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A CHARACTERIZATION OF RIEMANN ALGEBRAS

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A topological algebra \mathcal{A} is called Riemann algebra if it is topologically isomorphic to the Fréchet algebra $\mathcal{O}(R)$ of all holomorphic functions on some Riemann surface R .

One obtains a characterization of Riemann algebras by a theorem of R. L. Carpenter and the Oka-Weil-Cartan theorem; we show that: a uniform Fréchet algebra \mathcal{A} whose spectrum is locally compact and connected, is a Riemann algebra if and only if every closed maximal ideal is principal.

1. Let R be a Riemann surface and $\mathcal{O}(R)$ that algebra of all holomorphic functions on R . $\mathcal{O}(R)$ is a uniform Fréchet algebra, if it is endowed with the topology of uniform convergence on compact sets of R . A topological algebra \mathcal{A} which is topologically isomorphic to $\mathcal{O}(R)$, is called a *Riemann algebra*.

If R is compact, one obtains the trivial Riemann algebra C .

In 1953 S. Kakutani posed the problem of characterizing Riemann algebras by intrinsic properties. The most far reaching result is due to I. Richards [9], who considers algebraic properties only. More natural conditions are obtained if one considers topologically algebraic properties. Some special results are found in R. F. Arens [1], F. T. Birtel [3], I. Kra [8], R. L. Carpenter [5]; a summary, in a sense, is provided by I. Richards' paper [10]. This present paper is written in a selfcontained way.

In his paper [4] R. L. Carpenter, using earlier work of A. Gleason [7], proved the following—here cited with a slight modification.

THEOREM. *Let \mathcal{A} be a uniform Fréchet algebra whose spectrum $s\mathcal{A}$ is locally compact and connected and does not consist of a singleton. If every closed maximal ideal in \mathcal{A} is principal, then $s\mathcal{A}$ can be given the structure of a Riemann surface in such a way that \mathcal{A} is topologically isomorphic to a closed subalgebra of $\mathcal{O}(s\mathcal{A})$. Furthermore \mathcal{A} contains local coordinates for every point in $s\mathcal{A}$.*

In fact, we show that \mathcal{A} is even topologically isomorphic to $\mathcal{O}(s\mathcal{A})$ under the above hypotheses.

2. Let us fix some notations first.

A *Fréchet algebra* (F -algebra) is a commutative, locally convex, complete algebra over the complex field C with unit whose topology is generated by a countable number of semi-norms.

Now let \mathcal{A} be a F -algebra. By $\mathfrak{s}\mathcal{A}$ we denote the *spectrum* of \mathcal{A} , the set of all nontrivial continuous C -algebra homomorphisms $f: \mathcal{A} \rightarrow C$; as usual, it is given the Gelfand topology (=weak- $*$ topology).

The Gelfand representation. Let $\mathcal{E}(\mathfrak{s}\mathcal{A})$ denote the space of all continuous functions on $\mathfrak{s}\mathcal{A}$, endowed with the compact open topology. Then the standard Gelfand representation

$$G: \mathcal{A} \longrightarrow \mathcal{E}(\mathfrak{s}\mathcal{A}), \quad a \longmapsto \hat{a}$$

given by setting $\hat{a}(f) := f(a)$ for $a \in \mathcal{A}$, $f \in \mathfrak{s}\mathcal{A}$, is a continuous C -algebra-homomorphism.

Call \mathcal{A} a *uniform Fréchet algebra* (uF -algebra), if the Gelfand representation G induces a topological isomorphism of \mathcal{A} onto a closed subalgebra $G(\mathcal{A}) \subset \mathcal{E}(\mathfrak{s}\mathcal{A})$.

Let $K \subset \mathfrak{s}\mathcal{A}$ be a compact set. For $a \in \mathcal{A}$ we denote the K -sup norm as usual.

$$\|a\|_K := \sup_{f \in K} |\hat{a}(f)|.$$

By \mathcal{A}_K we denote the separated completion of \mathcal{A} under the seminorm $\|\cdot\|_K$. It is well known that a uF -algebra \mathcal{A} can be represented as an inverse limit of countably many uniform Banach algebras

$$\mathcal{A} = \varprojlim \mathcal{A}_{K_n},$$

where an appropriate sequence $\dots \subset K_n \subset K_{n+1} \subset \dots$ of compact subsets exhausts $\mathfrak{s}\mathcal{A}$ such that any compact $K \subset \mathfrak{s}\mathcal{A}$ is contained in some K_n .

It is well known (e.g. cf. [6] p. 377f.) that every Riemann algebra $\mathcal{O}(R)$ is a uF -algebra whose Gelfand space is topologically equivalent to R . The following gives a simple *characterization of Riemann algebras*.

THEOREM. *Let $\mathcal{A} \neq C$ be a uF -algebra such that $\mathfrak{s}\mathcal{A}$ is locally compact and connected. Then the following statements are equivalent:*

- (i) \mathcal{A} is a Riemann algebra;
- (ii) every closed maximal ideal in \mathcal{A} is principal;
- (iii) every closed ideal in \mathcal{A} is principal.

Proof. The implication (i) \Rightarrow (iii) follows e.g. from O. Forster [6], Satz 5.2. The implication (iii) \Rightarrow (ii) is trivial. Now we prove (ii) \Rightarrow (i)¹.

Carpenter's theorem states in particular: $s\mathcal{A}$ can be given the structure of a Riemann surface by coordinates which are the Gelfand transforms of certain elements of \mathcal{A} , such that $G(\mathcal{A})$ is a closed subalgebra of $\mathcal{O}(s\mathcal{A})$.

Now we want to prove that $G(\mathcal{A}) = \mathcal{O}(s\mathcal{A})$. We will do this by means of the Oka-Weil-Cartan theorem (cf. [2], p. 145) which we state here, together with the necessary definitions. Recall that $s\mathcal{A}$ is a Riemann surface, and $G(\mathcal{A})$ is an algebra of holomorphic functions on $s\mathcal{A}$. One calls $s\mathcal{A}$:

- (a) " $G(\mathcal{A})$ -convex" if, for every compact subset $K \subset s\mathcal{A}$, the set

$$\hat{K} = \{f \in s\mathcal{A} \mid |\hat{a}(f)| \leq \|a\|_K \text{ for all } \hat{a} \in G(\mathcal{A})\}$$

is also compact;

- (b) " $G(\mathcal{A})$ -separable" if $G(\mathcal{A})$ separates points on $s\mathcal{A}$;
- (c) " $G(\mathcal{A})$ -regular" if $G(\mathcal{A})$ contains local coordinates for every point of $s\mathcal{A}$.

The Oka-Weil-Cartan theorem states that for any Stein manifold (and hence in particular for any noncompact Riemann surface), the conditions (a), (b), (c) imply that the algebra $G(\mathcal{A})$ is dense in $\mathcal{O}(s\mathcal{A})$.

Thus we wish to show that $s\mathcal{A}$ is $G(\mathcal{A})$ -convex, $G(\mathcal{A})$ -separable, and $G(\mathcal{A})$ -regular.

$G(\mathcal{A})$ -separable. This is trivial, since by definition the algebra \mathcal{A} separates points on $s\mathcal{A}$, and $G(\mathcal{A})$ is an isomorphic copy of \mathcal{A} .

$G(\mathcal{A})$ -regular. The existence of local coordinates is part of Carpenter's theorem.

$G(\mathcal{A})$ -convex. This follows from the fact that \mathcal{A} is a uF -algebra and from Tychonov's theorem. Take any compact set $K \subset s\mathcal{A}$. We need to show that its $G(\mathcal{A})$ -convex hull \hat{K} defined in (a) above is also compact. The restriction map $G(\mathcal{A}) \rightarrow G(\mathcal{A})_K$ has dense image and therefore, the canonic map $s\mathcal{A}_K \rightarrow s\mathcal{A}$ is injective;

¹ There is a different, more complicated proof of the main theorem which uses the functional calculus for B -algebras instead of the Oka-Weil-Cartan theorem.

it is even a homomorphic embedding, as a direct calculation with a subbasis of the weak- $*$ topology of $\mathfrak{s}\mathcal{A}$ shows.

Now it is easily seen that \hat{K} coincides with $\mathfrak{s}\mathcal{A}_K$. Since \mathcal{A}_K is a commutative Banach algebra with unit, it follows from Tychonov's theorem that \mathcal{A}_K has compact spectrum. Thus \hat{K} is compact, and the $G(\mathcal{A})$ -convexity is proved.

Now the Oka-Weil-Cartan theorem implies that $G(\mathcal{A})$ is a dense subalgebra of $\mathcal{O}(\mathfrak{s}\mathcal{A})$. Since \mathcal{A} is complete and topologically isomorphic to $G(\mathcal{A})$, we conclude that $G(\mathcal{A}) = \mathcal{O}(\mathfrak{s}\mathcal{A})$, as desired.

3. We illustrate the proof of our theorem by three *examples*. Let $C^* := C - \{0\}$. Then $\mathcal{A}_1 := \mathcal{O}(C)$ is a (closed) uF -subalgebra of $\mathcal{O}(C^*)$. C^* is \mathcal{A}_1 -separable and \mathcal{A}_1 -regular but not \mathcal{A}_1 -convex, and indeed $\mathcal{A}_1 \neq \mathcal{O}(C^*)$.

It is easily seen that the algebra \mathcal{A}_2 of all holomorphic functions on the analytic set

$$\{(x, y) \in C^2 \mid x^3 - y^2 = 0\} \quad (\text{Neil's parabola})$$

may be identified algebraically and topologically with the uniform closure of $C[z^2, z^3]$ in $\mathcal{O}(C)$. C is \mathcal{A}_2 -separable and \mathcal{A}_2 -convex but not \mathcal{A}_2 -regular. For one cannot find in \mathcal{A}_2 any local coordinate for a neighbourhood of $0 \in C$ which is equivalent to $z \in \mathcal{O}(C)$. In fact, the maximal ideal in \mathcal{A}_2 correspondent to $0 \in C$ is not principal, and therefore \mathcal{A}_2 is not a Riemann algebra.

The uniform closure of $C[z^2, 1/z^2]$ in $\mathcal{O}(C^*)$ provides an example of a uF -subalgebra \mathcal{A}_3 such that C^* is \mathcal{A}_3 -convex and \mathcal{A}_3 -regular but not \mathcal{A}_3 -separable.

It can happen that a Riemann algebra contains a properly closed subalgebra which can be mapped topologically isomorphic onto the former algebra. For example, take the above algebras $\mathcal{A}_3 \subset \mathcal{O}(C^*)$. The map $\mathcal{O}(C^*) \rightarrow \mathcal{A}_3$ induced by $z \mapsto z^2$, is a topologic isomorphism, by the open mapping theorem for Fréchet spaces. The hence homeomorphic spectra are linked as a twosheeted covering, by the adjoint spectral map of the inclusion map $\mathcal{A}_3 \subset \mathcal{O}(C^*)$.

4. We assume in this section that all uF -algebras under consideration have locally compact and connected spectrum. "Generators" for algebras are understood to be topological generators. Call a Riemann algebra *planar*, if its associated Riemann surface can be realized as a domain in the complex plane. Then the above theorem and Runge's approximation theorem yield the

COROLLARY. *A Riemann algebra is planar if and only if it is*

a singly rationally generated uF -algebra, whose closed maximal ideals are principal.

Finally, we note two reformulations of classic function theory in the language of uF -algebras.

PROPOSITION.

(1) *There exist two and only two singly generated uF -algebras whose closed maximal ideals are principal (modulo topological isomorphism).*

(2) *Every uF -algebra such that every closed maximal ideal is principal, can be generated by three elements.*

Proof. (1) Use Riemann's mapping theorem.

(2) Use the embedding theorem for Riemann surfaces.

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