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**REARRANGEMENTS OF FUNCTIONS ON THE RING OF  
INTEGERS OF A  $p$ -SERIES FIELD**

BENJAMIN BAXTER WELLS, JR.

## REARRANGEMENTS OF FUNCTIONS ON THE RING OF INTEGERS OF A $p$ -SERIES FIELD

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**We show that every continuous function on the ring of  
 integers of a  $p$ -series field has a rearrangement that has  
 absolutely convergent Fourier series.**

**I. Introduction.** Let  $p$  be a rational prime fixed throughout.  $K$  will denote the  $p$ -series field of formal Laurent series in one variable with finite principal part and with coefficients in  $GF(p)$ . Thus, an element  $x \in K$  has representation as

$$x = \sum a_j p^j \quad (a_j = 0, 1, \dots, p - 1)$$

and  $a_j = 0$  for  $j$  sufficiently small. Addition and multiplication are defined by the usual formal sums and products of Laurent series.

The field  $K$  is topologized by taking as a basis the sets

$$V_{x,k} = \{ \sum b_j p^j : b_j = a_j, j < k \}$$

where  $x = \sum a_j p^j$ . With this topology,  $K$  is locally compact, totally disconnected and nondiscrete.

The ring of integers  $\mathfrak{D} = \{x : x = \sum_{j=0}^{\infty} a_j p^j\}$  is the unique maximal compact subring of  $K$ . Let  $dx$  denote Haar measure on  $K$  derived from the additive structure and normalized so that  $\mathfrak{D}$  has measure 1.

As a locally compact abelian group,  $\mathfrak{D}$  has a Pontryagin dual  $\hat{\mathfrak{D}}$  that may be identified with  $K/\mathfrak{D}$ . We choose the representatives of the form

$$\sum_{-1}^{-\infty} r_j p^j \quad (r_j = 0, 1, \dots, p - 1)$$

and use the lexicographic ordering to match the characters  $\chi_t$  to the nonnegative integers. Of course, if  $\chi$  is a continuous unitary character of  $K^+$ , then  $\chi(x)$  is a  $p$ th root of unity for all  $x \in K$ .

If  $f$  is an integrable function on  $\mathfrak{D}$ , its Fourier coefficients are given by

$$\hat{f}(t) = \int_{\mathfrak{D}} f(x) \bar{\chi}_t(x) dx \quad (t = 0, 1, \dots).$$

We define the class  $A(\mathfrak{D})$  of continuous complex-valued functions on  $\mathfrak{D}$  as those functions  $f$  for which the quantity

$$\sum_{t=0}^{\infty} |\hat{f}(t)|$$

is finite. Under the pointwise operations  $A(\mathfrak{D})$  is an algebra; it is, in fact, a Banach algebra with the above taken as the norm of  $f$ .

Suppose that  $h$  is a homeomorphism of  $\mathfrak{D}$ , and that  $f$  and  $g$  are two functions on  $\mathfrak{D}$  related by

$$g = f \circ h .$$

Then  $g$  is said to be a *rearrangement* of  $f$ . N. Lusin asked whether every continuous function on the circle group has a rearrangement that has absolutely convergent Fourier series (see [4] p. 8). This question is still open; however, see [3] for the best known result. Here we prove the following.

**THEOREM.** *Every continuous function  $f$  on  $\mathfrak{D}$  has a rearrangement  $g$  that has absolutely convergent Fourier series.*

It should be noted that the setting of the theorem contains as a special case ( $p = 2$ ) the classical dyadic group  $2^{\omega}$ .

**II. Preliminaries.** The principal ideal in  $\mathfrak{D}$  generated by  $\mathfrak{p}$ ,  $\mathfrak{P}$ , is the unique maximal ideal in  $\mathfrak{D}$ . There is a non-archimedean valuation  $|\cdot|$  on  $K$  given by setting

$$|\mathfrak{p}| = p^{-1} .$$

$|\cdot|$  satisfies  $|x + y| \leq \text{Max}\{|x|, |y|\}$  ( $x, y \in K$ ), and therefore defines a metric on  $K$ . The topology induced by this metric coincides with that defined earlier.

The fractional ideals  $\mathfrak{p}^{\nu}$  are given by

$$\mathfrak{P}^{\nu} = \{x: |x| \leq p^{-\nu}\} .$$

Now for each  $\nu$ ,  $\mathfrak{D}$  decomposes into  $p^{\nu}$  pairwise disjoint spheres  $\omega(\nu, j)$ , each of measure  $p^{-\nu}$ ,

$$\omega(\nu, j) = x_j + \mathfrak{P}^{\nu} \quad (j = 1, 2, \dots, p^{\nu}) .$$

We assume that the  $x_j$  are ordered lexicographically. Thus, consecutive blocks of length  $p^{\nu-1}$  have the same coefficient of the  $\mathfrak{P}^{\nu}$  term, consecutive blocks of length  $p^{\nu-2}$  have the same coefficient of the  $\mathfrak{P}^{\nu-1}$  and  $\mathfrak{P}^{\nu}$  terms, etc.

Consequently, we have the containments

$$\omega(\nu + 1, j) \subset \omega(\nu, k) , \quad ((k - 1)p + 1 \leq j \leq kp) .$$

In our construction of a homeomorphism of  $\mathfrak{D}$  it will be necessary to make repeated use of the fact that two compact, totally discon-

nected, metrizable, and perfect spaces are homeomorphic (see [1] p. 97).

From now on, since the prime number  $p$  will frequently occur exponentiated and subscripted, for typographical reasons we shall write  $p(\nu)$  for  $p^\nu$ .

III. LEMMA. *Suppose that  $g$  is a continuous complex-valued function defined on  $\mathfrak{D}$ . Then  $g$  is an  $A$ -function if the series whose  $n$ th term ( $n = 0, 1, 2, \dots$ ) is given by*

$$(1) \quad p(n)(p - 1) \sum_{i=1}^{p(n)} \min_{b_i} \int_{\omega(n, i)} |g(x) - b_i| dx$$

is convergent. If  $M$  denotes the sum of this series, then  $\|g\|_A \leq M + \|g\|_\infty$ .

*Proof.* Suppose that  $g$  is locally constant on  $\mathfrak{D}$  and takes the value  $a_j$  on  $\omega(\nu, j)$ , ( $j = 1, \dots, p(\nu)$ ). Then

$$(2) \quad \hat{g}(t) = \int_{\mathfrak{D}} g \bar{\chi}_t dx = \sum_{k=1}^{p(\nu)} a_k \int_{\omega(\nu, k)} \bar{\chi}_t dx.$$

Now, if  $t \geq p(\nu)$ , it follows from the orthogonality relations (see [2] p. 613) that  $\hat{g}(t) = 0$ . Suppose that  $0 \leq t \leq p(\nu)$ , and therefore that  $\chi_t$  is a character identified with a representative of  $K/\mathfrak{D}$  of the form

$$(3) \quad \sum_{j=-1}^{-\nu} r_j p^j \quad (r_j = 0, 1, \dots, p - 1).$$

There are  $p(n)(p - 1)$  characters corresponding to the representatives (3) with  $r_j = 0, j < -n - 2, r_{-n-1} \neq 0, -1 < n < \nu$ .

Consider the sum

$$(4) \quad \sum_{t=0}^{p(\nu)-1} |\hat{g}(t)|.$$

In order to estimate (4), let  $\chi_t$  be a character corresponding to (3) with  $r_{-\nu} = r_{-\nu+1} = \dots = r_{-n-2} = 0, r_{-n-1} \neq 0$ , and  $-1 < n$ . From (2) we see that

$$(5) \quad \begin{aligned} p(\nu)\hat{g}(t) &= \{A_1^1 z^{1+q_1} + \dots + A_p^1 z^{p+q_1}\} \\ &+ \dots \\ &+ \{A_1^{p(n)} z^{1+q} + \dots + A_p^{p(n)} z^{p+q}\}, \end{aligned}$$

where the  $A$ 's are the sums of consecutive blocks of the  $a$ 's of length  $p(\nu - (n + 1))$ .

$$\begin{aligned} A_1^1 &= a_1 + \dots + a_{p(\nu-(n+1))} \\ &\dots \\ A_p^{p(n)} &= a_{p(\nu)-p(\nu-(n+1))+1} + \dots + a_{p(\nu)}. \end{aligned}$$

Furthermore,  $z \neq 1$  is a  $p$ th root of unity, and  $q_1, \dots, q_{p^{(n)}}$  are positive integers which depend on  $\chi_t$ .

Since the sum of  $p$  successive powers of a  $p$ th root of unity  $\neq 1$  is zero, we see that for arbitrary complex numbers  $b_1, \dots, b_{p^{(n)}}$

$$(6) \quad \hat{g}(t) = \hat{g}(t) - p(\nu - (n + 1))b_1(z + \dots + z^p) - \dots - p(\nu - (n + 1))b_{p^{(n)}}(z + \dots + z^p).$$

Combining (5) and (6) and applying the triangle inequality, we see that

$$(7) \quad |\hat{g}(t)| \leq \left\{ \sum_{k=1}^{p^{(\nu-n)}} |a_k - b_1| + \dots + \sum_{k=p^{(\nu)}-p^{(\nu-n)}+1}^{p^{(\nu)}} |a_k - b_{p^{(n)}}| \right\} 1/p^{(\nu)}.$$

However, the right hand side of (7) is just

$$\sum_{i=1}^{p^{(n)}} \int_{\omega(n,i)} |g(x) - b_i| dx.$$

Since there are  $p^{(n)}(p - 1)$  characters  $\chi_t$  of the type under consideration, the lemma is proved in the case that  $g$  is locally constant.

Now assume that  $g$  is an arbitrary continuous function on  $\mathfrak{D}$  which satisfies the hypothesis of the lemma. Let  $N$  be a fixed positive integer, and approximate  $g$  uniformly on  $\mathfrak{D}$  by a sequence  $g_m$  of locally constant continuous functions. Now, for every choice of integer  $n$  and complex numbers  $b_j (1 \leq j \leq p^{(n)})$  we have

$$(8) \quad \sum_{j=1}^{p^{(n)}} \int_{\omega(n,j)} |g_m(x) - b_j| dx \longrightarrow \sum_{j=1}^{p^{(n)}} \int_{\omega(n,j)} |g(x) - b_j| dx$$

as  $m \rightarrow \infty$ . Since the left hand side of (8) bounds  $|\hat{g}_m(t)|$ , where  $\chi_t$  is a character corresponding to (3) with  $r_j = 0, j < -(n + 1), r_{-n-1} \neq 0$ , it follows that for arbitrary  $\varepsilon > 0$  that

$$\sum_{t=0}^N |\hat{g}_m(t)| < M + \|g\|_\infty + \varepsilon$$

when  $m$  is sufficiently large. Furthermore, since for each  $t, \hat{g}_m(t) \rightarrow \hat{g}(t)$  as  $m \rightarrow \infty$ , we conclude that

$$\sum_{t=0}^N |\hat{g}(t)| \leq M + \|g\|_\infty + \varepsilon.$$

Since  $N$  and  $\varepsilon$  are arbitrary, the lemma is proved.

**IV. Proof of the theorem.** Suppose without loss of generality that  $\|f\|_\infty = 1$ ; we show how to construct a homeomorphism  $h$  of  $\mathfrak{D}$  such that  $g = f \circ h$  satisfies the hypothesis of the lemma. Thus we will have rearrangement of  $f$  whose Fourier series converges absolutely.

We shall construct  $h$  as a limit of homeomorphisms  $H_n$

$$h = \lim_n H_n$$

where  $H_n$  is a composition of  $n$  homeomorphisms of  $\mathfrak{D}$ ,  $h_1 \circ h_2 \circ \dots \circ h_n$ . We begin by describing the construction of the  $h$ 's.

For  $U \subset \mathfrak{D}$ , it will be convenient to use the following notation

$$O_f(U) = \sup_{x, y \in U} |f(x) - f(y)|.$$

The quantity  $O_f(U)$  is referred to as the *oscillation* of  $f$  on  $U$ .

Choose a partition of  $\mathfrak{D}$  consisting of mutually disjoint, nonvoid, open and closed sets  $U_j (1 \leq j \leq p + 1)$  such that the oscillation of  $f$  on the union of the  $U_j (1 \leq j \leq p)$  is less than or equal  $1/p(3)$ . Thus,

$$O_f\left(\bigcup_{j=1}^p U_j\right) \leq 1/p(3).$$

Then take  $h_1$  to be a homeomorphism of  $\mathfrak{D}$  satisfying the following requirements

$$\begin{aligned} h_1(\omega(1, j)) &= U_j \quad (1 \leq j \leq p - 1) \\ h_1(\omega(1, p) \setminus \omega(3, p(3))) &= U_p \\ h_1(\omega(3, p(3))) &= U_{p+1}. \end{aligned}$$

Now suppose that  $h_1, \dots, h_{n-1}$  are homeomorphisms of  $\mathfrak{D}$  that have been defined. Set  $H_{n-1} = h_1 \circ h_2 \circ \dots \circ h_{n-1}$ .

We now turn to the definition of  $h_n$ . For  $i = 1, \dots, p(n - 1)$  let  $U_{i,j} (1 \leq j \leq p + 1)$  denote a partition of  $\omega(n - 1, i)$  into open and closed sets such that the following are satisfied.

$$(9) \quad O_{f \circ H_{n-1}}\left(\bigcup_{j=1}^p U_{i,j}\right) \leq 1/p(2n + 1) \quad (i = 1, 2, \dots, p(n - 1))$$

$$(10) \quad \omega(3(n - 1), ip(2(n - 1) + 1)) \subset U_{ip, p+1} \quad (i = 1, 2, \dots, p(n - 2)).$$

Then take  $h_n$  to be a homeomorphism of  $\mathfrak{D}$  satisfying the following requirements ( $i = 1, 2, \dots, p(n - 1)$ )

$$(11) \quad h_n(\omega(n, k)) = U_{i,j} \quad (k = (i - 1)p + j, 1 \leq j \leq p - 1)$$

$$(12) \quad h_n(\omega(n, ip) \setminus \omega(3n, ip(2n + 1))) = U_{i,p}$$

$$(13) \quad h_n(\omega(3n, ip(2n + 1))) = U_{i, p+1}.$$

Finally, we set  $H_n = H_{n-1} \circ h_n$ .

First, we observe that

$$(14) \quad h_n \omega(n - 1, i) = \omega(n - 1, i) \quad (i = 1, 2, \dots, p(n - 1)).$$

From (14) we see that for every neighborhood  $V$  of  $0$ ,  $h_n(x)$  and  $h_n^{-1}(x)$  belong to  $V + x$  for  $n$  sufficiently large. From this follows the existence of the limits

$$\lim_n H_n = h, \quad \lim_n H_n^{-1} = h^{-1}.$$

Again, from (14) the continuity of  $h$  is clear. Therefore  $h$  is a well-defined homeomorphism of  $\mathfrak{D}$ .

Set  $g = f \circ h$ . The function  $g$  is then our rearrangement of  $f$ , and it remains to check that series described in the lemma is convergent.

Now, the inequalities

$$O_{f \circ H_n}(\omega(n, j)) \leq 1/p(2n + 1) \quad (j = (i - 1)p + k, 1 \leq k \leq p - 1, \\ i = 1, \dots, p(n - 1))$$

follow immediately from (9) and (11). Successive application of (14) therefore yields

$$(15) \quad O_g(\omega(n, j)) \leq 1/p(2n + 1) \quad (j = (i - 1)p + k, 1 \leq k \leq p - 1, \\ i = 1, \dots, p(n - 1))$$

The inequalities

$$(16) \quad O_{f \circ H_n}(\omega(n, ip) \setminus \omega(3n, ip(2n + 1))) \leq 1/p(2n + 1) \\ (i = 1, 2, \dots, p(n - 1))$$

follow from (9) and (12). Relation (10) (with  $n - 1$  replaced by  $n$  and the fact that  $\omega(3n, ip(2n + 1)) \supset \omega(3(n + 1), i'p(2(n + 1) + 1))$ , where  $i' = ip$ , imply that (16) holds with  $H_n$  replaced by  $H_{n+1}$ . This last step may be successively repeated to obtain

$$(17) \quad O_{f \circ H_m}(\omega(n, ip) \setminus \omega(3n, ip(2n + 1))) \leq 1/p(2n + 1) \quad (n \leq m).$$

However, since  $f \circ H_m$  tends uniformly to  $g$  we see that

$$(18) \quad O_g(\omega(n, ip) \setminus \omega(3n, ip(2n + 1))) \leq 1/p(2n + 1).$$

From (15) and the fact that the measure of  $\omega(n, j)$  is  $p(-n)$  we obtain the inequalities

$$(19) \quad \min_{b_j} \int_{\omega(n, j)} |g(x) - b_j| dx \leq 1/\{p(2n + 1)p(n)\} \\ (j = (i - 1)p + k, 1 \leq k \leq p - 1, i = 1, \dots, p(n - 1)).$$

From (18) we deduce that

$$(20) \quad \min_{b_j} \int_{\omega(n, j)} |g(x) - b_j| dx \leq 1/\{p(2n + 1)p(n)\} + 2/p(3n) \\ (j = ip, i = 1, \dots, p(n - 1)).$$

We consider now the  $n$ th term of the series described in the lemma. Combining (19) and (20) we obtain the inequality

$$\begin{aligned} p(n)(p-1) \sum_{j=1}^{p(n)} \min_{b_j} \int_{\omega(n,j)} |g(x) - b_j| dx &\leq p(n)(p-1)\{1/p(2n+1) \\ &\quad + 2p(n-1)/p(3n)\} \\ &\leq 3/p(n). \end{aligned}$$

Therefore, the series of the lemma is convergent. The proof of the theorem is now complete.

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