

Pacific Journal of Mathematics

REARRANGEMENTS OF FUNCTIONS ON THE RING OF INTEGERS OF A p -SERIES FIELD

BENJAMIN BAXTER WELLS, JR.

REARRANGEMENTS OF FUNCTIONS ON THE RING OF INTEGERS OF A p -SERIES FIELD

BENJAMIN B. WELLS, JR.

We show that every continuous function on the ring of integers of a p -series field has a rearrangement that has absolutely convergent Fourier series.

I. Introduction. Let p be a rational prime fixed throughout. K will denote the p -series field of formal Laurent series in one variable with finite principal part and with coefficients in $GF(p)$. Thus, an element $x \in K$ has representation as

$$x = \sum a_j p^j \quad (a_j = 0, 1, \dots, p - 1)$$

and $a_j = 0$ for j sufficiently small. Addition and multiplication are defined by the usual formal sums and products of Laurent series.

The field K is topologized by taking as a basis the sets

$$V_{x,k} = \{ \sum b_j p^j : b_j = a_j, j < k \}$$

where $x = \sum a_j p^j$. With this topology, K is locally compact, totally disconnected and nondiscrete.

The ring of integers $\mathfrak{D} = \{x : x = \sum_{j=0}^{\infty} a_j p^j\}$ is the unique maximal compact subring of K . Let dx denote Haar measure on K derived from the additive structure and normalized so that \mathfrak{D} has measure 1.

As a locally compact abelian group, \mathfrak{D} has a Pontryagin dual $\hat{\mathfrak{D}}$ that may be identified with K/\mathfrak{D} . We choose the representatives of the form

$$\sum_{j=1}^v r_j p^j \quad (r_j = 0, 1, \dots, p - 1)$$

and use the lexicographic ordering to match the characters χ_t to the nonnegative integers. Of course, if χ is a continuous unitary character of K^+ , then $\chi(x)$ is a p th root of unity for all $x \in K$.

If f is an integrable function on \mathfrak{D} , its Fourier coefficients are given by

$$\hat{f}(t) = \int_{\mathfrak{D}} f(x) \bar{\chi}_t(x) dx \quad (t = 0, 1, \dots).$$

We define the class $A(\mathfrak{D})$ of continuous complex-valued functions on \mathfrak{D} as those functions f for which the quantity

$$\sum_{t=0}^{\infty} |\hat{f}(t)|$$

is finite. Under the pointwise operations $A(\mathfrak{D})$ is an algebra; it is, in fact, a Banach algebra with the above taken as the norm of f .

Suppose that h is a homeomorphism of \mathfrak{D} , and that f and g are two functions on \mathfrak{D} related by

$$g = f \circ h .$$

Then g is said to be a *rearrangement* of f . N. Lusin asked whether every continuous function on the circle group has a rearrangement that has absolutely convergent Fourier series (see [4] p. 8). This question is still open; however, see [3] for the best known result. Here we prove the following.

THEOREM. *Every continuous function f on \mathfrak{D} has a rearrangement g that has absolutely convergent Fourier series.*

It should be noted that the setting of the theorem contains as a special case ($p = 2$) the classical dyadic group 2^{ω} .

II. Preliminaries. The principal ideal in \mathfrak{D} generated by \mathfrak{p} , \mathfrak{P} , is the unique maximal ideal in \mathfrak{D} . There is a non-archimedian valuation $|\cdot|$ on K given by setting

$$|\mathfrak{p}| = p^{-1} .$$

$|\cdot|$ satisfies $|x + y| \leq \text{Max}\{|x|, |y|\}$ ($x, y \in K$), and therefore defines a metric on K . The topology induced by this metric coincides with that defined earlier.

The fractional ideals \mathfrak{p}^{ν} are given by

$$\mathfrak{P}^{\nu} = \{x: |x| \leq p^{-\nu}\} .$$

Now for each ν , \mathfrak{D} decomposes into p^{ν} pairwise disjoint spheres $\omega(\nu, j)$, each of measure $p^{-\nu}$,

$$\omega(\nu, j) = x_j + \mathfrak{P}^{\nu} \quad (j = 1, 2, \dots, p^{\nu}) .$$

We assume that the x_j are ordered lexicographically. Thus, consecutive blocks of length $p^{\nu-1}$ have the same coefficient of the \mathfrak{P}^{ν} term, consecutive blocks of length $p^{\nu-2}$ have the same coefficient of the \mathfrak{P}^{ν} and $\mathfrak{P}^{\nu-1}$ terms, etc.

Consequently, we have the containments

$$\omega(\nu + 1, j) \subset \omega(\nu, k) , \quad ((k - 1)p + 1 \leq j \leq kp) .$$

In our construction of a homeomorphism of \mathfrak{D} it will be necessary to make repeated use of the fact that two compact, totally discon-

nected, metrizable, and perfect spaces are homeomorphic (see [1] p. 97).

From now on, since the prime number p will frequently occur exponentiated and subscripted, for typographical reasons we shall write $p(\nu)$ for p^ν .

III. LEMMA. *Suppose that g is a continuous complex-valued function defined on \mathfrak{D} . Then g is an A -function if the series whose n th term ($n = 0, 1, 2, \dots$) is given by*

$$(1) \quad p(n)(p - 1) \sum_{i=1}^{p(n)} \min_{b_i} \int_{\omega(n, i)} |g(x) - b_i| dx$$

is convergent. If M denotes the sum of this series, then $\|g\|_A \leq M + \|g\|_\infty$.

Proof. Suppose that g is locally constant on \mathfrak{D} and takes the value a_j on $\omega(\nu, j)$, ($j = 1, \dots, p(\nu)$). Then

$$(2) \quad \hat{g}(t) = \int_{\mathfrak{D}} g \bar{\chi}_t dx = \sum_{k=1}^{p(\nu)} a_k \int_{\omega(\nu, k)} \bar{\chi}_t dx.$$

Now, if $t \geq p(\nu)$, it follows from the orthogonality relations (see [2] p. 613) that $\hat{g}(t) = 0$. Suppose that $0 \leq t \leq p(\nu)$, and therefore that χ_t is a character identified with a representative of K/\mathfrak{D} of the form

$$(3) \quad \sum_{j=-1}^{-\nu} r_j p^j \quad (r_j = 0, 1, \dots, p - 1).$$

There are $p(n)(p - 1)$ characters corresponding to the representatives (3) with $r_j = 0$, $j < -n - 2$, $r_{-n-1} \neq 0$, $-1 < n < \nu$.

Consider the sum

$$(4) \quad \sum_{t=0}^{p(\nu)-1} |\hat{g}(t)|.$$

In order to estimate (4), let χ_t be a character corresponding to (3) with $r_{-\nu} = r_{-\nu+1} = \dots = r_{-n-2} = 0$, $r_{-n-1} \neq 0$, and $-1 < n$. From (2) we see that

$$(5) \quad \begin{aligned} p(\nu)\hat{g}(t) &= \{A_1^1 z^{1+q_1} + \dots + A_p^1 z^{p+q_1}\} \\ &+ \dots \\ &+ \{A_1^{p(n)} z_{p(n)}^{1+q} + \dots + A_p^{p(n)} z_{p(n)}^{p+q}\}, \end{aligned}$$

where the A 's are the sums of consecutive blocks of the a 's of length $p(\nu - (n + 1))$.

$$\begin{aligned} A_1^1 &= a_1 + \dots + a_{p(\nu - (n + 1))} \\ &\dots \\ A_p^{p(n)} &= a_{p(\nu) - p(\nu - (n + 1)) + 1} + \dots + a_{p(\nu)}. \end{aligned}$$

Furthermore, $z \neq 1$ is a p th root of unity, and $q_1, \dots, q_{p(n)}$ are positive integers which depend on χ_t .

Since the sum of p successive powers of a p th root of unity $\neq 1$ is zero, we see that for arbitrary complex numbers $b_1, \dots, b_{p(n)}$

$$(6) \quad \hat{g}(t) = \hat{g}(t) - p(\nu - (n + 1))b_1(z + \dots + z^p) - \dots - p(\nu - (n + 1))b_{p(n)}(z + \dots + z^p).$$

Combining (5) and (6) and applying the triangle inequality, we see that

$$(7) \quad |\hat{g}(t)| \leq \left\{ \sum_{k=1}^{p(\nu-n)} |a_k - b_1| + \dots + \sum_{k=p(\nu)-p(\nu-n)+1}^{p(\nu)} |a_k - b_{p(n)}| \right\} 1/p(\nu).$$

However, the right hand side of (7) is just

$$\sum_{i=1}^{p(n)} \int_{\omega(n,i)} |g(x) - b_i| dx.$$

Since there are $p(n)(p - 1)$ characters χ_t of the type under consideration, the lemma is proved in the case that g is locally constant.

Now assume that g is an arbitrary continuous function on \mathfrak{D} which satisfies the hypothesis of the lemma. Let N be a fixed positive integer, and approximate g uniformly on \mathfrak{D} by a sequence g_m of locally constant continuous functions. Now, for every choice of integer n and complex numbers $b_j (1 \leq j \leq p(n))$ we have

$$(8) \quad \sum_{j=1}^{p(n)} \int_{\omega(n,j)} |g_m(x) - b_j| dx \longrightarrow \sum_{j=1}^{p(n)} \int_{\omega(n,j)} |g(x) - b_j| dx$$

as $m \rightarrow \infty$. Since the left hand side of (8) bounds $|\hat{g}_m(t)|$, where χ_t is a character corresponding to (3) with $r_j = 0, j < -(n + 1), r_{-n-1} \neq 0$, it follows that for arbitrary $\varepsilon > 0$ that

$$\sum_{t=0}^N |\hat{g}_m(t)| < M + \|g\|_\infty + \varepsilon$$

when m is sufficiently large. Furthermore, since for each $t, \hat{g}_m(t) \rightarrow \hat{g}(t)$ as $m \rightarrow \infty$, we conclude that

$$\sum_{t=0}^N |\hat{g}(t)| \leq M + \|g\|_\infty + \varepsilon.$$

Since N and ε are arbitrary, the lemma is proved.

IV. Proof of the theorem. Suppose without loss of generality that $\|f\|_\infty = 1$; we show how to construct a homeomorphism h of \mathfrak{D} such that $g = f \circ h$ satisfies the hypothesis of the lemma. Thus we will have rearrangement of f whose Fourier series converges absolutely.

We shall construct h as a limit of homeomorphisms H_n

$$h = \lim_n H_n$$

where H_n is a composition of n homeomorphisms of \mathfrak{D} , $h_1 \circ h_2 \circ \dots \circ h_n$. We begin by describing the construction of the h 's.

For $U \subset \mathfrak{D}$, it will be convenient to use the following notation

$$O_f(U) = \sup_{x, y \in U} |f(x) - f(y)|.$$

The quantity $O_f(U)$ is referred to as the *oscillation* of f on U .

Choose a partition of \mathfrak{D} consisting of mutually disjoint, nonvoid, open and closed sets $U_j (1 \leq j \leq p + 1)$ such that the oscillation of f on the union of the $U_j (1 \leq j \leq p)$ is less than or equal $1/p(3)$. Thus,

$$O_f\left(\bigcup_{j=1}^p U_j\right) \leq 1/p(3).$$

Then take h_1 to be a homeomorphism of \mathfrak{D} satisfying the following requirements

$$\begin{aligned} h_1(\omega(1, j)) &= U_j \quad (1 \leq j \leq p - 1) \\ h_1(\omega(1, p) \setminus \omega(3, p(3))) &= U_p \\ h_1(\omega(3, p(3))) &= U_{p+1}. \end{aligned}$$

Now suppose that h_1, \dots, h_{n-1} are homeomorphisms of \mathfrak{D} that have been defined. Set $H_{n-1} = h_1 \circ h_2 \circ \dots \circ h_{n-1}$.

We now turn to the definition of h_n . For $i = 1, \dots, p(n - 1)$ let $U_{i,j} (1 \leq j \leq p + 1)$ denote a partition of $\omega(n - 1, i)$ into open and closed sets such that the following are satisfied.

$$(9) \quad O_{f \circ H_{n-1}}\left(\bigcup_{j=1}^p U_{i,j}\right) \leq 1/p(2n + 1) \quad (i = 1, 2, \dots, p(n - 1))$$

$$(10) \quad \omega(3(n - 1), ip(2(n - 1) + 1)) \subset U_{ip, p+1} \quad (i = 1, 2, \dots, p(n - 2)).$$

Then take h_n to be a homeomorphism of \mathfrak{D} satisfying the following requirements ($i = 1, 2, \dots, p(n - 1)$)

$$(11) \quad h_n(\omega(n, k)) = U_{i,j} \quad (k = (i - 1)p + j, 1 \leq j \leq p - 1)$$

$$(12) \quad h_n(\omega(n, ip) \setminus \omega(3n, ip(2n + 1))) = U_{i,p}$$

$$(13) \quad h_n(\omega(3n, ip(2n + 1))) = U_{i, p+1}.$$

Finally, we set $H_n = H_{n-1} \circ h_n$.

First, we observe that

$$(14) \quad h_n \omega(n - 1, i) = \omega(n - 1, i) \quad (i = 1, 2, \dots, p(n - 1)).$$

From (14) we see that for every neighborhood V of 0, $h_n(x)$ and $h_n^{-1}(x)$ belong to $V + x$ for n sufficiently large. From this follows the existence of the limits

$$\lim_n H_n = h, \quad \lim_n H_n^{-1} = h^{-1}.$$

Again, from (14) the continuity of h is clear. Therefore h is a well-defined homeomorphism of \mathfrak{D} .

Set $g = f \circ h$. The function g is then our rearrangement of f , and it remains to check that series described in the lemma is convergent.

Now, the inequalities

$$O_{f \circ H_n}(\omega(n, j)) \leq 1/p(2n + 1) \quad (j = (i - 1)p + k, 1 \leq k \leq p - 1, \\ i = 1, \dots, p(n - 1))$$

follow immediately from (9) and (11). Successive application of (14) therefore yields

$$(15) \quad O_g(\omega(n, j)) \leq 1/p(2n + 1) \quad (j = (i - 1)p + k, 1 \leq k \leq p - 1, \\ i = 1, \dots, p(n - 1))$$

The inequalities

$$(16) \quad O_{f \circ H_n}(\omega(n, ip) \setminus \omega(3n, ip(2n + 1))) \leq 1/p(2n + 1) \\ (i = 1, 2, \dots, p(n - 1))$$

follow from (9) and (12). Relation (10) (with $n - 1$ replaced by n) and the fact that $\omega(3n, ip(2n + 1)) \supset \omega(3(n + 1), i'p(2(n + 1) + 1))$, where $i' = ip$, imply that (16) holds with H_n replaced by H_{n+1} . This last step may be successively repeated to obtain

$$(17) \quad O_{f \circ H_m}(\omega(n, ip) \setminus \omega(3n, ip(2n + 1))) \leq 1/p(2n + 1) \quad (n \leq m).$$

However, since $f \circ H_m$ tends uniformly to g we see that

$$(18) \quad O_g(\omega(n, ip) \setminus \omega(3n, ip(2n + 1))) \leq 1/p(2n + 1).$$

From (15) and the fact that the measure of $\omega(n, j)$ is $p(-n)$ we obtain the inequalities

$$(19) \quad \min_{b_j} \int_{\omega(n, j)} |g(x) - b_j| dx \leq 1/\{p(2n + 1)p(n)\} \\ (j = (i - 1)p + k, 1 \leq k \leq p - 1, i = 1, \dots, p(n - 1)).$$

From (18) we deduce that

$$(20) \quad \min_{b_j} \int_{\omega(n, j)} |g(x) - b_j| dx \leq 1/\{p(2n + 1)p(n)\} + 2/p(3n) \\ (j = ip, i = 1, \dots, p(n - 1)).$$

We consider now the n th term of the series described in the lemma. Combining (19) and (20) we obtain the inequality

$$\begin{aligned} p(n)(p-1) \sum_{j=1}^{p(n)} \min_{b_j} \int_{\omega(n,j)} |g(x) - b_j| dx &\leq p(n)(p-1)\{1/p(2n+1) \\ &\quad + 2p(n-1)/p(3n)\} \\ &\leq 3/p(n). \end{aligned}$$

Therefore, the series of the lemma is convergent. The proof of the theorem is now complete.

REFERENCES

1. J. C. Hocking and G. S. Young, *Topology*, Addison-Wesley (1961) Reading, Mass.
2. R. A. Hunt and M. H. Taibleson, *Almost everywhere convergence of Fourier series on the ring of integers of a local field*, SIAM J. Math. Anal., **2** (1971), 607-625.
3. J.-P. Kahane, *Sur les réarrangements de fonctions de la classe A*, Studia Math., **31** (1968), 287-293.
4. ———, *Séries, de Fourier Absolument Convergentes*, *Ergebnisse, Band 50*, Springer-Verlag, 1970.
5. M. H. Taibleson, *Fourier series on the ring of integers in a p -series field*, Bull. Amer. Math. Soc., **73** (1967), 623-629.

Received October 10, 1974.

UNIVERSITY OF HAWAII, HONOLULU

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

David Lee Armacost, <i>Compactly cogenerated LCA groups</i>	1
Sun Man Chang, <i>On continuous image averaging of probability measures</i>	13
J. Chidambaraswamy, <i>Generalized Dedekind ψ-functions with respect to a polynomial. II</i>	19
Freddy Delbaen, <i>The Dunford-Pettis property for certain uniform algebras</i>	29
Robert Benjamin Feinberg, <i>Faithful distributive modules over incidence algebras</i>	35
Paul Froeschl, <i>Chained rings</i>	47
John Brady Garnett and Anthony G. O'Farrell, <i>Sobolev approximation by a sum of subalgebras on the circle</i>	55
Hugh M. Hilden, José M. Montesinos and Thomas Lusk Thickstun, <i>Closed oriented 3-manifolds as 3-fold branched coverings of S^3 of special type</i>	65
Atsushi Inoue, <i>On a class of unbounded operator algebras</i>	77
Peter Kleinschmidt, <i>On facets with non-arbitrary shapes</i>	97
Narendrakumar Ramanlal Ladhawala, <i>Absolute summability of Walsh-Fourier series</i>	103
Howard Wilson Lambert, <i>Links which are unknottable by maps</i>	109
Kyung Bai Lee, <i>On certain g-first countable spaces</i>	113
Richard Ira Loeb, <i>A Hahn decomposition for linear maps</i>	119
Moshe Marcus and Victor Julius Mizel, <i>A characterization of non-linear functionals on W_1^p possessing autonomous kernels. I</i>	135
James Miller, <i>Subordinating factor sequences and convex functions of several variables</i>	159
Keith Pierce, <i>Amalgamated sums of abelian l-groups</i>	167
Jonathan Rosenberg, <i>The C^*-algebras of some real and p-adic solvable groups</i>	175
Hugo Rossi and Michele Vergne, <i>Group representations on Hilbert spaces defined in terms of ∂_b-cohomology on the Silov boundary of a Siegel domain</i>	193
Mary Elizabeth Schaps, <i>Nonsingular deformations of a determinantal scheme</i>	209
S. R. Singh, <i>Some convergence properties of the Bubnov-Galerkin method</i>	217
Peggy Strait, <i>Level crossing probabilities for a multi-parameter Brownian process</i>	223
Robert M. Tardiff, <i>Topologies for probabilistic metric spaces</i>	233
Benjamin Baxter Wells, Jr., <i>Rearrangements of functions on the ring of integers of a p-series field</i>	253
Robert Francis Wheeler, <i>Well-behaved and totally bounded approximate identities for $C_0(X)$</i>	261
Delores Arletta Williams, <i>Gauss sums and integral quadratic forms over local fields of characteristic 2</i>	271
John Yuan, <i>On the construction of one-parameter semigroups in topological semigroups</i>	285