

Pacific Journal of Mathematics

**ON ALMOST EVERYWHERE CONVERGENCE OF ABEL
MEANS OF CONTRACTION SEMIGROUPS**

STEPHEN ALLAN MCGRATH

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S. A. MCGRATH

Let (X, Σ, μ) be a σ -finite measure space and $L_p(X, \Sigma, \mu)$, $1 \leq p \leq \infty$, the usual Banach spaces of complex valued functions. Let $\{T_t: t \geq 0\}$ be a strongly continuous semigroup of contractions of $L_p(X, \Sigma, \mu)$ for some $1 \leq p < \infty$ and set $R_\lambda f = \int_0^\infty e^{-\lambda t} T_t f dt$, $\lambda > 0$. If $\|T_t\|_\infty \leq 1$ for all $t \geq 0$, then $\lim_{\lambda \rightarrow \infty} \lambda R_\lambda f(x) = f(x)$ a.e. for all $f \in L_p(X, \Sigma, \mu)$.

A strongly continuous contraction semigroup on $L_p(X, \Sigma, \mu)$ satisfies the following: (i) $T_{s+t} = T_s \cdot T_t$, $s, t \geq 0$; (ii) $\|T_t\|_p \leq 1$, $t \geq 0$; (iii) $\|T_t f - T_s f\|_p \rightarrow 0$ as $s \rightarrow t$ for any $f \in L_p = L_p(X, \Sigma, \mu)$. Merely as a notational convenience, we assume that $T_0 = I$. Before proceeding further it is necessary to clarify the definition of $R_\lambda f(x)$. By Theorem III.11.17 in [3], given $f \in L_p$ there exists a scalar function $g(t, x)$, measurable with respect to the usual product measure on $[0, \infty) \times X$, such that (i) for a.e. t , $g(t, \cdot) = T_t f$ and (ii) there exists a μ -null set $E(f)$, independent of λ , such that $x \notin E(f)$ implies $\int_0^\infty e^{-\lambda t} g(t, x) dt$, as a function of x , is in the equivalence class of $\int_0^\infty e^{-\lambda t} g(t, x) dt$. The scalar representation $g(t, x)$ is uniquely determined up to a set of product measure zero. Defining $R_\lambda f(x) = \int_0^\infty e^{-\lambda t} g(t, x) dt$, we see that $R_\lambda f(x)$ is in the equivalence class of $R_\lambda f = \int_0^\infty e^{-\lambda t} T_t f dt$ for all $\lambda > 0$. This justifies our definition of $R_\lambda f(x)$. Note that for $x \notin E(f)$, $R_\lambda f(x)$ is a continuous function of $\lambda > 0$.

The main result of this note (Theorem 4) extends a special case of a theorem of N. Dunford and J. T. Schwartz [2, p. 178]. If $p = 1$ in our theorem then the assumption $\|T_t\|_\infty \leq 1$ for $t \geq 0$ is unnecessary [5].

Preliminary results.

LEMMA 1. Let $\{T_t: t \geq 0\}$ be a strongly continuous semigroup of L_p contractions for some $1 \leq p < \infty$. Set $\mathcal{M} = \{\lambda R_\lambda f: 0 < \lambda < \infty, f \in L_p\}$. Then \mathcal{M} is dense in L_p and $\lim_{\lambda \rightarrow \infty} \lambda R_\lambda f(x) = f(x)$ a.e. for any $f \in L_p$.

The denseness of \mathcal{M} follows from the fact that $s - \lim_{\lambda \rightarrow \infty} \lambda R_\lambda f = f$ [4, p. 321], and the existence of the pointwise limit follows from the resolvent equation. The details appear in [5]. The next result is proved in [1].

LEMMA 2. Let $\{T_t; t \geq 0\}$ be a strongly continuous semigroup of L_p contractions for some $1 \leq p < \infty$. Suppose that $\|T_t\|_\infty \leq 1$ for all $t \geq 0$. Then

$$\lim_{\varepsilon \downarrow 0} \left(\frac{1}{\varepsilon} \right) \int_0^\varepsilon T_t f(x) dt = f(x) \text{ a.e.}$$

for every $f \in L_p$.

For a given L_p semigroup $\{T_t; t \geq 0\}$, define $T'_t = e^{-t} T_t$. Then $\{T'_t; t \geq 0\}$ is a semigroup; if $\{T_t; t \geq 0\}$ is strongly continuous so is $\{T'_t; t \geq 0\}$. We shall denote the resolvent of $\{T'_t\}$ by R'_λ . For $f \in L_p$, set $f^* = \sup_{\lambda > 0} |\lambda R'_\lambda f|$.

LEMMA 3. Suppose $\{T_t; t \geq 0\}$ is a strongly continuous contraction semigroup on L_p for some $1 \leq p < \infty$. If, in addition, $\|T_t\|_\infty \leq 1$ for all $t \geq 0$, then $f^* < \infty$ a.e. for any $f \in L_p$.

Proof. Fix $f \in L_p$ and choose $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$. Set

$$\begin{aligned} g_n &= \inf_{\varepsilon \leq \varepsilon_n} \left[\frac{1}{\varepsilon} \int_0^\varepsilon T'_t f(x) dt \right], \\ h_n &= \sup_{\varepsilon \leq \varepsilon_n} \left[\frac{1}{\varepsilon} \int_0^\varepsilon T'_t f(x) dt \right], \\ f_n^* &= \sup_{\delta \leq \varepsilon} \left| \frac{1}{\delta} \int_0^\delta T'_t f(x) dt \right|. \end{aligned}$$

Let A be a measurable subset of X with $0 < \mu(A) < \infty$. Since $\{T'_t; t \geq 0\}$ satisfies the conditions of Lemma 2, we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon T'_t f(x) dt = f(x) \text{ a.e. on } X.$$

Hence $\lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} h_n = f(x)$ a.e. By Egoroff's theorem, given $0 < \delta < \mu(A)/2$, there exists a measurable subset B of A such that $\mu(B) > \mu(A) - 2\delta$ and $\{g_n\}, \{h_n\}$ converge uniformly on B to $f(x)$. Therefore, for some K , $n \geq K$ implies $|g_n - f| \leq 1$ and $|h_n - f| \leq 1$ for all $x \in B$. Consequently $|g_n| \leq |f| + 1$ and $|h_n| \leq |f| + 1$ on B for all $n \geq K$. For given n , we have

$$g_n(x) \leq \frac{1}{\varepsilon} \int_0^\varepsilon T'_t f(x) dt \leq h_n(x)$$

for any $\varepsilon \leq \varepsilon_n$. Thus for any $x \in B$ and $n \geq K$,

$$\begin{aligned} f_n^*(x) &\leq |g_n(x)| + |h_n(x)| \\ &\leq 2|f(x)| + 2, \end{aligned}$$

provided $\varepsilon \leq \varepsilon_n$. For some fixed $n \geq K$, set $\delta = \varepsilon_n$.

By an integration by parts, we have

$$\lambda \int_0^\infty e^{-\lambda t} T'_t f(x) dt = \lambda^2 \int_0^\infty e^{-\lambda t} \left[\frac{1}{t} \int_0^t T'_s f(x) ds \right] dt \text{ a.e. on } X.$$

For $t \geq \delta$ we have

$$\left| \frac{1}{t} \int_0^t T'_s f(x) ds \right| \leq \frac{1}{\delta} \int_0^\infty |T'_s f(x)| ds < \infty \text{ a.e. on } X$$

since $\left\| \int_0^\infty |T'_s f(x)| ds \right\|_p \leq \|f\|_p$. Hence for a.e. $x \in B$,

$$\begin{aligned} & \left| \lambda^2 \int_0^\infty e^{-\lambda t} \left[\frac{1}{t} \int_0^t T'_s f(x) ds \right] dt \right| \\ & \leq \lambda^2 \int_0^\delta e^{-\lambda t} [2|f(x)| + 2] dt \\ & \quad + \left(\frac{\lambda^2}{\delta} \right) \int_0^\infty e^{-\lambda t} \left[\int_0^\infty |T'_s f(x)| ds \right] dt \\ & \leq [2|f(x)| + 2] \left[\lambda^2 \int_0^\infty t e^{-\lambda t} dt \right] \\ & \quad + \left[\frac{1}{\delta} \int_0^\infty |T'_s f(x)| ds \right] \left[\lambda^2 \int_0^\infty t e^{-\lambda t} dt \right] \\ & \leq \{2|f(x)| + 2\} + \left(\frac{1}{\delta} \right) \int_0^\infty |T'_s f(x)| ds \end{aligned}$$

for all $\lambda > 0$. Hence $f^* < \infty$ a.e. on B . Since the set A was an arbitrary set of finite measure and B is a measurable subset of A having positive measure, we conclude that $f^* < \infty$ a.e. on X .

Main results.

THEOREM 4. *Let $\{T_t; t \geq 0\}$ be a strongly continuous semigroup of L_p contractions for some $1 \leq p < \infty$. Suppose that $\|T_t\|_\infty \leq 1$ for all $t \geq 0$. If $f \in L_p$, then*

$$\lim_{\lambda \rightarrow \infty} \lambda R_\lambda f(x) = f(x) \text{ a.e.}$$

Proof. By Lemmas 1 and 3 and Banach's convergence theorem [3, p. 332-333], $\lim \lambda R'_\lambda f(x)$ exists and is finite a.e. as $\lambda \rightarrow \infty$ through some countable set, say Q^+ (= set of positive rationals). We recall that $\lambda R'_\lambda f(x)$ depends continuously on λ for x outside some null set. Since Q^+ is dense in R^+ it follows that $\lim_{\lambda \rightarrow \infty} \lambda R'_\lambda f(x)$ exists and is finite a.e. for all $f \in L_p$. Since $s - \lim_{\lambda \rightarrow \infty} \lambda R'_\lambda f = f$, we must have $\lim_{\lambda \rightarrow \infty} \lambda R'_\lambda f(x) = f(x)$ a.e. Upon noting that $\lim_{\lambda \rightarrow \infty} R_\lambda f(x) = 0$ a.e. for any $f \in L_p$, we see that

$$\begin{aligned}
 \lim_{\lambda \rightarrow \infty} \lambda R_\lambda f(x) &= \lim_{\lambda \rightarrow \infty} (\lambda + 1) R_{\lambda+1} f(x) \\
 &= \lim_{\lambda \rightarrow \infty} \lambda R'_\lambda f(x) \\
 &= f(x) \text{ a.e.}
 \end{aligned}$$

The following result which generalizes Theorem 4 follows from (4.9) in [1] and the arguments used in obtaining Theorem 4.

THEOREM 5. *Let $\{T_t: t \geq 0\}$ be a strongly continuous semigroup of L_p contractions for some $1 \leq p < \infty$. Suppose there exists a measurable function h on $[0, \infty) \times X$ such that*

- (i) $h > 0$ on $[0, \infty) \times X$, and
- (ii) $f \in L_p$, $|f(x)| \leq h(t, x)$ μ -a.e. implies

$$|T_s f(x)| \leq h(t + s, x) \text{ for all } s, t \geq 0.$$

Then $\lim_{\lambda \rightarrow \infty} \lambda R_\lambda f(x) = f(x)$ a.e. for $f \in L_p$.

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