AN APPROXIMATION THEOREM FOR MAPS INTO KAN FIBRATIONS

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In this note we prove that a semisimplicial map into the base of a Kan fibration having a continuous lifting to the total space also has a semisimplicial lifting, very “close” to a given continuous lifting. As a special case we obtain a new proof of the famous Milnor-Lamotke theorem that a Kan set is a strong deformation retract of the singular set of its geometric realization.

First we state our main

THEOREM. Let

\[ \begin{array}{ccc}
X & \xrightarrow{f} & E \\
\downarrow{i} & & \downarrow{p} \\
Y & \xrightarrow{h} & B 
\end{array} \]

be a commutative square in the category of semisimplicial sets with \( i \) an inclusion and \( p \) a Kan fibration. Further, suppose given a continuous \( \bar{g} : |Y| \to |E| \) with \( \bar{g} \circ i = |f| \) and \( |p| \circ \bar{g} = |h| \). Then there exists a homotopy \( \bar{g} \equiv g' \) rel. \( |X| \) and over \( |B| \) so that \( g' = |g| \) for some semisimplicial \( g \).

This theorem has an interesting special case. Take \( X = E \) a Kan set, \( Y = S|E| \), \( B \) a point, \( p, h \) the unique constant maps, \( f = \text{id}_E \), \( i \) the natural inclusion and \( \bar{g} \) the natural retraction. What comes out is the famous Milnor-Lamotke theorem saying \( E \) is strong deformation retract of \( S|E| \). Thus we get a new proof of this theorem which in contrast to the original one [4] avoids any reference to J.H.C. Whitehead’s theorems.

On the other hand, if \( B \) is a point, the statement is a trivial consequence of the Milnor-Lamotke theorem. An elementary proof for this case—avoiding the Milnor-Lamotke theorem—has been given by B. J. Sanderson [7] whose techniques are also important for our proceeding.

Proof of theorem. (For the technical details we use the notation explained in §0 of [1].) By an induction over skeletons, it is enough
to prove the theorem in the case \( y \) is an \( n \)-simplex \( \Delta[n] \) with \( n > 0 \) and \( X \) is its boundary \( \partial [n] \). Let \( \iota \) be the generating simplex of \( \Delta[n], y = \iota \in B \) and \( \overline{y} = S\overline{\iota} \in SBE \). We have to prove that \( \overline{y} \) is \( SBE \)-equivalent ([3] p. 123) to a simplex in \( E' \).

Decompose \( y = y^+y^0 \) with \( y^+ \) nondegenerate and \( y^0 \) surjective. We perform a further induction, over a (partial) ordering of the set of the possible \( y^0 \), that is the set \( D_n \) of surjective monotone maps with domain \([n]\). Choose\(^{3}\) an ordering of this set satisfying (i) and (ii):

(i) \( \beta \alpha \leq \alpha \) if \( \alpha, \beta \alpha \in D_m \); and

(ii) each nonconstant \( \alpha \in D_n \) admits an \( \alpha' < \alpha \) so that \( \alpha' \) is the surjective part of \( \alpha \sigma_i \delta_j \) for some suitable pair \( i,j \).

Evidently the constant map is the minimum of \( D_n \) with respect to this ordering.

First, assume \( y^0 \) is constant. Denote by \( F \) the fibre over \( y \) which is Kan. Now comes Sanderson's idea. Since the boundary of \( \overline{y} \) belongs to \( F \) we can choose the zeroth vertex \(*\) of \( \overline{y} \) for base point of \( F \). Then, form the path fibration \( q:W(F) \to F \) ([5] p. 196) and lift \( y \) to a filling \( \bar{u} \) in \( S|W(F)| \) of the horn \((-,-, \bar{y} \sigma_{i_0}, \ldots, \bar{y} \sigma_{i_n} \sigma_0)\) in \( W(F) \subset S|W(F)| \). By induction, \( \bar{u} \delta_0 = S|F| - \) equivalent to an \( u \in W(F) \). That gives a \( \bar{z} \in S|W(F)| \) with boundary \( (u, \bar{u} \delta_0, u \sigma_0 \delta_2, \ldots, u \sigma_j \delta_n) \) and \( S|q| \bar{z} = \bar{y} \delta_0 \sigma_0 \in F \) ([5] p. 25). Next we use that every sphere in \( W(F) \) can be filled ([5] p. 196) and also every sphere in \( S|W(F)| \) since \( W(F) \) is contractible. Take a filling \( v \in W(F) \) of the sphere \( (u, \bar{y} \sigma_0, \ldots, \bar{y} \sigma_n \sigma_0) \) and finally a filling \( \bar{v} \in S|W(F)| \) of the sphere \( (\bar{z}, v, \bar{u}, \bar{z} \sigma_0 \delta_2, \ldots, \bar{z} \sigma_0 \delta_n) \). Then \( S|q| \bar{v} \) is an \( S|B| \)-equivalence between \( \overline{y} \) and \( qv \in F \subset E \).

If \( y^0 \) is not constant, we choose \( i \) and \( j \) such that the surjective part of \( y^0 \sigma_i \delta_j \) is less than \( y^0 \). Set \( \varepsilon = 0 \) if \( j < i \) and \( \varepsilon = 1 \) if \( j > i + 1 \). Lift \( y \) to \( u \in E \) with \( u \delta_k = \bar{y} \delta_k \) if \( k \neq j - \varepsilon \) and lift \( y \sigma_i \) to \( \bar{u} \in S|E| \) with \( \bar{u} \delta_i = \bar{y}, \bar{u} \delta_{i+1} = u, \bar{u} \delta_k = \bar{y} \sigma_i \delta_k \) if \( k \neq i, i + 1, j \). By induction, \( \bar{u} \delta_j \) is \( |B|-\)equivalent to a \( v \in E \) and there is a \( \bar{v} \in S|E| \) with boundary \( (v \sigma_i \delta_0, \ldots, v, v \sigma_i \delta_n, \ldots, v \sigma_{i+1} \delta_{n+1}) \) and \( S|p| \bar{v} = y \sigma_i \sigma_{i+1} \delta_{j+1} \). Next, lift \( y \sigma_i \) to \( w \in E \) with \( w \delta_{i+1} = u, w \delta_j = v, w \delta_k = \bar{y} \sigma_i \delta_k \) if \( k \neq i, i + 1, j \) and lift \( y \sigma_i \sigma_{i+1} \) to \( \bar{w} \) with \( \bar{u} \delta_{i+1} = w, \bar{w} \delta_{i+2} = u, \bar{w} \delta_{j+1} = \bar{v}, \bar{w}_k = w \sigma_{i+1} \delta_k \) if \( k \neq i, i + 1, i + 2, j + \varepsilon \). Then \( \bar{w} \delta_i \) is an \( S|B| \)-equivalence between \( \overline{y} \) and \( w \delta_i \in E \).

This finishes the proof. As an application, we'll derive a strenge-

\(^{1}\) Note that \( S|p| \) is also a Kan fibration, by Quillen's result [6].

\(^{2}\) Cf. the proof of Lemma 4 in [2].
thening of this result which is based on the cartesian closedness of
the category of semisimplicial sets. Roughly speaking, it states the
semisimplicial set of semisimplicial diagonals of a square as in the
theorem is a strong deformation retract of the semisimplicial set of
its continuous diagonals.

To make this precise, we define the semisimplicial set \( D(Y, E) \)
of (semisimplicial) diagonals of a square (*) by means of the following
diagram where the squares involved are pullbacks

\[
\begin{array}{ccc}
D(Y, E) \\
& \searrow \swarrow & \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f} & X \\
& \searrow \swarrow & \\
& \downarrow & \\
& \downarrow & \\
& \downarrow & \\
\downarrow & & \downarrow \\
S \uparrow \ & \xrightarrow{E} & S \downarrow \uparrow \ \\
& \swarrow \searrow & \\
S \uparrow & & \downarrow S \downarrow \\
& \searrow \swarrow & \\
E & \xrightarrow{p} & Y \\
& \searrow \swarrow & \\
& \downarrow & \\
& \downarrow & \\
& \downarrow & \\
\downarrow & & \downarrow \\
S & \uparrow \ & \xrightarrow{h} & S \\
\end{array}
\]

Further, the semisimplicial set of continuous diagonals of (*) is defined
to be the semisimplicial set \( D(Y, S|E|) \) of semisimplicial diagonals
of the square

\[
\begin{array}{ccc}
X & \xrightarrow{i_{E\cap f}} & S|E| \\
\downarrow & & \downarrow \\
Y & \xrightarrow{i_{E\cap h}} & S|B| \\
\end{array}
\]

The following lemma gives another description of \( D(Y, S|E|) \).

**Lemma.** Let

\[
\begin{array}{ccc}
\bar{E} & \longrightarrow & S|E| \\
\downarrow & & \downarrow \\
\bar{B} & \longrightarrow & S|\bar{B}| \\
\end{array}
\]

be a pullback. Then the semisimplicial set \( D(Y, \bar{E}) \) of diagonals of
the induced square
is isomorphic to $D(Y, S|E|)$.

The proof of this lemma is evident. Note the universal property of $E$: The continuous $g: Y \rightarrow |E|$ so that $|p| \circ g$ is realized correspond bijectively to the semisimplicial maps $Y \rightarrow \tilde{E}$. If $B$ is a point, this is the adjunction between geometric realization and singular functor.

With these definitions we have the

**Corollary.** Under the assumptions of the theorem on the square (*) $D(Y, E)$ is an strong deformation retract of $D(Y, \tilde{E})$.

**Proof.** The map $|\tilde{E}| \rightarrow |E|$ corresponding to $\text{id}\tilde{E}$ is a continuous diagonal of the square

$$
\begin{array}{ccc}
E & \xrightarrow{\text{id}E} & E \\
\downarrow & & \downarrow p \\
\tilde{E} & \longrightarrow & B
\end{array}
$$

Thus, the theorem implies $E$ is a strong deformation retract of $\tilde{E}$. Let $G: \tilde{E} \times \Delta[1]$ be a suitable deformation. Further, let $e$ denote the evalution $Y \times \tilde{E}^\circ \rightarrow \tilde{E}$ and $\text{id}\tilde{E}$. Then, by adjointness $G \circ e$ corresponds to a map $K: \tilde{E}^\circ \times \Delta[1] \rightarrow \tilde{E}^\circ$. Its restriction to $D(Y, \tilde{E}) \times \Delta[1]$ factors through $D(Y, \tilde{E})$ and induces a deformation of the desired kind.

**References**

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Andrew Adler, *Weak homomorphisms and invariants: an example* ................. 293
Howard Anton and William J. Pervin, *Separation axioms and metric-like functions* ................................................................. 299
Ron C. Blei, *Sidon partitions and p-Sidon sets* ........................................ 307
T. J. Cheatham and J. R. Smith, *Regular and semisimple modules* ........... 315
Charles Edward Cleaver, *Packing spheres in Orlicz spaces* .................. 325
Le Baron O. Ferguson and Michael D. Rusk, *Korovkin sets for an operator on a space of continuous functions* .................................................. 337
Rudolf Fritsch, *An approximation theorem for maps into Kan fibrations* ...... 347
David Sexton Gilliam, *Geometry and the Radon-Nikodym theorem in strict Mackey convergence spaces* .................................................. 353
William Hery, *Maximal ideals in algebras of topological algebra valued functions* ................................................................. 365
Alan Hopenwasser, *The radical of a reflexive operator algebra* ............... 375
Bruno Kramm, *A characterization of Riemann algebras* .......................... 393
Peter K. F. Kuhfittig, *Fixed points of locally contractive and nonexpansive set-valued mappings* .................................................. 399
Stephen Allan McGrath, *On almost everywhere convergence of Abel means of contraction semigroups* ............................................. 405
Edward Peter Merkes and Marion Wetzel, *A geometric characterization of indeterminate moment sequences* ......................................... 409
John C. Morgan, II, *The absolute Baire property* ..................................... 421
Eli Aaron Passow and John A. Roulier, *Negative theorems on generalized convex approximation* .................................................. 437
Louis Jackson Ratliff, Jr., *A theorem on prime divisors of zero and characterizations of unmixed local domains* ................................. 449
Ellen Elizabeth Reed, *A class of T₁-compactifications* .............................. 471
Maxwell Alexander Rosenlicht, *On Liouville’s theory of elementary functions* ................................................................................. 485
Arthur Argyle Sagle, *Power-associative algebras and Riemannian connections* ................................................................................. 493
Chester Cornelius Seabury, *On extending regular holomorphic maps from Stein manifolds* .......................................................... 499
Elias Sai Wan Shiu, *Commutators and numerical ranges of powers of operators* ................................................................................. 517
Donald Mark Topkis, *The structure of sublattices of the product of n lattices* .... 525
John Bason Wagoner, *Delooping the continuous K-theory of a valuation ring* ................................................................................. 533
Ronson Joseph Warne, *Standard regular semigroups* .................................. 539
Anthony William Wickstead, *The centraliser of E ⊗₀ F* ............................ 563
R. Grant Woods, *Characterizations of some C*-embedded subspaces of βN* .... 573