FIXED POINTS OF LOCALLY CONTRACTIVE AND NONEXPANSIVE SET-VALUED MAPPINGS

Peter K. F. Kuhfittig
FIXED POINTS OF LOCALLY CONTRACTIVE AND NONEXPANSIVE SET-VALUED MAPPINGS

PETER K. F. KUHFITTIG

Let \((M, d)\) be a complete metric space and \(S(M)\) the set of all nonempty bounded closed subsets of \(M\). A set-valued mapping \(f: M \rightarrow S(M)\) will be called (uniformly) locally contractive if there exist \(\varepsilon > 0\), \(0 < \lambda < 1\) such that

\[ D(f(x), f(y)) \leq \lambda d(x, y) \]

whenever \(d(x, y) < \varepsilon\) and where \(D(f(x), f(y))\) is the distance between \(f(x)\) and \(f(y)\) in the Hausdorff metric induced by \(d\) on \(S(M)\). It is shown in the first theorem that if \(M\) is "well-chained," then \(f\) has a fixed point, that is, a point \(x \in M\) such that \(x \in f(x)\). This fact, in turn, yields a fixed-point theorem for locally nonexpansive set-valued mappings on a compact star-shaped subset of a Banach space. Both theorems are extensions of earlier results.

1. Locally contractive set-valued mappings. Following Assad and Kirk [1] we shall define \(D\) as follows: if \(r > 0\) and \(Y \in S(M)\), let

\[ Z(r, Y) = \{ x \in M : \text{dist} (x, Y) < r \} \]

Then for \(A, B \in S(M)\) we define

\[ D(A, B) = \inf \{ r : A \subset Z(r, B) \text{ and } B \subset Z(r, A) \} \]

Also noted in [1] are two lemmas:

**Lemma 1.** If \(A, B \in S(M)\) and \(x \in A\), then for each positive number \(\alpha\) there exists \(y \in B\) such that

\[ d(x, y) \leq D(A, B) + \alpha \]

**Lemma 2.** Let \(\{X_n\}\) be a sequence of sets in \(S(M)\), and assume that \(\lim_{n \to \infty} D(X_n, X_0) = 0\) (\(X_0 \in S(M)\)). Then if \(x_n \in X_n\) \((n = 1, 2, \cdots)\) and \(\lim_{n \to \infty} x_n = x_0\), it follows that \(x_0 \in X_0\).

Finally, suppose \(M\) is well-chained in the sense that for every \(\varepsilon > 0\) and \(x, y \in M\) there exists an \(\varepsilon\)-chain, that is, a finite set of points

\[ x = y_0, y_1, \cdots, y_n = z \]

\((n\) may depend on both \(x\) and \(z)\) such that \(d(y_i, y_{i+1}) < \varepsilon\) \((i = 0, 1, \cdots, n - 1)\).
**Theorem 1.** Suppose \((M, d)\) is a complete well-chained metric space and \(S(M)\) the set of all nonempty bounded closed subsets of \(M\). If \(f: M \to S(M)\) is locally contractive, then \(f\) has a fixed point.

**Proof.** Assume that \(\varepsilon < 1\) and let \(x_0, y_0 \in M\) such that \(d(x_0, y_0) < \varepsilon\). Then

\[
D(f(x_0), f(y_0)) \leq \lambda d(x_0, y_0).
\]

Now choose a positive number \(\eta < \varepsilon - \lambda \varepsilon < 1\). Let \(x_1\) be any element in \(f(x_0)\); then there exists by Lemma 1 an element \(y_1 \in f(y_0)\) such that

\[
d(x_1, y_1) \leq D(f(x_0), f(y_0)) + \eta.
\]

Hence

\[
d(x_1, y_1) < \lambda \varepsilon + \eta < \lambda \varepsilon + \varepsilon - \lambda \varepsilon = \varepsilon.
\]

Next, let \(x_2 \in f(x_1)\); then there exists \(y_2 \in f(y_1)\) such that

\[
d(x_2, y_2) \leq D(f(x_1), f(y_1)) + \eta^2
\]

\[
\leq \lambda d(x_1, y_1) + \eta^2.
\]

In general, for \(n > 0\)

\[
d(x_n, y_n) \leq D(f(x_{n-1}), f(y_{n-1})) + \eta^n,
\]

and we can show by induction that

\[
(1) \quad d(x_n, y_n) < \lambda^n \varepsilon + \lambda^{n-1} \eta + \lambda^{n-2} \eta^2 + \cdots + \eta^n.
\]

Indeed,

\[
\lambda^n \varepsilon + \lambda^{n-1} \eta + \lambda^{n-2} \eta^2 + \cdots + \eta^n
\]

\[
< \lambda^n \varepsilon + \lambda^{n-1} (\varepsilon - \lambda \varepsilon) + \lambda^{n-2} (\varepsilon - \lambda \varepsilon)^2 + \cdots + (\varepsilon - \lambda \varepsilon)^n
\]

\[
\leq \lambda^n \varepsilon + \lambda^{n-1} (\varepsilon - \lambda \varepsilon) + \lambda^{n-2} (\varepsilon - \lambda \varepsilon) + \cdots + (\varepsilon - \lambda \varepsilon)
\]

\[
= \lambda^n \varepsilon + (\lambda^{n-1} \varepsilon - \lambda^n \varepsilon) + (\lambda^{n-2} \varepsilon - \lambda^{n-1} \varepsilon) + \cdots + (\varepsilon - \lambda \varepsilon)
\]

\[
= \varepsilon.
\]

So if (1) is valid for \(n = N > 0\), let \(x_{N+1} \in f(x_N)\); then there exists \(y_{N+1} \in f(y_N)\) such that

\[
d(x_{N+1}, y_{N+1}) \leq D(f(y_N), f(y_N)) + \eta^{N+1} \leq \lambda d(x_N, y_N) + \eta^{N+1}
\]

\[
< \lambda (\lambda^N \varepsilon + \lambda^{N-1} \eta + \lambda^{N-2} \eta^2 + \cdots + \eta^N) + \eta^{N+1}
\]

\[
= \lambda^{N+1} \varepsilon + \lambda^N \eta + \lambda^{N-1} \eta^2 + \cdots + \lambda \eta^N + \eta^{N+1}.
\]

Using this information we now construct a sequence in \(M\) as follows: let \(y_{0,0}\) be an arbitrary element in \(M\) and let \(y_{1,0} \in f(y_{0,0})\).
Consider the $\varepsilon$-chain

$$y_{0,0}, y_{0,1}, \ldots, y_{0,n} = y_{1,0} \in f(y_{0,0}),$$

so that $d(y_{0,i}, y_{0,i+1}) < \varepsilon$ ($i = 0, 1, \ldots, n - 1$). Since $y_{1,0} \in f(y_{0,0})$, we may choose $y_{1,1} \in f(y_{0,1})$ such that

$$d(y_{1,0}, y_{1,1}) \leq D(f(y_{0,0}), f(y_{0,1})) + \eta. \tag{2}$$

Similarly, since $y_{1,1} \in f(y_{0,1})$, choose $y_{1,2} \in f(y_{0,2})$ such that

$$d(y_{1,1}, y_{1,2}) \leq D(f(y_{0,1}), f(y_{0,2})) + \eta.$$

Continuing along the $\varepsilon$-chain, since $y_{1,n-1} \in f(y_{0,n-1})$, there exists $y_{1,n} = y_{2,0} \in f(y_{0,n})$ (i.e., $y_{2,0} \in f(y_{1,0})$) such that

$$d(y_{1,n-1}, y_{1,n}) \leq D(f(y_{0,n-1}), f(y_{0,n})) + \eta.$$

Consequently,

$$d(y_{1,0}, y_{2,0}) = d(y_{1,0}, y_{1,n}) \leq \sum_{i=0}^{n-1} d(y_{1,i}, y_{1,i+1}) < n(\lambda \varepsilon + \eta).$$

Next, referring to (2), since $y_{2,0} \in f(y_{1,0})$, there exists $y_{2,1} \in f(y_{1,1})$ for which

$$d(y_{2,0}, y_{2,1}) \leq D(f(y_{1,0}), f(y_{1,1})) + \eta^2,$$

and for $y_{2,n-1} \in f(y_{1,n-1})$, we have $y_{2,n} = y_{3,0} \in f(y_{1,n})$ (i.e., $y_{3,0} \in f(y_{2,0})$) such that

$$d(y_{2,n-1}, y_{2,n}) \leq D(f(y_{1,n-1}), f(y_{1,n})) + \eta^2.$$

Proceeding in this manner, and making use of (1), we get (for $m > 0$

$$d(y_{m,l}, y_{m,l+1}) < \lambda^m \varepsilon + \lambda^{m-1} \eta + \lambda^{m-2} \eta^2 + \cdots + \eta^m$$

($l = 0, 1, \ldots, n - 1$). Now let $z_m = y_{m,0}$, so that $z_m \in f(z_{m-1})$, $m = 1, 2, \ldots$, and $z_{m+1} = y_{m+1,0} = y_{m,n}$. Then

$$d(z_m, z_{m+1}) \leq \sum_{l=0}^{n-1} d(y_{m,l}, y_{m,l+1})$$

$$< n(\lambda^m \varepsilon + \lambda^{m-1} \eta + \lambda^{m-2} \eta^2 + \cdots + \eta^m).$$

To show that $\{z_m\}$ is a Cauchy sequence, let $\beta = \max(\lambda, \eta)$. Then

$$d(z_m, z_{m+1}) < n(\lambda + 1)\beta^m,$$

and for $0 < i < j$
\[ d(z_i, z_j) \leq \sum_{k=i}^{j-1} d(z_k, z_{k+1}) \]
\[ < n \sum_{k=i}^{j-1} (k + 1)\beta^k \]
\[ \leq n \sum_{k=i}^{\infty} (k + 1)\beta^k. \]

It is easily checked that \( d(z_i, z_j) \to 0 \) as \( i \to \infty \), implying that \( \{z_m\} \) is a Cauchy sequence, which converges to some \( z \in M \) by the completeness of \( M \).

Finally, since \( z_m \in f(z_{m-1}) \) and \( z_m \to z, f(z_{m-1}) \to f(z) \) and, by Lemma 2, \( z \in f(z) \).

**Remark 1.** Nadler [4] proved a similar theorem by a different method under the additional assumption that each \( f(x) \) is compact.

2. Locally nonexpansive set-valued mappings. Let \( X \) be a Banach space and \( C \) a subset of \( X \). A mapping \( T: C \to S(C) \) will be called **locally nonexpansive** if there exists \( \varepsilon > 0 \) such that

\[ D(Tx, Ty) \leq ||x - y||, \]

whenever \( ||x - y|| < \varepsilon \) and where \( D \) is again the distance in the Hausdorff metric induced by \( d \) on \( S(M) \) (as usual, \( d(x, y) = ||x - y|| \) for all \( x, y \in X \)).

**Theorem 2.** Let \( X \) be a Banach space and \( C \) a compact star-shaped subset of \( X \). If \( T: C \to S(C) \) is locally nonexpansive, then there exists a point \( x \in C \) such that \( x \in Tx \).

**Proof.** Let \( c \) be the star-center of \( C \) and let \( \{k_n\} \) be an increasing sequence of real numbers converging to 1. Define \( U_n: C \to S(C) \) by

\[ U_n x = (1 - k_n)c + k_n Tx, \]

where \( k_n Tx = \{k_n y; y \in Tx\} \). Let \( z, y \in C \) such that \( ||z - y|| < \varepsilon \). Then \( D(Tz, Ty) \leq ||z - y||. \) Now for any two elements \( z' \in Tz \) and \( y' \in Ty \)

\[ ||(1 - k_n)c + k_n z' - (1 - k_n)c - k_n y'|| = k_n ||z' - y'||. \]

Hence

\[ D(U_n z, U_n y) \leq k_n ||z - y||. \]

Consequently, \( U_n \) has a fixed point \( x_n \in C \) by Theorem 1. Since \( C \) is
compact, there exists a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) converging to some \( x \in C \), and because \( T \) is continuous,

\[
T x_{n_i} \longrightarrow Tx.
\]

Now

\[
dist(x_{n_i}, Tx_{n_i}) \leq D(U_{n_i} x_{n_i}, Tx_{n_i})
= D((1 - k_{n_i})c + k_{n_i} x_{n_i}, Tx_{n_i}) \longrightarrow D(Tx, Tx) \text{ as } i \to \infty.
\]

Thus

\[
dist(x, Tx) = 0,
\]

which implies that \( x \in Tx \), \( Tx \) being closed.

Theorem 2 and its point-to-point analogue generalize an earlier theorem due to Dotson [2]:

**Corollary.** A nonexpansive self-mapping of a compact star-shaped subset of a Banach space has a fixed point.

**Remark 2.** Edelstein [3] has shown that a locally contractive (nonexpansive) point-to-point mapping need not be globally contractive (nonexpansive). On convex sets, however, a locally nonexpansive mapping is nonexpansive.

**References**


Received August 12, 1975.

Milwaukee School of Engineering
Andrew Adler, Weak homomorphisms and invariants: an example .............. 293
Howard Anton and William J. Pervin, Separation axioms and metric-like functions ................................................................. 299
Ron C. Blei, Sidon partitions and p-Sidon sets ................................... 307
T. J. Cheatham and J. R. Smith, Regular and semisimple modules .......... 315
Charles Edward Cleaver, Packing spheres in Orlicz spaces ................. 325
Le Baron O. Ferguson and Michael D. Rusk, Korovkin sets for an operator on a space of continuous functions ............................................. 337
Rudolf Fritsch, An approximation theorem for maps into Kan fibrations .... 347
David Sexton Gilliam, Geometry and the Radon-Nikodym theorem in strict Mackey convergence spaces .......................................... 353
William Hery, Maximal ideals in algebras of topological algebra valued functions .......................................................... 365
Alan Hopenwasser, The radical of a reflexive operator algebra ............ 375
Bruno Kramm, A characterization of Riemann algebras ....................... 393
Peter K. F. Kuhfittig, Fixed points of locally contractive and nonexpansive set-valued mappings ................................................. 399
Stephen Allan McGrath, On almost everywhere convergence of Abel means of contraction semigroups ........................................... 405
Edward Peter Merkes and Marion Wetzel, A geometric characterization of indeterminate moment sequences .................................... 409
John C. Morgan, II, The absolute Baire property ................................ 421
Eli Aaron Passow and John A. Roulier, Negative theorems on generalized convex approximation ................................................. 437
Louis Jackson Ratliff, Jr., A theorem on prime divisors of zero and characterizations of unmixed local domains ............................. 449
Ellen Elizabeth Reed, A class of $T_1$-compactifications .......................... 471
Maxwell Alexander Rosenlicht, On Liouville’s theory of elementary functions .......................................................... 485
Arthur Argyle Sagle, Power-associative algebras and Riemannian connections ............................................................... 493
Chester Cornelius Seabury, On extending regular holomorphic maps from Stein manifolds ......................................................... 499
Elias Sai Wan Shiu, Commutators and numerical ranges of powers of operators .......................................................... 517
Donald Mark Topkis, The structure of sublattices of the product of n lattices .... 525
John Bason Wagoner, Delooping the continuous $K$-theory of a valuation ring ........................................................................ 533
Ronson Joseph Warne, Standard regular semigroups ............................ 539
Anthony William Wickstead, The centraliser of $E \otimes F_\lambda$ .................. 563
R. Grant Woods, Characterizations of some $C^*$-embedded subspaces of $\beta N$.... 573