

Pacific Journal of Mathematics

**ON ALMOST EVERYWHERE CONVERGENCE OF ABEL
MEANS OF CONTRACTION SEMIGROUPS**

STEPHEN ALLAN MCGRATH

ON ALMOST EVERYWHERE COVERGENCE OF ABEL MEANS OF CONTRACTION SEMIGROUPS

S. A. McGRATH

Let (X, Σ, μ) be a σ -finite measure space and $L_p(X, \Sigma, \mu)$, $1 \leq p \leq \infty$, the usual Banach spaces of complex valued functions. Let $\{T_t: t \geq 0\}$ be a strongly continuous semigroup of contractions of $L_p(X, \Sigma, \mu)$ for some $1 \leq p < \infty$ and set $R_\lambda f = \int_0^\infty e^{-\lambda t} T_t f dt$, $\lambda > 0$. If $\|T_t\|_\infty \leq 1$ for all $t \geq 0$, then $\lim_{\lambda \rightarrow \infty} \lambda R_\lambda f(x) = f(x)$ a.e. for all $f \in L_p(X, \Sigma, \mu)$.

A strongly continuous contraction semigroup on $L_p(X, \Sigma, \mu)$ satisfies the following: (i) $T_{s+t} = T_s \cdot T_t$, $s, t \geq 0$; (ii) $\|T_t\|_p \leq 1$, $t \geq 0$; (iii) $\|T_t f - T_s f\|_p \rightarrow 0$ as $s \rightarrow t$ for any $f \in L_p = L_p(X, \Sigma, \mu)$. Merely as a notational convenience, we assume that $T_0 = I$. Before proceeding further it is necessary to clarify the definition of $R_\lambda f(x)$. By Theorem III.11.17 in [3], given $f \in L_p$ there exists a scalar function $g(t, x)$, measurable with respect to the usual product measure on $[0, \infty) \times X$, such that (i) for a.e. t , $g(t, \cdot) = T_t f$ and (ii) there exists a μ -null set $E(f)$, independent of λ , such that $x \notin E(f)$ implies $\int_0^\infty e^{-\lambda t} g(t, x) dt$, as a function of x , is in the equivalence class of $\int_0^\infty e^{-\lambda t} g(t, x) dt$. The scalar representation $g(t, x)$ is uniquely determined up to a set of product measure zero. Defining $R_\lambda f(x) = \int_0^\infty e^{-\lambda t} g(t, x) dt$, we see that $R_\lambda f(x)$ is in the equivalence class of $R_\lambda f = \int_0^\infty e^{-\lambda t} T_t f dt$ for all $\lambda > 0$. This justifies our definition of $R_\lambda f(x)$. Note that for $x \notin E(f)$, $R_\lambda f(x)$ is a continuous function of $\lambda > 0$.

The main result of this note (Theorem 4) extends a special case of a theorem of N. Dunford and J. T. Schwartz [2, p. 178]. If $p = 1$ in our theorem then the assumption $\|T_t\|_\infty \leq 1$ for $t \geq 0$ is unnecessary [5].

Preliminary results.

LEMMA 1. Let $\{T_t: t \geq 0\}$ be a strongly continuous semigroup of L_p contractions for some $1 \leq p < \infty$. Set $\mathcal{M} = \{\lambda R_\lambda f: 0 < \lambda < \infty, f \in L_p\}$. Then \mathcal{M} is dense in L_p and $\lim_{\lambda \rightarrow \infty} \lambda R_\lambda f(x) = f(x)$ a.e. for any $f \in L_p$.

The denseness of \mathcal{M} follows from the fact that $s - \lim_{\lambda \rightarrow \infty} \lambda R_\lambda f = f$ [4, p. 321], and the existence of the pointwise limit follows from the resolvent equation. The details appear in [5]. The next result is proved in [1].

LEMMA 2. Let $\{T_t; t \geq 0\}$ be a strongly continuous semigroup of L_p contractions for some $1 \leq p < \infty$. Suppose that $\|T_t\|_\infty \leq 1$ for all $t \geq 0$. Then

$$\lim_{\varepsilon \downarrow 0} \left(\frac{1}{\varepsilon} \right) \int_0^\varepsilon T_t f(x) dt = f(x) \text{ a.e.}$$

for every $f \in L_p$.

For a given L_p semigroup $\{T_t; t \geq 0\}$, define $T'_t = e^{-t}T_t$. Then $\{T'_t; t \geq 0\}$ is a semigroup; if $\{T_t; t \geq 0\}$ is strongly continuous so is $\{T'_t; t \geq 0\}$. We shall denote the resolvent of $\{T'_t\}$ by R'_λ . For $f \in L_p$, set $f^* = \sup_{\lambda > 0} |\lambda R'_\lambda f|$.

LEMMA 3. Suppose $\{T_t; t \geq 0\}$ is a strongly continuous contraction semigroup on L_p for some $1 \leq p < \infty$. If, in addition, $\|T_t\|_\infty \leq 1$ for all $t \geq 0$, then $f^* < \infty$ a.e. for any $f \in L_p$.

Proof. Fix $f \in L_p$ and choose $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$. Set

$$\begin{aligned} g_n &= \inf_{\varepsilon \leq \varepsilon_n} \left[\frac{1}{\varepsilon} \int_0^\varepsilon T'_t f(x) dt \right], \\ h_n &= \sup_{\varepsilon \leq \varepsilon_n} \left[\frac{1}{\varepsilon} \int_0^\varepsilon T'_t f(x) dt \right], \\ f_\varepsilon^* &= \sup_{\delta \leq \varepsilon} \left| \frac{1}{\delta} \int_0^\delta T'_t f(x) dt \right|. \end{aligned}$$

Let A be a measurable subset of X with $0 < \mu(A) < \infty$. Since $\{T'_t; t \geq 0\}$ satisfies the conditions of Lemma 2, we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon T'_t f(x) dt = f(x) \text{ a.e. on } X.$$

Hence $\lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} h_n = f(x)$ a.e. By Egoroff's theorem, given $0 < \delta < \mu(A)/2$, there exists a measurable subset B of A such that $\mu(B) > \mu(A) - 2\delta$ and $\{g_n\}, \{h_n\}$ converge uniformly on B to $f(x)$. Therefore, for some K , $n \geq K$ implies $|g_n - f| \leq \delta$ and $|h_n - f| \leq \delta$ for all $x \in B$. Consequently $|g_n| \leq |f| + \delta$ and $|h_n| \leq |f| + \delta$ on B for all $n \geq K$. For given n , we have

$$g_n(x) \leq \frac{1}{\varepsilon} \int_0^\varepsilon T'_t f(x) dt \leq h_n(x)$$

for any $\varepsilon \leq \varepsilon_n$. Thus for any $x \in B$ and $n \geq K$,

$$\begin{aligned} f_\varepsilon^*(x) &\leq |g_n(x)| + |h_n(x)| \\ &\leq 2|f(x)| + 2\delta, \end{aligned}$$

provided $\varepsilon \leq \varepsilon_n$. For some fixed $n \geq K$, set $\delta = \varepsilon_n$.

By an integration by parts, we have

$$\lambda \int_0^\infty e^{-\lambda t} T'_t f(x) dt = \lambda^2 \int_0^\infty e^{-\lambda t} t \left[\frac{1}{t} \int_0^t T'_s f(x) ds \right] dt \text{ a.e. on } X.$$

For $t \geq \delta$ we have

$$\left| \frac{1}{t} \int_0^t T'_s f(x) ds \right| \leq \frac{1}{\delta} \int_0^\infty |T'_s f(x)| ds < \infty \text{ a.e. on } X$$

since $\left\| \int_0^\infty |T'_s f(x)| ds \right\|_p \leq \|f\|_p$. Hence for a.e. $x \in B$,

$$\begin{aligned} & \left| \lambda^2 \int_0^\infty e^{-\lambda t} t \left[\frac{1}{t} \int_0^t T'_s f(x) ds \right] dt \right| \\ & \leq \lambda^2 \int_0^\delta e^{-\lambda t} [2|f(x)| + 2] dt \\ & \quad + \left(\frac{\lambda^2}{\delta} \right) \int_0^\infty e^{-\lambda t} \left[\int_0^\infty |T'_s f(x)| ds \right] dt \\ & \leq [2|f(x)| + 2] \left[\lambda^2 \int_0^\infty t e^{-\lambda t} dt \right] \\ & \quad + \left[\frac{1}{\delta} \int_0^\infty |T'_s f(x)| ds \right] \left[\lambda^2 \int_0^\infty t e^{-\lambda t} dt \right] \\ & \leq \{2|f(x)| + 2\} + \left(\frac{1}{\delta} \right) \int_0^\infty |T'_s f(x)| ds \end{aligned}$$

for all $\lambda > 0$. Hence $f^* < \infty$ a.e. on B . Since the set A was an arbitrary set of finite measure and B is a measurable subset of A having positive measure, we conclude that $f^* < \infty$ a.e. on X .

Main results.

THEOREM 4. *Let $\{T_t: t \geq 0\}$ be a strongly continuous semigroup of L_p contractions for some $1 \leq p < \infty$. Suppose that $\|T_t\|_\infty \leq 1$ for all $t \geq 0$. If $f \in L_p$, then*

$$\lim_{\lambda \rightarrow \infty} \lambda R_\lambda f(x) = f(x) \text{ a.e.}$$

Proof. By Lemmas 1 and 3 and Banach's convergence theorem [3, p. 332-333], $\lim \lambda R'_\lambda f(x)$ exists and is finite a.e. as $\lambda \rightarrow \infty$ through some countable set, say Q^+ (= set of positive rationals). We recall that $\lambda R'_\lambda f(x)$ depends continuously on λ for x outside some null set. Since Q^+ is dense in R^+ it follows that $\lim_{\lambda \rightarrow \infty} \lambda R'_\lambda f(x)$ exists and is finite a.e. for all $f \in L_p$. Since $s - \lim_{\lambda \rightarrow \infty} \lambda R'_\lambda f = f$, we must have $\lim_{\lambda \rightarrow \infty} \lambda R'_\lambda f(x) = f(x)$ a.e. Upon noting that $\lim_{\lambda \rightarrow \infty} R_\lambda f(x) = 0$ a.e. for any $f \in L_p$, we see that

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \lambda R_\lambda f(x) &= \lim_{\lambda \rightarrow \infty} (\lambda + 1) R_{\lambda+1} f(x) \\ &= \lim_{\lambda \rightarrow \infty} \lambda R'_\lambda f(x) \\ &= f(x) \text{ a.e.} \end{aligned}$$

The following result which generalizes Theorem 4 follows from (4.9) in [1] and the arguments used in obtaining Theorem 4.

THEOREM 5. *Let $\{T_t; t \geq 0\}$ be a strongly continuous semigroup of L_p contractions for some $1 \leq p < \infty$. Suppose there exists a measurable function h on $[0, \infty) \times X$ such that*

- (i) $h > 0$ on $[0, \infty) \times X$, and
- (ii) $f \in L_p$, $|f(x)| \leq h(t, x)$ μ -a.e. implies

$$|T_s f(x)| \leq h(t + s, x) \text{ for all } s, t \geq 0.$$

Then $\lim_{\lambda \rightarrow \infty} \lambda R_\lambda f(x) = f(x)$ a.e. for $f \in L_p$.

REFERENCES

1. J. R. Baxter and R. V. Chacon, *A local ergodic theorem on L_p* , *Canad. J. Math.*, **26** (1974), 1206-1216.
2. N. Dunford and J. T. Schwartz, *Convergence almost everywhere of operator averages*, *J. Math. and Mech.*, **5** (1956), 129-178.
3. ———, *Linear Operators*, part I, Interscience, New York, 1958.
4. E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, rev. ed., American Mathematical Society, Providence, R.I., 1957.
5. S. A. McGrath, *Abelian ergodic theorems for contraction semigroups*, *Studia Math.* (to appear).

Received April 14, 1976 and in revised form June 30, 1976.

U.S. NAVAL ACADEMY
ANNAPOLIS, MARYLAND 21402

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.),
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1975 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

Andrew Adler, <i>Weak homomorphisms and invariants: an example</i>	293
Howard Anton and William J. Pervin, <i>Separation axioms and metric-like functions</i>	299
Ron C. Blei, <i>Sidon partitions and p-Sidon sets</i>	307
T. J. Cheatham and J. R. Smith, <i>Regular and semisimple modules</i>	315
Charles Edward Cleaver, <i>Packing spheres in Orlicz spaces</i>	325
Le Baron O. Ferguson and Michael D. Rusk, <i>Korovkin sets for an operator on a space of continuous functions</i>	337
Rudolf Fritsch, <i>An approximation theorem for maps into Kan fibrations</i>	347
David Sexton Gilliam, <i>Geometry and the Radon-Nikodym theorem in strict Mackey convergence spaces</i>	353
William Hery, <i>Maximal ideals in algebras of topological algebra valued functions</i>	365
Alan Hopenwasser, <i>The radical of a reflexive operator algebra</i>	375
Bruno Kramm, <i>A characterization of Riemann algebras</i>	393
Peter K. F. Kuhfittig, <i>Fixed points of locally contractive and nonexpansive set-valued mappings</i>	399
Stephen Allan McGrath, <i>On almost everywhere convergence of Abel means of contraction semigroups</i>	405
Edward Peter Merkes and Marion Wetzel, <i>A geometric characterization of indeterminate moment sequences</i>	409
John C. Morgan, II, <i>The absolute Baire property</i>	421
Eli Aaron Passow and John A. Roulier, <i>Negative theorems on generalized convex approximation</i>	437
Louis Jackson Ratliff, Jr., <i>A theorem on prime divisors of zero and characterizations of unmixed local domains</i>	449
Ellen Elizabeth Reed, <i>A class of T_1-compactifications</i>	471
Maxwell Alexander Rosenlicht, <i>On Liouville's theory of elementary functions</i>	485
Arthur Argyle Sagle, <i>Power-associative algebras and Riemannian connections</i>	493
Chester Cornelius Seabury, <i>On extending regular holomorphic maps from Stein manifolds</i>	499
Elias Sai Wan Shiu, <i>Commutators and numerical ranges of powers of operators</i>	517
Donald Mark Topkis, <i>The structure of sublattices of the product of n lattices</i>	525
John Bason Wagoner, <i>Delooping the continuous K-theory of a valuation ring</i>	533
Ronson Joseph Warne, <i>Standard regular semigroups</i>	539
Anthony William Wickstead, <i>The centraliser of $E \otimes_{\lambda} F$</i>	563
R. Grant Woods, <i>Characterizations of some C^*-embedded subspaces of $\beta\mathbb{N}$</i>	573