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KOROVKIN SETS FOR AN OPERATOR ON A SPACE OF CONTINUOUS FUNCTIONS

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We characterize Korovkin sets for sequences of either positive operators or contractive operators converging to an operator T . Properties of both the Korovkin sets and the operator T are given which were previously known only in the case T was the identity operator.

Let $C = C(Q)$ be the Banach space of continuous functions on a first countable, compact Hausdorff space Q . Let \mathcal{L} be a subset of the bounded linear operators $\mathcal{B}(C)$ on C . A subspace X of C is said to be a \mathcal{L} -Korovkin set for an operator T in \mathcal{L} if for any sequence of operators $\{T_n\}$ in \mathcal{L} the convergence of $T_n f$ to Tf in the uniform norm for all f in X implies the convergence of $T_n f$ to Tf for all f in C . This paper is concerned with \mathcal{L} -Korovkin sets when \mathcal{L} consists of either the positive ($f \geq 0$ implies $Tf \geq 0$) operators or the contractive ($\|T\| \leq 1$) operators in $\mathcal{B}(C)$.

The case where T is the identity operator is now classical. See, for instance, Lorentz [3]. In this same paper the extension of the classical theory to the case of arbitrary operators T is mentioned as an open problem. This extension is the subject of the present paper. In case T is the identity operator our results reduce to the classical ones. A number of authors have considered the case where T is a lattice homomorphism between (possibly distinct) vector lattices. The present situation is different since we consider operators with the same domain and range and assume the weaker condition that T either be positive or have norm one. In addition, Cavaretta [1] and Micchelli [4, 5] have considered the case where T is a positive operator, but not necessarily a lattice homomorphism.

Many of the following results have obvious analogues in the case where the operators are assumed to be both positive and contractive.

1. General theory. Korovkin-type theorems are usually stated for either sequences of positive operators or sequences of contractive operators. Some results about Korovkin sets can be shown in a more general setting. This observation has also been made by Micchelli [5].

For a bounded linear operator T , let T^* denote the adjoint of T . For a point q in Q , let \hat{q} denote the functional in C^* given by evaluation at the point q . Let \mathcal{L} be a subset of C^* and \mathcal{L} be

the set of all bounded linear operators T on C such that $T^*\hat{q}$ is in \mathcal{L} for all q in Q . For example, if \mathcal{L} were the positive functionals on C , then \mathcal{F} would be the positive operators.

Let λ be a functional in \mathcal{L} and X be a subspace of C . We say X is an \mathcal{L} -Korovkin set for λ if for any sequence $\{\lambda_n\}$ in \mathcal{L} the fact that $\lambda_n(f) \rightarrow \lambda(f)$ for all f in X implies $\lambda_n(f) \rightarrow \lambda(f)$ for all f in C . We say X is an \mathcal{L} -determining set for λ if for any μ in \mathcal{L} the equality $\mu(f) = \lambda(f)$ for all f in X implies $\mu = \lambda$. The latter concept was first introduced for operators by Šaškin [8]. These two concepts are equivalent in the following sense.

THEOREM 1.1. *Let \mathcal{L} be a weak* closed subset of C^* and let λ be a functional in \mathcal{L} . A subset X of C has the property that for any norm bounded sequence $\{\lambda_n\}$ in \mathcal{L} the fact that $\lambda_n(f) \rightarrow \lambda(f)$ for all f in X implies $\lambda_n(f) \rightarrow \lambda(f)$ for all f in C (X is a Korovkin set for norm bounded sequences in \mathcal{L} converging to λ) if and only if X is an \mathcal{L} -determining set for λ .*

Proof. Suppose X is a Korovkin set for norm bounded sequences in \mathcal{L} converging to λ . Let μ be any functional in \mathcal{L} such that $\mu(f) = \lambda(f)$ for all f in X . Define $\lambda_n = \mu$ for $n = 1, 2, \dots$. Then $\{\lambda_n\}$ is a norm bounded sequence in \mathcal{L} such that $\lambda_n(f) \rightarrow \lambda(f)$ for all f in X . By assumption $\lambda_n(f) \rightarrow \lambda(f)$ for all f in C . Therefore $\mu(f) = \lambda(f)$ for all f in C . Hence, X is an \mathcal{L} -determining set for λ .

On the other hand, suppose X is an \mathcal{L} -determining set for λ . Let $\{\lambda_n\}$ be a norm bounded sequence in \mathcal{L} such that $\lambda_n(f) \rightarrow \lambda(f)$ for all f in X . By the Banach-Alaoglu theorem there exists a weak* limit point μ of $\{\lambda_n\}$. Since $\mu(f) = \lambda(f)$ for all f in X and X is an \mathcal{L} -determining set for λ , we have $\mu = \lambda$. The sequence $\{\lambda_n\}$ is contained in a compact set and has the unique limit point λ , hence the sequence converges to λ . Thus $\lambda_n(f) \rightarrow \lambda(f)$ for all f in C and the theorem is proved.

Another fundamental result is the relation between \mathcal{F} - and \mathcal{L} -Korovkin sets.

THEOREM 1.2. *For a convex subset \mathcal{L} of C^* we define \mathcal{F} as above. Let X be a subspace of C and T an operator in \mathcal{F} . Then X is a \mathcal{F} -Korovkin set for T if and only if X is an \mathcal{L} -Korovkin set for $T^*\hat{q}$ where q is any point in Q .*

Proof. Suppose X is a \mathcal{F} -Korovkin set for T . Let q be a point in Q . Suppose there is a sequence $\{\lambda_n\}$ in \mathcal{L} such that $\lambda_n(f) \rightarrow (T^*\hat{q})(f)$ for f in X . We show $\lambda_n(f) \rightarrow (T^*\hat{q})(f)$ for all f in C by constructing a sequence of operators in \mathcal{F} . Let the sequence $\{U_n\}$

be a decreasing neighborhood base at the point q . This is possible since Q is first countable. By Urysohn's lemma there exists, for each $n \geq 1$, a continuous function $h_n: Q \rightarrow [0, 1]$ such that $h_n(q) = 1$ and $h_n(p) = 0$ for p not in U_n . Define the operator T_n in $\mathcal{B}(C)$ by

$$(T_n f)(p) = h_n(p)\lambda_n(f) + (1 - h_n(p))(Tf)(p).$$

It follows from the definition of \mathcal{L} and the convexity of \mathcal{L} that T_n is in \mathcal{L} for all $n \geq 1$. Fix $\varepsilon > 0$ and f in X . Since Tf is a continuous function, there exists a set, say U_m , in the neighborhood base of the point q such that

$$|(Tf)(q) - (Tf)(p)| < \varepsilon/2$$

for all p in U_m . Also $h_n(p) = 0$ for any p not in U_m and any $n \geq m$. Hence, for any p in Q and any $n \geq m$ we have

$$h_n(p)|(Tf)(p) - (Tf)(q)| < \varepsilon/2.$$

By assumption there exists $j \geq 1$ such that for all $n \geq j$

$$|\lambda_n(f) - (Tf)(q)| < \varepsilon/2.$$

Using the definition of T_n and letting k be the maximum of j and m , we have for all $n \geq k$

$$\begin{aligned} |(T_n f)(p) - (Tf)(p)| &= |h_n(p)\lambda_n(f) - h_n(p)(Tf)(p)| \\ &= h_n(p)|\lambda_n(f) - (Tf)(p)| \\ &\leq h_n(p)(|\lambda_n(f) - (Tf)(q)| + |(Tf)(q) - (Tf)(p)|) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since X is a \mathcal{L} -Korovkin set for T , $T_n f \rightarrow Tf$ for all f in C . In particular,

$$\lambda_n(f) = (T_n f)(q) \rightarrow (Tf)(q) = (T^* \hat{q})(f).$$

This shows that X is an \mathcal{L} -Korovkin set for $T^* \hat{q}$ for all q in Q .

On the other hand, suppose X is an \mathcal{L} -Korovkin set for $T^* \hat{q}$ for all q in Q . Let $\{T_n\}$ be a sequence of operators in \mathcal{L} such that $T_n f \rightarrow Tf$ for all f in X . As in Šaškin [7] we use the well-known result that a sequence $\{f_n\}$ in C converges uniformly to f in C if and only if for any sequence $\{q_n\}$ converging to a point q in Q it follows that $f_n(q_n) \rightarrow f(q)$. Let $\{q_n\}$ be a sequence in Q converging to a point q in Q . For all f in X , $(T_n f)(q_n) \rightarrow (Tf)(q)$ or $(T_n^* \hat{q}_n)(f) \rightarrow (T^* \hat{q})(f)$. By assumption this implies $(T_n^* \hat{q}_n)(f) \rightarrow (T^* \hat{q})(f)$ for all f in C . By the same result we now have $T_n f \rightarrow Tf$ for all f in C . Hence, X is a \mathcal{L} -Korovkin set for T and the theorem is proved.

2. Positive operators. In this section we give two characterizations of Korovkin sets for positive operators.

Let \mathcal{L}_+ denote the positive bounded linear operators on C and let \mathcal{L}_+ denote the positive bounded linear functionals on C . For a subspace X of C and a functional μ in C^* , let $\mu|_X$ be the functional in X^* obtained by restricting μ to X . We define $M = \{\hat{p}|_X; p \in Q\}$.

In Micchelli [5] and Grossman [2], the authors assume that a \mathcal{L}_+ -Korovkin set contains a strictly positive function. The following theorem (compare to Šaškin [7, Lemma 1]) shows this hypothesis to be unnecessary.

THEOREM 2.1. *If a subspace X of C is an \mathcal{L}_+ -Korovkin set for a functional μ in \mathcal{L}_+ , then X contains a strictly positive function.*

Proof. Suppose X is an \mathcal{L}_+ -Korovkin set for a functional μ in \mathcal{L}_+ . Let $\overline{\text{co}}(M)$ be the weak* closure of the convex hull of M . We claim $\overline{\text{co}}(M)$ does not contain $0|_X$, the zero functional restricted to X . We assume the claim is false and arrive at a contradiction. Let $\{\mu_\alpha\}$ be a net in $\text{co}(M)$, the convex hull of M , such that $\mu_\alpha \rightarrow 0|_X$ in the weak* topology of X^* . For each α there exists a positive integer n , $\beta_i \geq 0$ and q_i in Q , $1 \leq i \leq n$, where $\sum_{i=1}^n \beta_i = 1$, such that

$$\mu_\alpha = \sum_{i=1}^n \beta_i \hat{q}_i|_X.$$

We define the natural extension $\nu_\alpha = \sum_{i=1}^n \beta_i \hat{q}_i$ in C^* . Note that ν_α is a positive functional where $\nu_\alpha(1) = 1$ and therefore $\|\nu_\alpha\| = 1$. By the Banach-Alaoglu theorem there exists a weak* limit point ν of $\{\nu_\alpha\}$. Clearly, ν is a positive functional in C^* with $\nu|_X = 0|_X$, but $\nu \neq 0$ since $\nu(1) = 1$. Then for the sequence $\{\lambda_n\}$ in \mathcal{L}_+ given by $\lambda_n = \nu + \mu$, we have $\lambda_n(f) \rightarrow \mu(f)$ for all f in X . However, $\lambda_n(1) = \nu(1) + \mu(1) = 1 + \mu(1)$ does not converge to $\mu(1)$. This contradicts the hypothesis of the theorem. So the claim is true. By a standard separation theorem there exists a weak closed hyperplane H separating $0|_X$ and $\overline{\text{co}}(M)$, i.e., there exists g in X and $\beta > 0$ such that $g(p) > \beta$ for all p in Q . The theorem is proved.

THEOREM 2.2. *A subspace X of C is a \mathcal{L}_+ -Korovkin set for an operator T in \mathcal{L}_+ if and only if X is an \mathcal{L}_+ -determining set for $T^*\hat{q}$ for all q in Q .*

Proof. Suppose X is a \mathcal{L}_+ -Korovkin set for T . By Theorem 1.2, X is an \mathcal{L}_+ -Korovkin set for $T^*\hat{q}$ for all q in Q . Therefore, as in Theorem 1.1, X is also an \mathcal{L}_+ -determining set for $T^*\hat{q}$ for all

q in Q .

Conversely, suppose X is an \mathcal{L}_+ -determining set for $T^*\hat{q}$ for all q in Q . By Theorem 2.1, there exists g in X such that $g(q) \geq 1$ for all q in Q . Fix a point q in Q . We claim X is an \mathcal{L}_+ -Korovkin set for $T^*\hat{q}$. Let $\{\lambda_n\}$ be a sequence in \mathcal{L}_+ such that $\lambda_n(f) \rightarrow (T^*\hat{q})(f)$ for all f in X . In particular, $\lambda_n(g) \rightarrow (T^*\hat{q})(g) = (Tg)(q)$. Since λ_n is a positive functional for each $n \geq 1$

$$\|\lambda_n\| = \lambda_n(1) \leq \lambda_n(g) \rightarrow (Tg)(q).$$

Therefore, the sequence $\{\lambda_n\}$ is norm bounded. By Theorem 1.1, the claim is true. The result now follows from Theorem 1.2. The theorem is proved.

If the subspace X is finite dimensional, then a more geometric condition is possible. When T is the identity operator, Corollary 2.3 is essentially Lorentz's Theorem 4 [3].

COROLLARY 2.3. *An m -dimensional subspace X is a \mathcal{L}_+ -Korovkin set for an operator T in \mathcal{L}_+ if and only if X satisfies the condition that $0|_X$ is not in $\text{co}(M)$ and for any subset $\{q, q_1, q_2, \dots, q_{m+1}\}$ in Q and $\beta_i \geq 0$ for $1 \leq i \leq m + 1$ the equality $\sum_{i=1}^{m+1} \beta_i \hat{q}_i|_X = (T^*\hat{q})|_X$ implies $\sum_{i=1}^{m+1} \beta_i \hat{q}_i = T^*\hat{q}$.*

Proof. The necessity of the condition follows directly from Theorem 2.2.

We now show the condition is sufficient. Suppose X satisfies the condition of the theorem, but X is not a \mathcal{L}_+ -Korovkin set for T . By Theorem 2.2 there exists a positive functional μ in C^* such that for some q in Q we have $\mu(f) = (T^*\hat{q})(f)$ for all f in X , but for some g in C we have $\mu(g) \neq (T^*\hat{q})(g)$. Let Y be the subspace of C spanned by X and g . Let M_1 be the set of point evaluations in Y^* . The functional $0|_Y$ is not in $\text{co}(M_1)$, since $0|_X$ is not in $\text{co}(M)$. By a known theorem there exists $\alpha_i \geq 0$ and q_i in Q for $1 \leq i \leq m + 1$ such that

$$\mu(f) = \sum_{i=1}^{m+1} \alpha_i \hat{q}_i(f)$$

for all f in Y . Thus,

$$\sum_{i=1}^{m+1} \alpha_i \hat{q}_i(f) = (T^*\hat{q})(f)$$

for all f in X , but not for $f = g$. This contradicts our assumption. Hence, X must be a \mathcal{L}_+ -Korovkin set for T . The corollary is proved.

If a positive operator T has an m -dimensional \mathcal{L}_+ -Korovkin set,

then Micchelli [5] has shown that T must be *finitely defined of order* m . This means that for every q in Q there exists $\alpha_i \geq 0$ and q_i in Q for $1 \leq i \leq m$ such that $T^*\hat{q} = \sum_{i=1}^m \alpha_i q_i$. We shall show a similar result for contractive operators in the next section.

3. Contractive operators. In this section we prove two characterizations of contractive operator Korovkin sets. We also establish two properties of any operator T having a contractive operator Korovkin set. Finally, we give conditions under which positive operator Korovkin sets are equivalent to contractive operator Korovkin sets. These results are all stated for an operator T of norm one. If T has norm $c > 0$, it is easy to verify the corresponding theorems for sequences of operators of norm at most c .

Let \mathcal{L}^1 denote the bounded linear operators on C of norm at most one and let \mathcal{L}^1 denote the bounded linear functionals on C of norm at most one. The following result can be compared with Šaškin [9].

THEOREM 3.1. *Let T be a norm one bounded linear operator. A subspace X of C is a \mathcal{L}^1 -Korovkin set for T if and only if X is an \mathcal{L}^1 -determining set for $T^*\hat{q}$ for all q in Q .*

Proof. The proof follows directly from Theorems 1.1 and 1.2.

The following is the analogue of Šaškin's Theorem 2 [9]. This condition seems to be necessarily more complicated than Šaškin's condition since for q in Q the functional $T^*\hat{q}$ is not necessarily a point evaluation.

COROLLARY 3.2. *Let T be an operator in $\mathcal{B}(C)$ of norm one. An m -dimensional subspace X of C is a \mathcal{L}^1 -Korovkin set for T if and only if X satisfies the condition that, for any subset $\{q, q_1, \dots, q_{m+2}\}$ in Q and for any functional $\mu = \sum_{i=1}^{m+2} \beta_i \hat{q}_i$ where β_i is in \mathbf{R} and $\sum_{i=1}^{m+2} |\beta_i| = 1$, the equality $\mu|_X = (T^*\hat{q})|_X$ implies $\mu = T^*\hat{q}$.*

The proof of this corollary is similar to the proof of Corollary 2.3.

If an operator T has a \mathcal{L}^1 -Korovkin set X , then $T^*\hat{q}$ for each q in Q must have certain properties. First, we note that T must be a finitely defined operator if X is finite dimensional.

COROLLARY 3.3. *If the m -dimensional subspace X of C is a \mathcal{L}^1 -Korovkin set for the norm one operator T in $\mathcal{B}(C)$, then T is finitely defined of order $m + 1$.*

Proof. Suppose T and X satisfy the hypotheses of the theorem.

For each q in Q there exist q_1, q_2, \dots, q_{m+1} in Q and α_i in \mathbf{R} where $\sum_{i=1}^{m+1} |\alpha_i| = 1$ such that for any f in X

$$(T^*\hat{q})(f) = \sum_{i=1}^{m+1} \alpha_i \hat{q}_i(f).$$

Since X is a Korovkin set, by Theorem 3.1, the above equality holds for all f in C . Therefore T is finitely defined of order $m + 1$. The corollary is proved.

In the following Corollary we have another condition on $(T^*\hat{q})|_X$.

COROLLARY 3.4. *Let T be a norm one operator in $\mathcal{B}(X)$. Let X be a closed proper subspace of C . If X is a \mathcal{L}^1 -Korovkin set for T , then for all q in Q we have*

$$\|T^*\hat{q}|_X\|_{X^*} = \|T^*\hat{q}\| = 1$$

where

$$\|(T^*\hat{q})|_X\|_{X^*} = \sup \{ |(T^*\hat{q})(f)| : f \in X, \|f\| \leq 1 \}.$$

Proof. Let $\alpha = \|T^*\hat{q}|_X\|_{X^*}$. Then $\alpha \leq \|T^*\hat{q}\| \leq 1$. Suppose $\alpha < 1$. Then by the Hahn-Banach theorem $(T^*\hat{q})|_X$ has an extension to C of norm α . By a modification (see Rusk [6]) of the proof of the Hahn-Banach theorem one shows that $(T^*\hat{q})|_X$ has an extension to C of norm β where $\alpha < \beta \leq 1$. This contradicts Theorem 3.1. Therefore $\alpha = 1$, which proves the corollary.

A natural question arises about subspaces which are contractive Korovkin sets for any finitely defined operator of a given order. Cavaretta [1] has given such sets for the positive operators. The next corollary shows that there are no such finite dimensional subspaces for contractive operators.

COROLLARY 3.5. *Let Q also be nondiscrete. Then C has no finite dimensional \mathcal{L}^1 -Korovkin set for all finitely defined operators T of order 2 such that $\|T^*\hat{q}\| = 1$ for all q in Q .*

Proof. Suppose the theorem is false. Then there exists a finite dimensional subspace X of C which is such a set. For any distinct points q_1 and q_2 in Q and β_1 and β_2 in \mathbf{R} such that $|\beta_1| + |\beta_2| = 1$ there is some operator T as above such that $T^*\hat{q} = \beta_1\hat{q}_1 + \beta_2\hat{q}_2$. By Corollary 3.4, $\|(T^*\hat{q})|_X\|_{X^*} = 1$. Since the closed unit ball of X is compact, there exists f in X such that $(T^*\hat{q})(f) = 1 = \beta_1 f(q_1) + \beta_2 f(q_2)$ and $\|f\| \leq 1$. This can happen only if $f(q_1) = \text{sgn } \beta_1$ and $f(q_2) = \text{sgn } \beta_2$, where $\text{sgn } x = x/|x|$ if $x \neq 0$. Since Q is not discrete, there exists a sequence $\{q_n\}$ in Q such that $q_n \rightarrow q$ in Q , and such that $q_n \neq q$ for $n \geq 1$. Then according to the previous argument there

exist f_n in C of norm one such that $f_n(q_n) = 1$, but $f_n(q) = -1$ for $n \geq 1$. Again since the closed unit ball of X is compact, there exists a subsequence $\{n_j\}$ such that $f_{n_j} \rightarrow f$ uniformly as $j \rightarrow \infty$ for some f in X . From the uniform convergence we have $1 = f_{n_j}(q_{n_j}) \rightarrow f(q)$ as $j \rightarrow \infty$. On the other hand $-1 = f_{n_j}(q) \rightarrow f(q)$ as $j \rightarrow \infty$. This contradiction implies the corollary is true.

Suppose X is a finite dimensional subspace of C containing the constants. Lorentz [3, Theorem 8] has shown that X is a \mathcal{L}_+ -Korovkin set for the identity operator if and only if X is a \mathcal{L}^1 -Korovkin set for the identity. We extend this result in the next theorem.

THEOREM 3.6. *Let T be a norm one positive operator in $\mathcal{B}(C)$ and let X be a subspace of C containing the constants. The subspace X is a \mathcal{L}_+ -Korovkin set for T and $T1 = 1$ if and only if X is a \mathcal{L}^1 -Korovkin set for T .*

Proof. Suppose X is a \mathcal{L}_+ -Korovkin set for T and $T1 = 1$. Suppose for a point q in Q and a functional μ in \mathcal{L}^1 that $\mu|_X = (T^*\hat{q})|_X$. From

$$\mu(1) = (T1)(q) = 1 = \|\mu\|$$

it follows that μ is a positive functional. By Theorem 2.2 we have $\mu = T^*\hat{q}$. Therefore, by Theorem 3.1, X is a \mathcal{L}^1 -Korovkin set for T .

Conversely, suppose X is a \mathcal{L}^1 -Korovkin set for T . By Corollary 3.4 and since T is a positive operator $1 = \|T^*\hat{p}\| = (T1)(p)$ for all p in Q , i.e., $T1 = 1$. Suppose for a point q in Q and a functional μ in \mathcal{L}_+ that $\mu|_X = (T^*\hat{q})|_X$. Then

$$\|\mu\| = \mu(1) = (T1)(q) = 1.$$

By Theorem 3.1 we have $\mu = T^*\hat{q}$. Therefore by Theorem 2.2, X is a \mathcal{L}_+ -Korovkin set for T . The theorem is proved.

The hypothesis that $T1 = 1$ cannot be omitted in Theorem 3.6. Consider the positive norm one operator T in $\mathcal{B}(C)$ where $Q = [0, 1]$ defined by

$$(Tf)(q) = (1 + q)f(q)/3, \quad q \in [0, 1].$$

If X is spanned by $\{1, x, x^2\}$, then X is a \mathcal{L}_+ -Korovkin set for T (see Cavaretta [1, Theorem 2]). However, since $\|T^*\hat{q}\| = 1/3$ when $q = 0$, by Corollary 3.4, X is not a \mathcal{L}^1 -Korovkin set for T .

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