^{-1/2} \right]\right) \\ &\quad \cdot \left(\frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_2}{t_2}\right)^{1/2} \right]\right) \end{aligned}$$

and for $a \neq 0$, τ_1/t_1 small, τ_2/t_2 small,

$$(40) \quad \begin{aligned} \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) &\geq \left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - 4\sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}\right) \\ &\cdot \left[\Phi\left(-\frac{6a}{t_1 t_2}\right) - \frac{1}{\pi} \left(\frac{3\tau_1}{t_1}\right)^{1/2} e^{-a^2/3t_1 t_2} O\left(\frac{3\tau_1}{t_1}\right)\right] \\ &\cdot \left[\Phi\left(-\frac{6a}{t_1 t_2}\right) - \frac{1}{\pi} \left(\frac{3\tau_2}{t_2}\right)^{1/2} e^{-a^2/3t_1 t_2} + O\left(\frac{3\tau_2}{t_2}\right)\right], \end{aligned}$$

where $\alpha = \frac{a}{\sqrt{3t_1 t_2}}$

Proof of Theorem 1. Apply Lemma 2 to Lemma 5 to obtain

$$\begin{aligned}
 & \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \\
 (41) \quad & \geq P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > \frac{a}{3} \text{ for } (i, j) \in C^*\right) \\
 & \cdot P\left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > \frac{a}{3} \text{ for } j = 1, 2, \dots, n\right) \\
 & \cdot P\left(\eta_3 + \sum_{h=2}^i \zeta_{h,1} > \frac{a}{3} \text{ for } i = 1, 2, \dots, m\right).
 \end{aligned}$$

Then use Lemmas 3 and 4 to obtain

$$\begin{aligned}
 & \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(0, t_1, t_2, \tau_1, \tau_2) \\
 (42) \quad & \geq \left(1 - 4\sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}\right) \left(\frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_1}{t_1}\right)^{1/2}\right]\right) \left(\frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_2}{t_2}\right)^{-1/2}\right]\right)
 \end{aligned}$$

and for $a \neq 0$,

$$\begin{aligned}
 & \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \\
 (43) \quad & \geq \left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - 4\sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}\right) \\
 & \cdot \left[\Phi\left(-\frac{6a}{t_1 t_2}\right) - \frac{1}{\pi} \left(\frac{3\tau_1}{t_1}\right)^{1/2} e^{-a^2/3t_1 t_2} + O\left(\frac{3\tau_1}{t_1}\right)\right] \\
 & \cdot \left[\Phi\left(-\frac{6a}{t_1 t_2}\right) - \frac{1}{\pi} \left(\frac{3\tau_2}{t_2}\right)^{1/2} e^{-a^2/3t_1 t_2} + O\left(\frac{3\tau_2}{t_2}\right)\right]
 \end{aligned}$$

where $\alpha = \frac{a}{\sqrt{3t_1 t_2}}$.

THEOREM 2. *An upper bound.*

$$\begin{aligned}
 & \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(0, t_1, t_2, \tau_1, \tau_2) \\
 (44) \quad & \leq \min \left\{ \left(\frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_1}{t_1}\right)^{-1/2}\right]\right), \right. \\
 & \left. \left(\frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_2}{t_2}\right)^{-1/2}\right]\right) \right\}
 \end{aligned}$$

and for $a \neq 0$, τ_1/t_1 small, and τ^2/t_2 small,

$$\begin{aligned}
 & \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \\
 (45) \quad & \leq \min \left\{ \left[\Phi\left(-a\sqrt{\frac{6}{t_1 t_2}}\right) - \frac{1}{\pi} \left(\frac{3\tau_1}{t_1}\right)^{1/2} e^{-a^2/3t_1 t_2} + O\left(\frac{3\tau_1}{t_1}\right)\right], \right. \\
 & \left. \left[\Phi\left(-a\sqrt{\frac{6}{t_1 t_2}}\right) - \frac{1}{\pi} \left(\frac{3\tau_2}{t_2}\right)^{1/2} e^{-a^2/3t_1 t_2} + O\left(\frac{3\tau_2}{t_2}\right)\right] \right\}.
 \end{aligned}$$

Proof of Theorem 2. Observe that

$$(46) \quad \begin{aligned} &P(X_{i,j} > a \text{ for } (i, j) \in C) \\ &\leq \min \{P(X_{1,j} > a, j = 1, \dots, n), P(X_{i,1} > a, i = 1, \dots, m)\}. \end{aligned}$$

For the case $a = 0$, we apply equation (29) to obtain

$$(47) \quad \begin{aligned} &\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P(X_{i,j} > 0 \text{ for } (i, j) \in C) \\ &\leq \min \left\{ \left(\frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_1}{t_1} \right)^{-1/2} \right] \right), \left(\frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_2}{t_2} \right)^{-1/2} \right] \right) \right\}. \end{aligned}$$

For the case $a \neq 0$, apply equation (32) to obtain

$$(48) \quad \begin{aligned} &\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P(X_{i,j} > a \text{ for } (i, j) \in C) \\ &\leq \min \left[\left\{ \Phi \left(-a \sqrt{\frac{6}{t_1 t_2}} \right) - \frac{1}{\pi} \left(\frac{3\tau_1}{t_1} \right)^{1/2} e^{-3a^2/t_1 t_2} + O \left(\frac{3\tau_1}{t_1} \right) \right\}, \right. \\ &\quad \left. \left\{ \Phi \left(-a \sqrt{\frac{6}{t_1 t_2}} \right) - \frac{1}{\pi} \left(\frac{3\tau_2}{t_2} \right)^{1/2} e^{-3a^2/t_1 t_2} + O \left(\frac{3\tau_2}{t_2} \right) \right\} \right] \end{aligned}$$

for small $\frac{\tau_1}{t_1}, \frac{\tau_2}{t_2}$.

4. **Remark.** The method used here may be generalized to apply to the N -parameter Brownian process.

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Received August 19, 1975.

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