AN APPROXIMATION THEOREM FOR MAPS INTO KAN FIBRATIONS

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In this note we prove that a semisimplicial map into the base of a Kan fibration having a continuous lifting to the total space also has a semisimplicial lifting, very "close" to a given continuous lifting. As a special case we obtain a new proof of the famous Milnor-Lamotke theorem that a Kan set is a strong deformation retract of the singular set of its geometric realization.

First we state our main

THEOREM. Let

\[
\begin{array}{ccc}
X & \xrightarrow{f} & E \\
\downarrow i & & \downarrow p \\
Y & \xrightarrow{h} & B
\end{array}
\] (*)

be a commutative square in the category of semisimplicial sets with \(i\) an inclusion and \(p\) a Kan fibration. Further, suppose given a continuous \(\bar{g}: |Y| \rightarrow |E|\) with \(\bar{g} \circ i = |f|\) and \(p \circ \bar{g} = |h|\). Then there exists a homotopy \(\bar{g} \equiv g'\) rel. \(|X|\) and over \(|B|\) so that \(g' = |g|\) for some semisimplicial \(g\).

This theorem has an interesting special case. Take \(X = E\) a Kan set, \(Y = S|E|\), \(B\) a point, \(p, h\) the unique constant maps, \(f = \text{id}_E\), \(i\) the natural inclusion and \(\bar{g}\) the natural retraction. What comes out is the famous Milnor-Lamotke theorem saying \(E\) is strong deformation retract of \(S|E|\). Thus we get a new proof of this theorem which in contrast to the original one [4] avoids any reference to J.H.C. Whitehead's theorems.

On the other hand, if \(B\) is a point, the statement is a trivial consequence of the Milnor-Lamotke theorem. An elementary proof for this case—avoiding the Milnor-Lamotke theorem—has been given by B. J. Sanderson [7] whose techniques are also important for our proceeding.

Proof of theorem. (For the technical details we use the notation explained in §0 of [1].) By an induction over skeletons, it is enough
to prove the theorem in the case \( y \) is an \( n \)-simplex \( \Delta[n] \) with \( n > 0 \) and \( X \) is its boundary \( \partial[n] \). Let \( \tau \) be the generating simplex of \( \Delta[n] \), \( y = h \tau \in B \) and \( \bar{y} = S\bar{g}(\epsilon) \in S|E| \). We have to prove that \( \bar{y} \) is \( S|B| \)-equivalent (\([3]\) p. 123) to a simplex in \( E^0 \).

Decompose \( y = y^+y^o \) with \( y^+ \) nondegenerate and \( y^o \) surjective. We perform a further induction, over a (partial) ordering of the set of the possible \( y^o \), that is the set \( D_n \) of surjective monotone maps with domain \([n]\). Choose\(^1\) an ordering of this set satisfying (i) and (ii):

(i) \( \beta \alpha \leq \alpha \) if \( \alpha, \beta \alpha \in D_n \); and

(ii) each nonconstant \( \alpha \in D_n \) admits an \( \alpha' < \alpha \) so that \( \alpha' \) is the surjective part of \( \alpha \sigma_i \delta_j \) for some suitable pair \( i, j \).

Evidently the constant map is the minimum of \( D_n \) with respect to this ordering.

First, assume \( y^o \) is constant. Denote by \( F \) the fibre over \( y \) which is Kan. Now comes Sanderson's idea. Since the boundary of \( \bar{y} \) belongs to \( F \) we can choose the zeroth vertex \( ^0 \) of \( \bar{y} \) for base point of \( F \). Then, form the path fibration \( q: W(F) \to F \) (\([5]\) p. 196) and lift \( \bar{y} \) to a filling \( \bar{u} \) in \( S|W(F)| \) of the horn \( (-, \bar{y} \delta_i \sigma_j, ..., \bar{y} \delta_n \sigma_0) \) in \( W(F) \subset S|W(F)| \). By induction, \( \bar{u} \delta_0 \) is \( S|F| \)-equivalent to an \( u \in W(F) \). That gives a \( \bar{z} \in S|W(F)| \) with boundary \( (u, \bar{u} \delta_0, u_\sigma \delta_1, ..., u_\sigma \delta_n) \) and \( S|q| \bar{z} = \bar{y} \delta \sigma_0 \in F \) (\([5]\) p. 25). Next we use that every sphere in \( W(F) \) can be filled (\([5]\) p. 196) and also every sphere in \( S|W(F)| \) since \( W(F) \) is contractible. Take a filling \( v \in W(F) \) of the sphere \( (u, \bar{y} \delta \sigma_0, ..., \bar{y} \delta_n \sigma_0) \) and finally a filling \( \bar{v} \in S|W(F)| \) of the sphere \( (\bar{z}, v, \bar{u}, \bar{z} \sigma \delta_0, ..., \bar{z} \sigma \delta_{n+1}) \). Then \( S|q| \bar{v} \) is an \( S|B| \)-equivalence between \( \bar{y} \) and \( qv \in F \subset E \).

If \( y^o \) is not constant, we choose \( i \) and \( j \) such that the surjective part of \( y^o \sigma_i \delta_j \) is less than \( y^o \). Set \( \epsilon = 0 \) if \( j < i \) and \( \epsilon = 1 \) if \( j > i + 1 \). Lift \( y \) to \( u \in E \) with \( \bar{u} \delta_k = \bar{y} \delta_k \) if \( k \neq j - \epsilon \) and lift \( y \sigma_i \) to \( \bar{u} \in S|E| \) with \( \bar{u} \delta_i = \bar{y} \), \( \bar{u} \delta_{i+1} = u \), \( \bar{u} \delta_k = \bar{y} \sigma_i \delta_k \) if \( k \neq i, i + 1, j \). By induction, \( \bar{u} \delta_j \) is \( |B| \)-equivalent to a \( v \in E \) and there is a \( \bar{v} \in S|E| \) with boundary \( (v \sigma_i \delta_i, ..., v \bar{u} \delta_j, ..., v \sigma_i \delta_{n+1}) \) and \( S|p| \bar{v} = y \sigma_i \sigma_{i+1} \delta_{j+1} \).

Next, lift \( y \sigma_i \) to \( w \in E \) with \( w \delta_{i+1} = u \), \( w \delta_j = v \), \( w \delta_k = \bar{y} \sigma_i \delta_k \) if \( k \neq i, i + 1, j \) and lift \( y \sigma_i \sigma_{i+1} \) to \( \bar{w} \) with \( \bar{w} \delta_{i+1} = w \), \( \bar{w} \delta_{i+1} = w \), \( \bar{w} \delta_{i+1} = v \), \( \bar{w} \delta_k = w \sigma_i \delta_k \) if \( k \neq i, i + 1, i + 2, j + \epsilon \). Then \( \bar{w} \delta_i \) is an \( S|B| \)-equivalence between \( \bar{y} \) and \( w \sigma_i \delta_i \in E \).

This finishes the proof. As an application, we'll derive a streng-

\(^1\) Note that \( S|p| \) is also a Kan fibration, by Quillen’s result \([6]\).

\(^2\) Cf. the proof of Lemma 4 in \([2]\).
thening of this result which is based on the cartesian closedness of the category of semisimplicial sets. Roughly speaking, it states the semisimplicial set of semisimplicial diagonals of a square as in the theorem is a strong deformation retract of the semisimplicial set of its continuous diagonals.

To make this precise, we define the semisimplicial set $D(Y, E)$ of (semisimplicial) diagonals of a square (*) by means of the following diagram where the squares involved are pullbacks

Further, the semisimplicial set of continuous diagonals of (*) is defined to be the semisimplicial set $D(Y, S|E|)$ of semisimplicial diagonals of the square

\[
\begin{array}{ccc}
X & \xrightarrow{i \circ f} & S|E| \\
\downarrow{i} & & \downarrow{|S|p|} \\
Y & \xrightarrow{ib \circ h} & S|B|
\end{array}
\]

The following lemma gives another description of $D(Y, S|E|)$.

**Lemma.** Let

\[
\begin{array}{ccc}
\bar{E} & \rightarrow & S|E| \\
\downarrow{\bar{p}} & & \downarrow{|S|p|} \\
B & \xrightarrow{ib} & S|\bar{B}|
\end{array}
\]

be a pullback. Then the semisimplicial set $D(Y, \bar{E})$ of diagonals of the induced square
is isomorphic to $D(Y, S|E|)$.

The proof of this lemma is evident. Note the universal property of $\bar{E}$: The continuous $\bar{g}: |Y| \to |E|$ so that $|p| \circ \bar{g}$ is realized correspond bijectively to the semisimplicial maps $Y \to \bar{E}$. If $B$ is a point, this is the adjunction between geometric realization and singular functor.

With these definitions we have the

**Corollary.** Under the assumptions of the theorem on the square (*) $D(Y, E)$ is a strong deformation retract of $D(Y, \bar{E})$.

**Proof.** The map $|\bar{E}| \to |E|$ corresponding to $\text{id}\bar{E}$ is a continuous diagonal of the square

$$
\begin{array}{ccc}
E & \xrightarrow{\text{id}E} & E \\
\downarrow & & \downarrow p \\
\bar{E} & \to & B
\end{array}
$$

Thus, the theorem implies $E$ is a strong deformation retract of $\bar{E}$. Let $G: \bar{E} \times \Delta[1] \to \bar{E}$ be a suitable deformation. Further, let $e$ denote the evalulation $Y \times \bar{E}^r \to \bar{E}$ and $\text{id}\bar{E}$. Then, by adjointness $G \circ e$ corresponds to a map $K: \bar{E}^r \times \Delta[1] \to \bar{E}^r$. Its restriction to $D(Y, \bar{E}) \times \Delta[1]$ factors through $D(Y, \bar{E})$ and induces a deformation of the desired kind.

**References**

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