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The theory of Gauss sums is developed for integral quadratic forms over a local field of characteristic 2, and Gauss sums are used to characterize these forms. For a character χ and an integral lattice L, the Gauss sum $\chi(L)$ is either zero, a nonnegative power of two, or the negative of a positive power of two. Gauss sums alone characterize the integral equivalence classes for modular lattices. For arbitrary lattices, other invariants are required.

The classification given in this paper is an alternate to the one by C.-H. Sah [6]. The notation and terminology of [6] is used except when stated to the contrary. O. T. O'Meara [4] used Gauss sums to characterize local integral quadratic forms over a field of characteristic not 2, and R. Jacobowitz [3] classified hermitian forms over the integers of a local field of characteristic not 2 by Gauss sums. When needed, results from these papers are referred to when the proofs hold for the characteristic 2 case.

After a few preliminaries, we introduce Gauss sums and prove some results for Gauss sums of lines and planes that will in turn be used to study more complicated lattices. In Theorem 5.4 we show that Gauss sums alone are sufficient to characterize modular lattices, and in Theorem 4.2 we show that for nondefective lattices only a finite number of Gauss sums need be considered.

1. Preliminaries. Throughout this paper k denotes a local field of characteristic 2 with fixed prime element π, ring of integers $\mathfrak{o}$, and residue class field of order 2^f. We let Ω denote a complete set of representatives for the residue class field. We refer the reader to [6] for a discussion of the Arf invariant ΔV for a quadratic space V and the additive group $\mathscr{P}\Omega$. As in [6] we let $\{0, \lambda\}$ be a fixed set of representatives of $\Omega/\mathscr{P}\Omega$. The letter e always denotes a unit of k. For a nonnegative integer s, H_s and H'_s denote s-hyperbolic lattices.

Let L be a lattice. $K(L) = \{x \in L \mid \langle x, y\rangle = 0 \text{ for all } y \in L\}$. If $K(L) = 0$, then L is nondefective. Otherwise, it is defective. We assume that if $x \in K(L)$ and $Q(x) = 0$, then $x = 0$.

We now state some lemmas and definitions from [6] in the form in which they are used in this paper.

LEMMA 1.1. *Let L be a lattice with $(qL)\mathfrak{o} = \pi^u\mathfrak{o}$.*

(1) *If rank* ${}_{\mathfrak{o}^2}q(L) = 1$, *then* $L = K(L)$ *and* $q(L) = e\pi^u\mathfrak{o}^2$ *with* e *a unit.*

(2) *If rank* ${}_{\mathfrak{o}^2}q(L) \geqq 2$, *then* $v = \min\{\mathrm{ord}\, a \mid a \in q(L)$ *and* $\mathrm{ord}\, a \equiv u + 1$ *mod* $2\}$ *is a rational integer with* $u < v$. *If, in addition,* $L \neq K(L)$, $v \leqq s + 1$ *with* $sL = \pi^s\mathfrak{o}$. *Moreover, for any unit* e *with* $e\pi^u \in Q(L)$, $q(L) = e(\pi^u\mathfrak{o}^2 + \pi^v\mathfrak{o}^2)$; *and there is at least one such* e.

If more that one lattice is under consideration, the invariants u and v for L are denoted by u_L and v_L respectively.

As in [6], for a rational integer i, $u(i)$ and $v(i)$ denote the u and v of Lemma 1.1 for the lattice $L(i) \otimes H_i$.

LEMMA 1.2. *Let L be an i-modular lattice such that* rank $L \geqq 4$ *and* $q(L) = e(\pi^u\mathfrak{o}^2 + \pi^v\mathfrak{o}^2)$ *with* $u < v \leqq i + 1$, $u + v \equiv 1$ mod 2, *and* $e\pi^u \in Q(L)$ (as in Lemma 1.1). *Then* $L \simeq \begin{pmatrix} & \pi^i \\ e\pi^v & e^{-1}\pi^{2i-u}\delta + e^{-1}\pi^v a^2 \end{pmatrix} \oplus \begin{pmatrix} & \pi^i \\ e\pi^v & e^{-1}\pi^{2i-v}\delta' \end{pmatrix} \oplus H$ *with* $\delta, \delta' \in \{0, \lambda\}$, H *i-hyperbolic or* 0, *and* $\delta' = 0$ *when* $v = i + 1$. *Moreover,* $q(L) = Q(L)$.

For a canonical decomposition $L = (\sum_{i=0}^t L_i) \oplus K(L)$, $s(i)$ always denotes $s(L_i)$.

DEFINITION 1.3. Let $L = (\sum_{i=0}^t L_i) \oplus K(L)$ be a canonical decomposition. Define $I = s(0)$ and $T = s(t) + \max\{u', v'\}$, where

$$u' = \begin{cases} u_{K(L)} - u(s(t)) & \text{if } L \text{ is defective} \\ 0 & \text{otherwise} \end{cases}$$

and

$$v' = \begin{cases} v_{K(L)} - v(s(t)) & \text{if rank } {}_{\mathfrak{o}^2}q(K(L)) = 2 \\ 0 & \text{otherwise} \end{cases},$$

and defined $R = (r(I), r(I + 1), \cdots, r(T))$ where

$$r(j) = \begin{cases} \text{rank } L_i & \text{if } s(i) - j \text{ for some } i \in \{0, \cdots, t\} \\ 0 & \text{otherwise} \end{cases}.$$

Then L is said to be of *type* (I, T, R) and *length* $(t + 1)$.

DEFINITION 1.4. For $L \neq K(L)$, let $L = (\sum_{i=0}^t L_i) \oplus K(L)$ be a canonical decomposition of L. If $s(0), \cdots, s(t)$ are consecutive integers, rank $L_i \geqq 8$, and $u' = v' = 0$, then L_i is a *normal lattice* of type $(s(0), s(t), R)$ and $R = (\text{rank } L_0, \cdots, \text{rank } L_t)$. Observe that the concept of normal is independent of the canonical decomposition.

LEMMA 1.5. *Let* $L = \sum_{i=0}^{t} L_i$ *and* $M = \sum_{i=0}^{t'} M_i$ *be canonical decomposition of the non-defective lattices L and M respectively. If L and M are of the same type, then* $t = t'$, $\operatorname{rank} L_i = \operatorname{rank} M_i$, *and* $s_L(i) = s_M(i)$.

2. Characters. A *character* of $\mathfrak{o}$ is a map χ from $\mathfrak{o}$ into the complex numbers such that $\chi(a+b) = \chi(a)\chi(b)$ for $a, b \in \mathfrak{o}$, and $\chi(\mathfrak{o}(r)) = 1$ for some nonnegative rational integer r. Let $m(\chi)$ denote the smallest $r \geqq 0$ such that $\chi(\mathfrak{o}(r)) = 1$. Then $\mathfrak{o}(m(\chi))$ is called the *maximal support* of χ and $m(\chi)$ is called the *maximal support ordinal* of χ. When a single character χ is being considered, we use m for the maximal support ordinal of χ without stating that $m = m(\chi)$.

Note that if χ is a character and $a \in \mathfrak{o}$, then $\chi(a) = \pm 1$. In fact, the image of χ is $\{-1, 1\}$ except when $m(\chi) = 0$.

The set of all characters of $\mathfrak{o}$, denoted by X, together with the operation of function multiplication is a group. For $r \geqq 0$ define $X(r) = \{\chi \in X \mid \chi(\mathfrak{o}(r)) = 1\} = \{\chi \in X \mid m(\chi) \leqq r\}$. $X(r)$ is a subgroup of X and will play an important role in the study of Gauss sums.

For each $r \geqq 0$ there is a one-to-one correspondence between $X(r)$ and the set of group characters on $\mathfrak{o}/\mathfrak{o}(r)$. This permits us to obtain information about $X(r)$ from the theory of group characters. (For details see [3].)

3. Gauss sums. A lattice L is said to be *integral* if $Q(L) \subseteqq \mathfrak{o}$. If L is integral, then $q(L) \subseteqq \mathfrak{o}$, since for $x, y \in L$, $\langle x, y\rangle = Q(x) + Q(y) + Q(x+y) \in \mathfrak{o}$. All lattices will be assumed to be integral. This causes no loss of generality in studying the classification problem.

For $r \geqq 0$, $L(r)$ will denote $\pi^r L$ and a *summation over* x mod $L(r)$ will mean that x runs over a complete set of representatives for the additive group $L/L(r)$. The order of $L/L(r)$ is $2^{fr(\operatorname{rank} L)}$.

For a lattice L and a character $\chi \in X(r)$, $\chi(L; \pi^r) = \sum_{x \bmod L(r)} \chi(Q(x))$ is called a *Gauss sum*. Write $\chi(L)$ for $\chi(L; \pi^m)$. Since L is integral and $L/L(r)$ is finite, all elements of $Q(L)$ are in the domain of χ and each Gauss sum is a finite sum of ones and negative-ones.

For a plane $P \cong \begin{pmatrix} & b \\ a & c\end{pmatrix}$ (or a line $P \cong (a)$), we will sometimes write $\chi\begin{pmatrix} & b \\ a & c\end{pmatrix}$ (or $\chi(a)$) for $\chi(P)$.

In view of [4, Propositions 1 and 2], it is clear that a great deal can be learned about Gauss sums by studying $X(L)$ when L is a line or plane. Our investigation can be further simplified by noting:

$$\begin{pmatrix} & b \\ a & c \end{pmatrix} \oplus \begin{pmatrix} & b \\ a & c \end{pmatrix} \cong \begin{pmatrix} & b \\ a & 0 \end{pmatrix} \oplus \begin{pmatrix} & b \\ c & 0 \end{pmatrix} \tag{3.1}$$

and

$$\chi\begin{pmatrix} & \pi^m \\ d & 0 \end{pmatrix} = 2^{fm} X(d)\,. \tag{3.2}$$

Let $L = \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_n$. It follows from the uniqueness of the representation of elements with respect to a basis that $a_1x_1 + \cdots + a_nx_n$ runs through a complete set of representatives for $L/L(m)$ as $a_1, \cdots, a_n$ each runs through a complete set of representatives for $\mathfrak{o}/\mathfrak{o}(m)$.

LEMMA 3.1. *For $X \in X$ and $L \cong \begin{pmatrix} & \pi^s \\ a & 0 \end{pmatrix}$,*

$$X(L) = 2^{fm} \sum_{\alpha \bmod \mathfrak{o}(s)} \chi(\alpha^2 a \pi^{2m-2s})$$

if $s \leqq m$ and $X(L) = 2^{fm} \sum_{\alpha \bmod \mathfrak{o}(m)} \chi(\alpha^2 a)$ if $m < s$.

Proof. Keeping in mind the preceding comment, $\chi(L) = \sum_{\alpha,\beta} \chi(\alpha^2 a + \alpha\beta\pi^s)$ with $\alpha \bmod \mathfrak{o}(m)$ and $\beta \bmod \mathfrak{o}(m)$. For a fixed α, by [3, (1.6)], $\sum_\beta \chi(\alpha\beta\pi^s) = \begin{cases} 2^{fm} \text{ if } \operatorname{ord}\alpha + s \geqq m \\ 0 \text{ otherwise} \end{cases}$ with $\beta \bmod \mathfrak{o}(m)$. Therefore $\chi(L) = 2^{fm} \sum \chi(\alpha^2 a)$ with $\alpha \bmod \mathfrak{o}(m)$ and $\operatorname{ord}\alpha \geqq m - s$.

Since the case $s > m$ is immediate, assume $s \leqq m$. As γ runs through a complete set of representatives for $\mathfrak{o}/\mathfrak{o}(s)$, $\alpha = \gamma\pi^{m-s}$ runs through a complete set of those representatives for $\mathfrak{o}/\mathfrak{o}(m)$ such that $\operatorname{ord}\alpha \geqq m - s$. Thus $\chi(L) = 2^{fm} \sum_\gamma \chi(\gamma^2 a \pi^{2m-2s})$.

PROPOSITION 3.2. *For $\chi \in X$ and H an i-hyperbolic plane, $\chi(H) = 2^{f \min(2m, m+i)}$.*

Proof. This follows from Lemma 3.1 and the fact that $\chi(0) = 1$.

LEMMA 3.3. *Let $\chi \in X$, $L \cong \begin{pmatrix} & \pi^s \\ a & 0 \end{pmatrix}$, and $M \cong \begin{pmatrix} & \pi^{s+w} \\ a\pi^{2w} & 0 \end{pmatrix}$ with $0 \leqq w \leqq s + w \leqq m$. Then $\chi(M) = 2^{fw}\chi(L)$.*

Proof. Using Lemma 3.1, $\chi(M) = 2^{fm} \sum \sum \chi((\alpha + \beta\pi^s)^2 a \pi^{2m-2s}) = 2^{f(m+w)} \sum \chi(\alpha^2 a \pi^{2m-2s}) = 2^{fw}\chi(L)$ with $\alpha \bmod \mathfrak{o}(s)$ and $\beta \bmod \mathfrak{o}(w)$.

Proposition 3.4. *Let* $L \cong \begin{pmatrix} & \pi^s \\ e\pi^w & 0 \end{pmatrix}$ *and* $m \geqq 0$. *Then the following hold:*

(1) *If* $m \leqq w$ *or* $2s - w \leqq m$, *then* $\chi(L) = 2^{f \min(2m, m+s)}$.

(2) *If* $w < m < 2s - w$ *with* w *and* m *of opposite parity, then* $\chi(L) = 0$.

(3) *If* $w < m < 2s - w$ *with* w *and* m *of the same parity, then there exists* χ_1 *and* χ_2 *in* X *with* $m(\chi_1) = m(\chi_2) = m$ *such that* $\chi_1(L) = 0$ *and* $\chi_2(L) \neq 0$. *Furthermore, if* $m(\chi) = m$ *and* $\chi(L) \neq 0$, *then* $\chi(L) = 2^{f \min(2m, m+s)}$.

Proof. (1) If $m \leqq w$, $\chi(L) = \chi(H_s)$; otherwise use Lemma 3.1.

(2) *Claim.* If χ has maximal support $\mathfrak{o}(m)$, $r + m \equiv 1 \bmod 2$, and $j \geqq (m - r + 1)/2 > 0$, then $\sum_\alpha \chi(\alpha^2 e\pi^r) = 0$ with $\alpha \bmod \mathfrak{o}(j)$.

Justification. With $\beta \bmod \mathfrak{o}((m - r - 1)/2)$, $\gamma \bmod \mathfrak{o}(1)$, and $\delta \bmod \mathfrak{o}(j - (m - r + 1)/2)$, $\sum \chi(\alpha^2 e\pi^r) = \sum_\beta \sum_\gamma \sum_\delta \chi(\beta + \pi^{(m-r-1)/2}\gamma + \pi^{(m-r+1)/2}\delta)^2 e\pi^r) = [\sum_\beta \chi(\beta^2 e\pi^r)][\sum_\gamma \chi(\gamma^2 e\pi^{m-1})][\sum_\delta \chi(\delta^2 e\pi^{m+1})] = 0$ since $\sum_\gamma \chi(\gamma^2 e\pi^{m-1}) = \sum_\gamma \chi_{(m-1)}(\gamma) = 0$, and the claim holds.

If $s \leqq m$, take $r = 2m - 2s + w$ and $j = s$; and if $m < s$, take $r = w$ and $j = m$.

(3) If $s < m$, then by Lemma 3.3 L may be replaced by $\begin{pmatrix} & \pi^m \\ e\pi^{w+2m-2s} & 0 \end{pmatrix}$. Hence we assume $m \leqq s$. Suppose $\chi(L) = 0$ for all χ with $m(\chi) = m$. Let $M \cong \begin{pmatrix} & \pi^s \\ e\pi^w & \pi^{m-1} \end{pmatrix}$. For $\chi \in X(m)$, $\chi(M) = \chi(L)$. By [4, Proposition 3], $L(\pi^{m-1}; \pi^m) = M(\pi^{m-1}; \pi^m)$. Thus $\pi^{m-1} \in Q(L)$ which is impossible. Hence there exists χ_2 with $m(\chi_2) = m$ and $\chi_2(L) \neq 0$.

Now suppose $\chi(L)$ is nonzero for all χ with $m(\chi) = m$. Write $N \cong \begin{pmatrix} & \pi^s \\ e\pi^{m-1} & 0 \end{pmatrix}$ and $N' \cong \begin{pmatrix} & \pi^s \\ e\pi^w + e\pi^{m-1} & 0 \end{pmatrix}$. Using [6, LT_2 b)], $N \oplus L \cong N' \oplus L$. Now $e\pi^w + e\pi^{m-1}$ is a unit times π^w and therefore by the argument just gone through for L, $\chi(N') \neq 0$ for some χ with $m(\chi) = m$. Thus $\chi(N) \neq 0$; but this contradicts (2). Therefore there exists χ_1 with $m(\chi_1) = m$ and $\chi_1(L) = 0$.

If $\chi(L) \neq 0$, setting $H \cong \begin{pmatrix} & \pi^s \\ 0 & 0 \end{pmatrix}$, $L \oplus L \cong H \oplus L$ and $\chi(L) = \chi(H) = 2^{f \min(2m, m+s)}$.

It now follows that if $\chi(L) \neq 0$ then $\chi(L) = \pm 2^n$ for some non-negative integer n.

4. A reduction.

Lemma 4.1. *Let* $L \cong \begin{pmatrix} & \pi^s \\ a & c \end{pmatrix}$ *and* $M \cong \begin{pmatrix} & \pi^r \\ a\pi^{2r-2s-2n} & c\pi^{2n} \end{pmatrix}$ *with* $r \geqq s + n$ *and* $n \geqq 0$. *For any* $\chi \in X$ *such that* $\chi\begin{pmatrix} & \pi^{s+n} \\ a & 0 \end{pmatrix}$, $\chi\begin{pmatrix} & \pi^s \\ 0 & c \end{pmatrix}$, *and* $\chi\begin{pmatrix} & \pi^r & \\ 0 & & c\pi^{2n}\end{pmatrix}$ *are all nonzero,* $sgn\,(\chi(L)) = sgn\,(\chi(M))$, *and* $\chi(L)$ *and* $\chi(M)$ *are both nonzero.*

Proof. Now $\begin{pmatrix} & \pi^s & \\ a & & c\end{pmatrix} \oplus \begin{pmatrix} & \pi^{s+n} & \\ a & & c\pi^{2n}\end{pmatrix} \cong \begin{pmatrix} & \pi^s & \\ 0 & & c\end{pmatrix} \oplus \begin{pmatrix} & \pi^{s+n} & \\ a & & 0\end{pmatrix}$ and $\begin{pmatrix} & \pi^{s+n} & \\ a & & c\pi^{2n}\end{pmatrix} \oplus \begin{pmatrix} & \pi^r & \\ a\pi^{2r-2s-2n} & & c\pi^{2n}\end{pmatrix} \cong \begin{pmatrix} & \pi^{s+n} & \\ a & & 0\end{pmatrix} \oplus \begin{pmatrix} & \pi^r & \\ 0 & & c\pi^{2n}\end{pmatrix}$. Since the Gauss sum of the lattice on the right-hand side in each isometry is positive, the result follows.

THEOREM 4.2. *Let* $L = \sum_{i=0}^{t} L_i$ *and* $M = \sum_{i=0}^{t'} M_i$ *be canonical decompositions for non-defective spaces* L *and* M *respectively, and let* $u = \min\,(u_L(s(t)), u_M(s(t')))$. *If* L *and* M *are of the same type* (I, T, R), *and if* $\chi(L) = \chi(M)$ *for all* $\chi \in X(2T - u + 1)$, *then* $\chi(L) = \chi(M)$ *for all* $\chi \in X$.

Proof. Fix $\chi \in X$ such that $m > 2T - u + 1$. Write $m = 2T - u + 2n + j$ where $j = 0$ or $j = 1$. In view of Lemma 1.5 $\operatorname{rank} L = \operatorname{rank} M$ and $s_L(i) = s_M(i)$ for $0 \leqq i \leqq t$. Fix i. Adjoining hyperbolic planes if necessary, decompose L according to Lemma 1.2. Applying Proposition 3.2 to the hyperbolic part and Lemma 4.1 and Proposition 3.4 to the rest, we conclude that $\chi(L) = 2^{fn \operatorname{rank} L}\chi_{(2n)}(L)$. Similarly $\chi(M) = 2^{fn \operatorname{rank} M}\chi_{(2n)}(M)$.

5. Classification of modular lattices.

PROPOSITION 5.1. *If* $sL = sM = \mathfrak{o}(s)$ *and* $\chi(L) = \chi(M)$ *for all* $\chi \in X(s)$, *then* $q(L) = q(M)$.

Proof. Case 1. $s = 0$. Using the fact that L has an 0-modular orthogonal summand and Lemma 1.2, $q(L) = \mathfrak{o}^2 + \pi\mathfrak{o}^2 = q(M)$.

Case 2. $s \geqq 1$. First assume there exists $\chi \in X$ such that $m(\chi) = s$ and $\chi(L) \neq 0$. Then $\operatorname{sgn}(\chi(L))2^{fm \operatorname{rank} L} = \chi(L) = \chi(M) = \operatorname{sgn}(\chi(M))2^{fm \operatorname{rank} M}$. Hence $\operatorname{rank} L = \operatorname{rank} M$, and by [4, Proposition 3] $L(a; \pi^s)$ $M(a; \pi^s)$ for all $a \in \mathfrak{o}$. It follows that $q(L) = q(M)$.

Now assume $\chi(L) = \chi(M) = 0$ for all $\chi \in X$ with $m(\chi) = s$. Assume without loss of generality that $\operatorname{rank} L \leqq \operatorname{rank} M$. Let $L' \cong L \oplus H \oplus N$ such that $\operatorname{rank} L' = \operatorname{rank} M$, H is an s-hyperbolic lattice or 0, and $N \cong (\pi^s)$ or 0. Then $q(L') = q(L)$ and $\chi(L') = 0$ for all $\chi \in X(s)$. Hence $\chi(L') = \chi(M)$ for all $\chi \in X(s)$. Then [4, Proposition 3] is applicable and it follows that $q(L) = q(L') = q(M)$.

PROPOSITION 5.2. *Let* $\chi \in X$ *with* $m \geqq 1$, *and let* $L \cong$

$\begin{pmatrix} & \pi^{m-1} \\ e\pi^{m-1} & e^{-1}\pi^{m-1}\delta \end{pmatrix}$. *Then* $\chi(L) < 0$ *if* $\delta = \lambda$ *and* $\chi(L) > 0$ *if* $\delta = 0$.

Proof. If $e \neq 1$ set $J \cong L \oplus H_{m-1}$. By Lemma 1.2 and the uniqueness of $\Delta(kJ)$, $J \cong \begin{pmatrix} & \pi^{m-1} \\ \pi^{m-1} & \pi^{m-1}\delta \end{pmatrix} \oplus H_{m-1}$. Thus we may reduce to the case in which $e = 1$. And since $\chi(L) = 2^{2f(m-1)}\chi_{(m-1)}\begin{pmatrix} & 1 \\ 1 & \lambda \end{pmatrix}$, we assume $m = 1$.

If $\delta = 0$ use Proposition 3.4. Assume $\delta = \lambda$. Fix a basis $\{p_1, \cdots, p_f\}$ for Ω over $\{0, 1\}$. With $a_1, \cdots, a_f$ running through $\{0, 1\}$, $\chi(L) = \sum_{a_1} \cdots \sum_{a_f} [\chi((a_1p_1 + \cdots + a_fp_f)^2\lambda)]$

$$[1 + \chi(p_1^2 + a_1p_1^2 + \cdots + a_fp_1p_f)] \cdots [1 + \chi(p_f^2 + a_1p_1p_f + \cdots + a_fp_f^2)] .$$

Since $m = 1$, at least one element in each basis for Ω must be mapped to -1 by χ. Since $\lambda \notin \mathscr{P}\Omega$, it follows from elementary linear algebra that if $\{a_1, \cdots, a_f\} \subseteq \{0, 1\}$, $\{(a_1p_1 + \cdots + a_fp_f)^2, p_1^2 + a_1p_1^2 + \cdots + a_fp_1p_f, \cdots, p_f^2 + a_1p_1p_f + \cdots + a_fp_f^2\}$ contains a basis for Ω. Thus $\chi(L) < 0$.

PROPOSITION 5.3. *Let L be an s-modular lattice with* rank $L \geqq 4$ *and* $q(L) = e(\pi^u\mathfrak{o}^2 + \pi^v\mathfrak{o}^2)$. *Write* $M \cong \begin{pmatrix} & \pi^s \\ e\pi^u & 0 \end{pmatrix}$ *and* $N \cong \begin{pmatrix} & \pi^s \\ e\pi^v & 0 \end{pmatrix}$. *For* $\chi \in X$, $\chi(L) = 0$ *iff* $\chi(M) = 0$ *or* $\chi(N) = 0$.

Proof. Write $L \cong \begin{pmatrix} & \pi^s \\ e\pi^u & c \end{pmatrix} \oplus \begin{pmatrix} & \pi^s \\ e\pi^v & e^{-1}\pi^{2s-v}\delta' \end{pmatrix} \oplus H_s$ with $c = e^{-1}\pi^{2s-u}\delta + e^{-1}\pi^v a^2$ as in Lemma 1.2.

Sufficiency is obvious, so assume $\chi(L) = 0$. Set $M' \cong \begin{pmatrix} & \pi^s \\ c & 0 \end{pmatrix}$ and $N' \cong \begin{pmatrix} & \pi^s \\ e^{-1}\pi^{2s-v}\delta' & 0 \end{pmatrix}$. By Propositions 3.2 and 3.4 $\chi(N') \neq 0$. Thus at least one of $\chi(M)$, $\chi(N)$, and $\chi(M')$ is zero. If $\chi(M') = 0$, by Proposition 3.4 either $\chi(M) = 0$ or $\chi(N) = 0$.

THEOREM 5.4. *Let* L_1 *and* L_2 *be s-modular lattices.* $L_1 \cong L_2$ *iff* $\chi(L_1) = \chi(L_2)$ *for all* $\chi \in X(2_s - \max(u_{L_1}, u_{L_2}) + 1)$.

Proof. Necessity is obvious, so we assume $\chi(L_1) = \chi(L_2)$ for $\chi \in X(2s - \max u_{L_1}, u_{L_2}) + 1)$. By Proposition 5.1, $u_{L_1} = u_{L_2}$. Call the common value u. By Propositions 3.4 and 5.3, $\chi(L_i) = \operatorname{sgn}(\chi(L_i))2^{f(m+s)(\operatorname{rank} L_i)/2}$ when $m = 2s - u + 1$. Thus $\operatorname{rank} L_1 = \operatorname{rank} L_2$. Adjoining the appropriate hyperbolic plane to L_1 and L_2 and applying Lemma 1.2 and Proposition 5.1, $Q(L_1) = q(L_1) = q(L_2) = Q(L_2) = e(\pi^u\mathfrak{o}^2 + \pi^v\mathfrak{o}^2)$ and

$$L_i \cong \begin{pmatrix} & \pi^s \\ e\pi^u & e^{-1}\pi^{2s-u}\delta_i + e^{-1}\pi^{w_i}a_i^2 \end{pmatrix} \oplus \begin{pmatrix} & \pi^s \\ e\pi^v & e^{-1}\pi^{2s-v}\delta_i' \end{pmatrix} \oplus H_s$$

with a_i a unit or zero, $w_i \geqq v$, and $w_i + v \equiv 0 \bmod 2$. By showing that $\operatorname{ord}(\pi^{w_1}a_1^2 + \pi^{w_2}a_2^2) > 2s - u$ and $\delta_1 = \delta_2$, it will follow from [6, Lemma 4.1] that $\begin{pmatrix} & \pi^s \\ e\pi^u & e^{-1}\pi^{2s-u}\delta_1 + e^{-1}\pi^{w_1}a_1^2 \end{pmatrix} \cong \begin{pmatrix} & \pi^s \\ e\pi^u & e^{-1}\pi^{2s-u}\delta_2 + e^{-1}\pi^{w_2}a_2^2 \end{pmatrix}$. If we also show that $\delta_1' = \delta_2'$, we will have $L_1 \cong L_2$.

By [6, Theorem 4.6],

$$L_1 \oplus L_2 \cong \begin{pmatrix} & \pi^s \\ e\pi^u & e^{-1}\pi^{2s-u}(\delta_1 + \delta_2) \end{pmatrix} \oplus \begin{pmatrix} & \pi^s \\ e\pi^u & e^{-1}\pi^{w}a^2 \end{pmatrix}$$
$$\oplus \begin{pmatrix} & \pi^s \\ e\pi^v & e^{-1}\pi^{2s-v}(\delta_1' + \delta_2') \end{pmatrix} \oplus H_s'$$

where $\pi^w a^2 = \pi^{w_1}a_1^2 + \pi^{w_2}a_2^2$ with $w \geqq v$, $w + v \equiv 0 \bmod 2$, and a is a unit or zero. Let $K \cong \begin{pmatrix} & \pi^s \\ e\pi^u & e^{-1}\pi^{2s-u}(\delta_1 + \delta_2) \end{pmatrix}$, $M \cong \begin{pmatrix} & \pi^s \\ e\pi^u & e^{-1}\pi^{w}a^2 \end{pmatrix}$, and $N \cong \begin{pmatrix} & \pi^s \\ e\pi^v & e^{-1}\pi^{2s-v}(\delta_1' + \delta_2') \end{pmatrix}$. For $\chi \in X$ with $m = 2s - u$, $\chi(K \oplus M \oplus N \oplus H_s') = \chi(L_1 \oplus L_2) > 0$ and $\chi(K \oplus H_s) > 0$. Thus $\operatorname{sgn}(\chi(M)) = \operatorname{sgn}(\chi(N))$.

Claim. If $\operatorname{ord}(e^{-1}\pi^w a^2) \leqq 2s - u$, then there exists $\chi', \chi'' \in X$ with $m(\chi') = m(\chi'') = 2s - u$ such that $\chi'(M) > 0$ and $\chi''(M) < 0$.

Justification. Since $w + u \equiv 1 \bmod 2$, $\operatorname{ord}(e^{-1}\pi^w a^2) \neq 2s - u$. Write $J \cong \begin{pmatrix} & \pi^{2s-u-1} \\ e\pi^{w-1} & e^{-1}\pi^{2s-u-1}a^2 \end{pmatrix}$, $J' \cong \begin{pmatrix} & \pi^{2s-u-1} \\ e\pi^{w-1} & e^{-1}\pi^{2s-u+1}a^2 \end{pmatrix}$, $J'' \cong \begin{pmatrix} & \pi^{2s-u-1} \\ 0 & e^{-1}\pi^{2s-u-1}a^2 \end{pmatrix}$, and $J''' \cong \begin{pmatrix} & \pi^{2s-u-1} \\ e\pi^{w-1} & 0 \end{pmatrix}$. $J \oplus J' \cong J'' \oplus J'''$. There exists $\chi' \in X$ with $m = 2s - u$ such that $\chi'(J') > 0$ and $\chi'(J'' \oplus J''') > 0$. Hence $\chi'(J) > 0$. Taking $P \cong \begin{pmatrix} & \pi^{2s-u-1} \\ e\pi^{w} & 0 \end{pmatrix}$, by an argument like that given in the proof of Proposition 3.4 (3), $\chi''(J) \leqq 0$ for some $\chi'' \in X$ with $m = 2s - u$. Now compare M with J by Lemma 4.1 to establish the claim.

It is easily seen that $\chi'(N) = \chi''(N)$. Thus $\operatorname{ord}(e^{-1}\pi^w a^2) > 2s - u$ and $\delta_1' = \delta_2'$.

By another application of Lemma 4.1, $\delta_1 = \delta_2$.

6. Classification when $L = K(L)$. If $\operatorname{rank} L = 1$, then $L \cong (e\pi^u)$. Otherwise $\operatorname{rank} L = 2$, in which case $\operatorname{rank}_{o_2} q(L) = 2$ and $L \cong (e\pi^u) \oplus (e\pi^v)$.

THEOREM 6.1. *Let* $L = K(L)$, $M = K(L)$, *and* $\operatorname{rank} L \leqq \operatorname{rank} M$. *Write* $w = \begin{cases} \max(u_L, u_M) \text{ if } \operatorname{rank} L = \operatorname{rank} M = 1 \\ \max(u_L, u_M, v_M) \text{ if } \operatorname{rank} L = 1 \text{ and } \operatorname{rank} M = 2 \\ \max(u_L, v_L, u_M, v_M) \text{ if } \operatorname{rank} L = \operatorname{rank} M = 2 \end{cases}$

(1) *If* $\chi(L) = \chi(M)$ *for* $\chi \in X(w+1)$, *then* $\operatorname{rank} L = \operatorname{rank} M$.

(2) *For* $L \cong (e\pi^{u_L})$ *and* $M \cong (e'\pi^{u_M})$, $L \cong M$ *if and only if* $e\pi^{u_L} \in e'\pi^{u_M}k^2$ *and* $\chi(L) = \chi(M)$ *for all* $\chi \in X(w)$.

(3) *If* $\operatorname{rank} L = \operatorname{rank} M = 2$, *then* $L \cong M$ *if and only if* $\chi(L) = \chi(M)$ *for all* $\chi \in X(w)$.

Proof. Write $M \cong (e'\pi^{u_M})$ or $M \cong (e'\pi^{u_M}) \oplus (e'\pi^{v_M})$ and $L \cong (e\pi^{u_L})$ or $L \cong (e\pi^{u}) \oplus (e\pi^{v_L})$ depending on the case at hand. Assume that $\chi(L) = \chi(M)$ for all $\chi \in X(w)$ and that $u_L \leqq u_M$. Setting $L' \cong (e\pi^{u_L})$ and using (3.2) and Proposition 3.4, $u_M = u_L$.

(1) Assume $\operatorname{rank} M = 2$. Then v_M exists, and comparing L' and $(e'\pi^{v_M})$ with appropriate lattices of rank 2, we see that $\chi(L') \neq 0$ for some χ with $m = v_M + 1$ and $\chi(e'\pi^{v_M}) = 0$ for all χ with $m = v_M + 1$. Hence L' and L are not isometric, and $\operatorname{rank} L = 2$.

(2) Since $u_L = u_M$, this is obvious.

(3) Necessity is obvious. $L(a; \pi^w) = M(a; \pi^w)$ for all $a \in \mathfrak{o}$. By a straightforward calculation $q(L) = q(M)$, and by [6, Lemma 1.2] $v_L = v_M$. It now follows that $L \cong M$.

7. Classification when $L \neq K(L)$.

PROPOSITION 7.1. *Let* $L = (\sum_{i=0}^{t} L_i) \oplus K(L)$ *be a saturated decomposition for the normal lattice* L. *For any* $\chi \in X$ *such that* $\chi(L_0) = 0$, $\chi(L_0^{\perp}) \geqq 0$.

Proof. Suppose $\chi(L_0) = 0$ and $\chi(L_0^{\perp}) \neq 0$ for some $\chi \in X$. Using the hypothesis that the given decomposition is saturated and [6, Definition 5.1], there exists a unit e such that $Q(L_j) = e(\pi^{u(s(j))})\mathfrak{o}^2 + (\pi^{v(s(j))}\mathfrak{o}^2)$ for $j = 0, 1$. By Propositions 5.3 and 3.4, $u(s(0)) < m$ $2s(0) - u(s(0))$.

We show that $m \leqq v(s(1))$. Then $m \leqq v(s(i))$ and hence $\chi(L_i) \geqq 0$ for $1 \leqq i \leqq t$. Since $\chi(K(L)) \geqq 0$, we then have $\chi(L_0^{\perp}) > 0$.

Assume $v(s(1)) < m$. Since $\chi(L_1) \neq 0$ and $u(s(1)) < m < 2s(1) - u(s(1))$, $2s(0) - v(s(0)) \leqq 2s(1) - v(s(1)) \leqq m$. So $\chi\begin{pmatrix} & \pi^{s(0)} \\ e\pi^{u(s(0))} & 0 \end{pmatrix} = 0$. By [6, Definition 5.1], one of $u(s(1))$ and $v(s(1))$ equals either $u(s(0))$ or $u(s(0)) + 2$. Since $\begin{pmatrix} & \pi^{s(0)} \\ e\pi^{u(s(0))} & 0 \end{pmatrix} \oplus \begin{pmatrix} & \pi^{s(1)} \\ e\pi^{u(s(0))} & 0 \end{pmatrix} \cong \begin{pmatrix} & \pi^{s(0)} \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} & \pi^{s(1)} \\ e\pi^{u(s(0))} & 0 \end{pmatrix}$, $\chi\begin{pmatrix} & \pi^{s(1)} \\ e\pi^{u(s(0))} & 0 \end{pmatrix} = 0$; and by Lemma 3.3, if $s(1) \leqq m$, $\chi\begin{pmatrix} & \pi^{s(1)} \\ e\pi^{u(s(0))+2} & 0 \end{pmatrix} = 0$. Thus $\chi(L_1) \neq 0$ implies $s(1) > m$. Hence $s(1) > m \geqq 2s(1) - v(s(1)) \geqq s(1) - 1$. So $v(s(1)) \geqq m$.

PROPOSITION 7.2. *Let* $L = (\sum_{i=0}^{t} L_i) \oplus K(L) = (\sum_{i=0}^{t} M_i) \oplus K(L)$ *be canonical decompositions of a normal lattice* L. *If* $u(j) =$

$u(j-1)+2$ and $v(j)=v(j-1)+2$ for some j with $I<j\leqq T$, write $\sum_{i=0}^{j-I-1}(\Delta(kL_i)+\Delta(kM_i))=\beta^2\pi^{-1}+\delta+\mathscr{P}k$ with $\beta\in\Omega[\pi^{-1}]$ and $\delta\in\{0,\lambda\}$. Then β and δ are uniquely determined by the two decompositions and $\operatorname{ord}(\beta^2\pi^{-1}+\delta)\geqq u(j)+v(j)-2j-1$.

Proof. The uniqueness of β and δ follows from [6, Lemma 1.1].

By [6, Theorems 2.2 and 2.3 and Lemma 1.2], a reduction can be made to the case in which the two canonical decompositions are complete and the second is obtained from the first by an elementary lattice transformation.

Denote $u(j-1)$, $v(j-1)$, and $j-I$ by u, v, and w, respectively. Let T_r denote the elementary lattice transformation used to obtain the second decomposition from the first. Only the cases for which $r=4$ and $r=9$ need be considered.

Case 1. $r=9$. If the two planes effected by T_9 are of the same modularity, the result holds trivially. If not, a reduction can be made to the case in which T_9 acts on L_{w-1} and L_w. Then for some c, $d\in Q(L_{w-1})$, c', $d'\in Q(L_w)$, and $\alpha\in\mathfrak{o}$, T_9 sends $\begin{pmatrix} & \pi^{j-1} \\ c & d\end{pmatrix}\oplus\begin{pmatrix} & \pi^{j} \\ c' & d'\end{pmatrix}$ to $\begin{pmatrix} & \pi^{j-1} \\ c+\alpha^2c' & d\end{pmatrix}\oplus\begin{pmatrix} & \pi^{j} \\ c' & d'+\alpha^2\pi^2 d\end{pmatrix}$ and leaves the other planes fixed.

By a direct computation, $\sum_{i=0}^{w-1}(\Delta(kL_i)+\Delta(kM_i))=\alpha^2c'd\pi^{-2j+2}+\mathscr{P}k=\beta^2\pi^{-1}+\delta+\mathscr{P}k$ with $\operatorname{ord}(\beta^2\pi^{-1}+\delta)\geqq u(j)+v(j)-2j-1$, where $\beta\in\Omega[\pi^{-1}]$ and $\delta\in\{0,\lambda\}$.

Case 2. $r=4$. If the plane altered by the application of T_4 has modularity greater that $j-1$, the result follows. Otherwise there exists a sequence of complete decompositions from the first given decomposition to the second, each of which can be obtained from its predecessor by either an application of T_9 or an application of T_4 to a plane of modularity $j-1$. Assume T_4 is applied to a plane of modularity $j-1$. For some c, $d\in Q(L_{w-1})$, $c'\in Q(K(L))$, and $\alpha\in\mathfrak{o}$, T_4 sends $\begin{pmatrix} & \pi^{j-1} \\ c & d\end{pmatrix}$ to $\begin{pmatrix} & \pi^{j-1} \\ c+\alpha^2c' & d\end{pmatrix}$ and leaves all else fixed. Now $c'\in Q(L_w)$; so the proof follows exactly as in Case 1.

NOTATION. Let L and M be lattices of the same type (I, T, R), j be a rational integer such that $I<j<T$, and $H=\sum_{i=1}^{T}H_i$ with H_i i-hyperbolic of rank 8. $L\oplus H$ and $M\oplus H$ are normal lattice of type $(I, T, R+R(H))$; and if $(\sum_{i=0}^{t}L_i)\oplus K(L\oplus H)$ and $(\sum_{i=0}^{t'}M_i)\oplus K(M\oplus H)$ are canonical decompositions of $L\oplus H$ and $M\oplus H$ respectively, then $t=t'=T-I$ and $s(L_i)=s(M_i)=I+i$ for $0\leqq i\leqq t$. Write

$$\operatorname{Ord}(L, M, j) = \min\{\operatorname{ord}(\beta^2\pi^{-1} + \delta) \mid \beta^2\pi^{-1} + \delta + \mathscr{P}k = \sum_{i=0}^{j-I-1} (\Delta(kL_i) + \Delta(kM_i))$$

for some canonical decompositions $(\sum_{i=0}^{T-I} L_i) \oplus K(L \oplus H)$ and $(\sum_{i=0}^{T-I} M_i) \oplus K(M \oplus H)$ of $L \oplus H$ and $M \oplus H$ respectively$\}$.

LEMMA 7.3. *Let $L \cong \begin{pmatrix} & \pi^s \\ a & c \end{pmatrix}$ and M be an $s + 1$-modular lattice such that* rank $M \geqq 8$ *and $\pi^2 q(L) \subseteq q(M)$. It $a + a' \in q(M)$, then $L \oplus M \cong \begin{pmatrix} & \pi^s \\ a' & c \end{pmatrix} \oplus M'$, where M' is $s + 1$-modular and $q(M') = q(M)$.*

The proof is a straightforward manipulation of lattices.

LEMMA 7.4. *Let s, w, and w' be nonnegative integers such that either $w' = w$ or $w' = w + 2$; and let e be a unit. For any $\chi \in X$ with $m > s$ and $\chi\begin{pmatrix} & \pi^s \\ e\pi^w & 0 \end{pmatrix} = 0$, $\chi\begin{pmatrix} & \pi^{s+1} \\ e\pi^{w'} & 0 \end{pmatrix} = 0$.*

Proof. If $w' = w + 2$, use Lemma 3.3. If $w' = w$, write $\begin{pmatrix} & \pi^s \\ e\pi^w & 0 \end{pmatrix} \oplus \begin{pmatrix} & \pi^{s+1} \\ e\pi^{w'} & 0 \end{pmatrix} \cong \begin{pmatrix} & \pi^s \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} & \pi^{s+1} \\ e\pi^{w'} & 0 \end{pmatrix}$.

THEOREM 7.5. *For lattices L and M, $L \cong M$ if and only if*

(1) *L and M are of the same type (I, T, R);*

(2) *For $I \leqq i \leqq T, q(L(i) \oplus H_i) = q(M(i) \oplus H_i)$. That is, the* ith *norm groups of L and M are the same.*

(3) *$\chi(L) = \chi(M)$ for all $\chi \in X(2T - u(T) + 2)$;*

(4) *If for some j with $I < j \leqq T$, $u(j) = u(j-1) + 2$ and $v(j) = v(j-1) + 2$, then* $\operatorname{Ord}(L, M, j) \geqq u(j) + v(j) - 2j - 1$;

(5) *If $K(L) \cong (d_1)$ and $K(M) \cong (d_2)$, then $d_1 \in d_2 k^2$.*

Proof. Necessity. Condition (4) follows from [6, Lemma 5.2] and Proposition 7.2. The others hold trivially.

Sufficiency. Replacing L and M by $L \oplus (\sum_{i=I}^{T} H_i)$ and $M \oplus (\sum_{i=I}^{T} H_i)$ respectively, we may assume L and M are normal lattices. Proceed by induction on n = length d = length M.

Part I. $n = 1$. Then $T = I$. Write $u(I) = u$ and $v(I) = v$.

Case 1. Rank $(K(L)) = 0$. Assume $K(M) \neq 0$. Then $u = u_{K(M)}$ by normality; and for any unit e and any χ with $m = 2T - u + 1$, $\chi(e\pi^{u_{K(M)}}) = 0$. Thus $\chi(L) = \chi(M) = 0$ for such χ. But $\chi(L) = 0$ by Propositions 3.4 and 5.3. Therefore $K(M) = 0$. It now follows from

Theorem 5.4 that $L \cong M$.

Case 2. Rank $(K(L)) = 1$. By normality, $K(L) \cong (e\pi^u)$ for some unit e. Write $L \cong L_0 \oplus K(L)$ with $q(L_0) = e(\pi^u \mathfrak{o}^2 + \pi^v \mathfrak{o}^2)$. There exists $\chi \in X$ with $m = 2I - v + 3$ and $\chi(L) \neq 0$. Since $\chi(e\pi^v) = 0$, rank $(K(M)) \neq 2$. Using condition (5), $K(M) \cong (e\pi^u)$. Using [6, Lemma 4.5, Theorem 4.6, and LT_0 d)], $L \cong \begin{pmatrix} & \pi^I \\ e\pi^v & e^{-1}\pi^{2I-v}\delta \end{pmatrix} \oplus H_I \oplus (e\pi^u)$ and $M \cong \begin{pmatrix} & \pi^I \\ e\pi^v & e^{-I}\pi^{2s-v}\delta' \end{pmatrix} \oplus H_I' \oplus (e\pi^u)$. By an argument already used on several occasions (see e.g., Theorem 5.4) $L \cong M$.

Case 3. Rank $(K(L)) = 2$. $K(L) \cong (e\pi^u) \oplus (e\pi^v)$ for some unit e. By an argument similar to the one given in Case 2, $L \cong H_I \oplus (e\pi^v) \oplus (e\pi^v) \cong M$.

Part II. Assume $n > 1$ and that the result holds for normal lattices of length $n - 1$. By [6, Lemma 5.3], L and M admit saturated decompositions $(\sum_{i=0}^{n-1} L_i) \oplus K(L)$ and $(\sum_{i=0}^{n-1} M_i) \oplus K(M)$ respectively. Our strategy is to show the following:

(7.1) For $j = 0, 1$, there exists $s(j)$-modular lattices L_j^* and M_j^* with $q(L_j^*) = q(L_j)$ and $q(M_j^*) = q(M_j)$ such that $L_0 \oplus L_1 \cong L_0^* \oplus L_1^*$, $M_0 \oplus M_1 \cong M_0^* \oplus M_1^*$, and $L_0^* \cong M_0^*$.

Then $(L_0^*)^\perp$ and $(M_0^*)^\perp$ are normal lattices of length $n - 1$ which satisfy conditions (1), (2), (4), and (5). For any $\chi \in X(2T - u(T) + 2)$ such that $\chi(L_0^*) \neq 0$, $\chi((L^*)^\perp) = \chi((M_0^*)^\perp)$. If $\chi \in X(2T - u(T) + 2)$ and $\chi(L_0^*) = 0$, then using Lemma 7.4 and Proposition 5.3, $\chi((L_0^*)^\perp) = 0 = \chi((M_0^*)^\perp)$. Thus condition (3) is satisfied. The result follows easily by induction.

We verify (7.1) by cases. But first write $L_0 \cong L_0' \oplus L_0'' \oplus H_I$ where $L_0' \cong \begin{pmatrix} & \pi^I \\ e\pi^{u(I)} & e^{-1}\pi^{2I-u(I)}\delta + e^{-1}\pi^{v(I)}a^2 \end{pmatrix} \oplus \begin{pmatrix} & \pi^I \\ e\pi^{v(I)} & e^{-1}\pi^{2I-v(I)}\delta' \end{pmatrix}$ and $L_0'' \cong \begin{pmatrix} & \pi^I \\ e\pi^{u(I)} & 0 \end{pmatrix} \oplus \begin{pmatrix} & \pi^I \\ e\pi^{v(I)} & 0 \end{pmatrix}$ with $\delta, \delta' \in \{0, \lambda\}$, $a \in \mathfrak{o}$, and e a unit such that $q_{s(j)}(L) = e(\pi^{u(s(j))}\mathfrak{o}^2 + \pi^{v(s(j))}\mathfrak{o}^2)$ for $j = 0, 1$.

Case A. $v(I + 1) = v(I)$. By [6, Definition 5.1], $u(I + 1) = u(I)$ or $u(I) + 2$. Applying Lemma 7.3 twice, $L_0 \oplus L_1 \cong L_0^* \oplus L_i^*$ with $L_0^* \cong L_0'' \oplus H_I'$. M_0^* and M_1^* exist with $M_0^* \cong L_0'' \oplus H_I'$, and these lattices satisfy (7.1).

Case B. $v(I + 1) = v(I) + 1$. Then $u(I + 1) = v(I) = u(I) + 1$. If $u(I) < I$, the result follows as in Case A. Suppose $u(I) = I$. $L_0^* = L_0 \cong \begin{pmatrix} & \pi^I \\ \pi^I & \pi^I\delta \end{pmatrix} \oplus H_I'$ and $M_0^* \cong \begin{pmatrix} & \pi^I \\ \pi^I & \pi^I\delta_1 \end{pmatrix} \oplus H_I'$ and the result follows.

Case C. $v(I+1) = v(I) + 2$. If $v(I+1) \leqq I+1$ and $u(I+1) = u(I)$, $L_0^* \cong L_0'' \oplus H_I'$.

If $v(I+1) \leqq I+1$ and $u(I+1) = u(I) + 2$, write $a = a_1 + a_2$ *with* $a_1 \in \Omega$ and $a_2 \in \mathfrak{o}(1)$. In this case $L_0^* \cong \begin{pmatrix} & \pi^I \\ e\pi^{u(I)} & e^{-1}\pi^{v(I)}a_1^2 \end{pmatrix} \oplus L_0'' \oplus H_I'$ and $M_0^* \cong \begin{pmatrix} & \pi^I \\ e\pi^{u(I)} & e^{-1}\pi^{v(I)}c^2 \end{pmatrix} \oplus L_0'' \oplus H_I'$. By condition (4) $a_1 = c$.

We are left with the situation in which $v(I+1) = I+2$. If $u(I+1) = u(I)$, take $L_0^* \cong \begin{pmatrix} & \pi^I \\ e\pi^{v(I)} & e^{-1}\pi^{2I-v(I)}\delta' \end{pmatrix} \oplus L_0'' \oplus H_I'$ and $M_0^* \cong \begin{pmatrix} & \pi^I \\ e\pi^{v(I)} & e^{-1}\pi^{2s-v(I)}\delta_1 \end{pmatrix} \oplus L_0'' \oplus H_I'$. If $u(I+1) = u(I) + 2$, by the argument used in the case where $v(I+1) = v(I) + 2 \leqq I+1$ and $u(I+1) = u(I) + 2$, $L_0^* \cong \begin{pmatrix} & \pi^I \\ e\pi^{u(I)} & e^{-1}\pi^{v(I)}a_1^2 \end{pmatrix} \oplus \begin{pmatrix} & \pi^I \\ e\pi^{v(I)} & e^{-1}\pi^{2I-v(I)}\delta' \end{pmatrix} \oplus L_0'' \oplus H_I$ and $M_0^* \cong \begin{pmatrix} & \pi^I \\ e\pi^{u(I)} & e^{-1}\pi^{v(I)}a_1^2 \end{pmatrix} \oplus \begin{pmatrix} & \pi^I \\ e\pi^{v(I)} & e^{-1}\pi^{2I-v(I)}\delta_1 \end{pmatrix} \oplus L_0'' \oplus H_I$ with $a_1 \in \Omega$. We need only show that $\delta' = \delta_1$. The difficulty here is in checking that there exists $\chi \in X$ with $m = I+1$ and $\chi(L) \neq 0$. Let r be the largest integer such that $0 \leqq r \leqq n-1$ and $u(I+r) < I+1$. Since $u(I) < I+1$, r exists. $u(I+r) + (I+1) \equiv 0 \bmod 2$. So there exists χ with $m = I+1$ and $\chi\begin{pmatrix} & \pi^{s(r)} \\ e_{s(r)}\pi^{u(s(r))} & 0 \end{pmatrix} \neq 0$. It follows that $\chi(L_r) \neq 0$ and by induction on r, $\chi(\sum_{i=0}^{r} L_i) \neq 0$. Thus $\chi(L) \neq 0$.

References

1. C. Arf, *Untersuchungen über quadratische formen in körpern der charakteristik* 2, I. J. Reine Angew. Math., **183** (1940), 148-167.

2. ———, *Uber arithmetische aequivalenz quadratisher formen in palenzreihenkörpern über einem vollkommen körper der charakterisk* 2, Rev. Fac. Sci. Univ. Istanbul, **8** (1943), 297-327.

3. R. Jacobowitz, *Gauss sums and the local classification of hermitian forms,* Amer. J. Math., **90** (1968), 528-552.

4. O. T. O'Meara, *Local characterization of integral quadratic forms by Gauss sums.* Amer. J. Math., **79** (1957), 687-709.

5. ———, *Introduction to Quadratic Forms,* Springer-Verlag, Berlin (1963).

6. C.-H. Sah, *Quadratic forms over fields of characteristic* 2, Amer. J. Math., **82** (1960), 812-830.

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