

# Pacific Journal of Mathematics

## **FIXED POINTS OF LOCALLY CONTRACTIVE AND NONEXPANSIVE SET-VALUED MAPPINGS**

PETER K. F. KUHFITIG

## FIXED POINTS OF LOCALLY CONTRACTIVE AND NONEXPANSIVE SET-VALUED MAPPINGS

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Let  $(M, d)$  be a complete metric space and  $S(M)$  the set of all nonempty bounded closed subsets of  $M$ . A set-valued mapping  $f: M \rightarrow S(M)$  will be called (uniformly) *locally contractive* if there exist  $\varepsilon$  and  $\lambda$  ( $\varepsilon > 0, 0 < \lambda < 1$ ) such that  $D(f(x), f(y)) \leq \lambda d(x, y)$  whenever  $d(x, y) < \varepsilon$  and where  $D(f(x), f(y))$  is the distance between  $f(x)$  and  $f(y)$  in the Hausdorff metric induced by  $d$  on  $S(M)$ . It is shown in the first theorem that if  $M$  is "well-chained," then  $f$  has a fixed point is, that is, a point  $x \in M$  such that  $x \in f(x)$ . This fact, in turn, yields a fixed-point theorem for locally nonexpansive set-valued mappings on a compact star-shaped subset of a Banach space. Both theorems are extensions of earlier results.

1. *Locally contractive set-valued mappings.* Following Assad and Kirk [1] we shall define  $D$  as follows: if  $r > 0$  and  $Y \in S(M)$ , let

$$Z(r, Y) = \{x \in M: \text{dist}(x, Y) < r\}.$$

Then for  $A, B \in S(M)$  we define

$$D(A, B) = \inf \{r: A \subset Z(r, B) \text{ and } B \subset Z(r, A)\}.$$

Also noted in [1] are two lemmas:

LEMMA 1. *If  $A, B \in S(M)$  and  $x \in A$ , then for each positive number  $\alpha$  there exists  $y \in B$  such that*

$$d(x, y) \leq D(A, B) + \alpha.$$

LEMMA 2. *Let  $\{X_n\}$  be a sequence of sets in  $S(M)$ , and assume that  $\lim_{n \rightarrow \infty} D(X_n, X_0) = 0$  ( $X_0 \in S(M)$ ). Then if  $x_n \in X_n$  ( $n = 1, 2, \dots$ ) and  $\lim_{n \rightarrow \infty} x_n = x_0$ , it follows that  $x_0 \in X_0$ .*

Finally, suppose  $M$  is well-chained in the sense that for every  $\varepsilon > 0$  and  $x, y \in M$  there exists an  $\varepsilon$ -chain, that is, a finite set of points

$$x = y_0, y_1, \dots, y_n = z$$

( $n$  may depend on both  $x$  and  $z$ ) such that  $d(y_i, y_{i+1}) < \varepsilon$  ( $i = 0, 1, \dots, n - 1$ ).

**THEOREM 1.** *Suppose  $(M, d)$  is a complete well-chained metric space and  $S(M)$  the set of all nonempty bounded closed subsets of  $M$ . If  $f: M \rightarrow S(M)$  is locally contractive, then  $f$  has a fixed point.*

*Proof.* Assume that  $\varepsilon < 1$  and let  $x_0, y_0 \in M$  such that  $d(x_0, y_0) < \varepsilon$ . Then

$$D(f(x_0), f(y_0)) \leq \lambda d(x_0, y_0).$$

Now choose a positive number  $\eta < \varepsilon - \lambda\varepsilon < 1$ . Let  $x_1$  be any element in  $f(x_0)$ ; then there exists by Lemma 1 an element  $y_1 \in f(y_0)$  such that

$$d(x_1, y_1) \leq D(f(x_0), f(y_0)) + \eta.$$

Hence

$$d(x_1, y_1) < \lambda\varepsilon + \eta < \lambda\varepsilon + \varepsilon - \lambda\varepsilon = \varepsilon.$$

Next, let  $x_2 \in f(x_1)$ ; then there exists  $y_2 \in f(y_1)$  such that

$$\begin{aligned} d(x_2, y_2) &\leq D(f(x_1), f(y_1)) + \eta^2 \\ &\leq \lambda d(x_1, y_1) + \eta^2. \end{aligned}$$

In general, for  $n > 0$

$$d(x_n, y_n) \leq D(f(x_{n-1}), f(y_{n-1})) + \eta^n,$$

and we can show by induction that

$$(1) \quad d(x_n, y_n) < \lambda^n \varepsilon + \lambda^{n-1} \eta + \lambda^{n-2} \eta^2 + \cdots + \eta^n.$$

Indeed,

$$\begin{aligned} &\lambda^n \varepsilon + \lambda^{n-1} \eta + \lambda^{n-2} \eta^2 + \cdots + \eta^n \\ &< \lambda^n \varepsilon + \lambda^{n-1} (\varepsilon - \lambda\varepsilon) + \lambda^{n-2} (\varepsilon - \lambda\varepsilon)^2 + \cdots + (\varepsilon - \lambda\varepsilon)^n \\ &\leq \lambda^n \varepsilon + \lambda^{n-1} (\varepsilon - \lambda\varepsilon) + \lambda^{n-2} (\varepsilon - \lambda\varepsilon) + \cdots + (\varepsilon - \lambda\varepsilon) \\ &= \lambda^n \varepsilon + (\lambda^{n-1} \varepsilon - \lambda^n \varepsilon) + (\lambda^{n-2} \varepsilon - \lambda^{n-1} \varepsilon) + \cdots + (\varepsilon - \lambda\varepsilon) \\ &= \varepsilon. \end{aligned}$$

So if (1) is valid for  $n = N > 0$ , let  $x_{N+1} \in f(x_N)$ ; then there exists  $y_{N+1} \in f(y_N)$  such that

$$\begin{aligned} d(x_{N+1}, y_{N+1}) &\leq D(f(y_N), f(y_N)) + \eta^{N+1} \leq \lambda d(x_N, y_N) + \eta^{N+1} \\ &< \lambda(\lambda^N \varepsilon + \lambda^{N-1} \eta + \lambda^{N-2} \eta^2 + \cdots + \eta^N) + \eta^{N+1} \\ &= \lambda^{N+1} \varepsilon + \lambda^N \eta + \lambda^{N-1} \eta^2 + \cdots + \lambda \eta^N + \eta^{N+1}. \end{aligned}$$

Using this information we now construct a sequence in  $M$  as follows: let  $y_{0,0}$  be an arbitrary element in  $M$  and let  $y_{1,0} \in f(y_{0,0})$ .

Consider the  $\varepsilon$ -chain

$$y_{0,0}, y_{0,1}, \dots, y_{0,n} = y_{1,0} \in f(y_{0,0}),$$

so that  $d(y_{0,i}, y_{0,i+1}) < \varepsilon$  ( $i = 0, 1, \dots, n - 1$ ). Since  $y_{1,0} \in f(y_{0,0})$ , we may choose  $y_{1,1} \in f(y_{0,1})$  such that

$$(2) \quad d(y_{1,0}, y_{1,1}) \leq D(f(y_{0,0}), f(y_{0,1})) + \eta.$$

Similarly, since  $y_{1,1} \in f(y_{0,1})$ , choose  $y_{1,2} \in f(y_{0,2})$  such that

$$d(y_{1,1}, y_{1,2}) \leq D(f(y_{0,1}), f(y_{0,2})) + \eta.$$

Continuing along the  $\varepsilon$ -chain, since  $y_{1,n-1} \in f(y_{0,n-1})$ , there exists  $y_{1,n} = y_{2,0} \in f(y_{0,n})$  (i.e.,  $y_{2,0} \in f(y_{1,0})$ ) such that

$$d(y_{1,n-1}, y_{1,n}) \leq D(f(y_{0,n-1}), f(y_{0,n})) + \eta.$$

Consequently,

$$d(y_{1,0}, y_{2,0}) = d(y_{1,0}, y_{1,n}) \leq \sum_{i=0}^{n-1} d(y_{1,i}, y_{1,i+1}) < n(\lambda\varepsilon + \eta).$$

Next, referring to (2), since  $y_{2,0} \in f(y_{1,0})$ , there exists  $y_{2,1} \in f(y_{1,1})$  for which

$$d(y_{2,0}, y_{2,1}) \leq D(f(y_{1,0}), f(y_{1,1})) + \eta^2,$$

and for  $y_{2,n-1} \in f(y_{1,n-1})$ , we have  $y_{2,n} = y_{3,0} \in f(y_{1,n})$  (i.e.,  $y_{3,0} \in f(y_{2,0})$ ) such that

$$d(y_{2,n-1}, y_{2,n}) \leq D(f(y_{1,n-1}), f(y_{1,n})) + \eta^2.$$

Proceeding in this manner, and making use of (1), we get (for  $m > 0$ )

$$d(y_{m,l}, y_{m,l+1}) < \lambda^m \varepsilon + \lambda^{m-1} \eta + \lambda^{m-2} \eta^2 + \dots + \eta^m$$

( $l = 0, 1, \dots, n - 1$ ). Now let  $z_m = y_{m,0}$ , so that  $z_m \in f(z_{m-1})$ ,  $m = 1, 2, \dots$ , and  $z_{m+1} = y_{m+1,0} = y_{m,n}$ . Then

$$\begin{aligned} d(z_m, z_{m+1}) &\leq \sum_{l=0}^{n-1} d(y_{m,l}, y_{m,l+1}) \\ &< n(\lambda^m \varepsilon + \lambda^{m-1} \eta + \lambda^{m-2} \eta^2 + \dots + \eta^m). \end{aligned}$$

To show that  $\{z_m\}$  is a Cauchy sequence, let  $\beta = \max(\lambda, \eta)$ . Then

$$d(z_m, z_{m+1}) < n(m + 1)\beta^m,$$

and for  $0 < i < j$

$$\begin{aligned}
 d(z_i, z_j) &\leq \sum_{k=i}^{j-1} d(z_k, z_{k+1}) \\
 &< n \sum_{k=i}^{j-1} (k+1)\beta^k \\
 &\leq n \sum_{k=i}^{\infty} (k+1)\beta^k .
 \end{aligned}$$

It is easily checked that  $d(z_i, z_j) \rightarrow 0$  as  $i \rightarrow \infty$ , implying that  $\{z_m\}$  is a Cauchy sequence, which converges to some  $z \in M$  by the completeness of  $M$ .

Finally, since  $z_m \in f(z_{m-1})$  and  $z_m \rightarrow z$ ,  $f(z_{m-1}) \rightarrow f(z)$  and, by Lemma 2,  $z \in f(z)$ .

REMARK 1. Nadler [4] proved a similar theorem by a different method under the additional assumption that each  $f(x)$  is compact.

2. **Locally nonexpansive set-valued mappings.** Let  $X$  be a Banach space and  $C$  a subset of  $X$ . A mapping  $T: C \rightarrow S(C)$  will be called *locally nonexpansive* if there exists  $\varepsilon > 0$  such that

$$D(Tx, Ty) \leq \|x - y\| ,$$

whenever  $\|x - y\| < \varepsilon$  and where  $D$  is again the distance in the Hausdorff metric induced by  $d$  on  $S(M)$  (as usual,  $d(x, y) = \|x - y\|$  for all  $x, y \in X$ ).

THEOREM 2. *Let  $X$  be a Banach space and  $C$  a compact star-shaped subset of  $X$ . If  $T: C \rightarrow S(C)$  is locally nonexpansive, then there exists a point  $x \in C$  such that  $x \in Tx$ .*

*Proof.* Let  $c$  be the star-center of  $C$  and let  $\{k_n\}$  be an increasing sequence of real numbers converging to 1. Define  $U_n: C \rightarrow S(C)$  by

$$U_n x = (1 - k_n)c + k_n T x ,$$

where  $k_n T x = \{k_n y: y \in T x\}$ . Let  $z, y \in C$  such that  $\|z - y\| < \varepsilon$ . Then  $D(Tz, Ty) \leq \|z - y\|$ . Now for any two elements  $z' \in Tz$  and  $y' \in Ty$

$$\|(1 - k_n)c + k_n z' - (1 - k_n)c - k_n y'\| = k_n \|z' - y'\| .$$

Hence

$$D(U_n z, U_n y) \leq k_n \|z - y\| .$$

Consequently,  $U_n$  has a fixed point  $x_n \in C$  by Theorem 1. Since  $C$  is

compact, there exists a subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$  converging to some  $x \in C$ , and because  $T$  is continuous,

$$Tx_{n_i} \longrightarrow Tx .$$

Now

$$\begin{aligned} \text{dist}(x_{n_i}, Tx_{n_i}) &\leq D(U_{n_i}x_{n_i}, Tx_{n_i}) \\ &= D((1 - k_{n_i})c + k_{n_i}Tx_{n_i}, Tx_{n_i}) \longrightarrow D(Tx, Tx) \text{ as } i \longrightarrow \infty . \end{aligned}$$

Thus

$$\text{dist}(x, Tx) = 0 ,$$

which implies that  $x \in Tx$ ,  $Tx$  being closed.

Theorem 2 and its point-to-point analogue generalize an earlier theorem due to Dotson [2]:

**COROLLARY.** *A nonexpansive self-mapping of a compact star-shaped subset of a Banach space has a fixed point.*

**REMARK 2.** Edelstein [3] has shown that a locally contractive (nonexpansive) point-to-point mapping need not be globally contractive (nonexpansive). On convex sets, however, a locally nonexpansive mapping is nonexpansive.

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