

# Pacific Journal of Mathematics

## **POWER-ASSOCIATIVE ALGEBRAS AND RIEMANNIAN CONNECTIONS**

ARTHUR ARGYLE SAGLE

## POWER-ASSOCIATIVE ALGEBRAS AND RIEMANNIAN CONNECTIONS

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Let  $G/H$  be a reductive homogeneous space with the corresponding Lie algebra decomposition  $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$  where the complementary subspace  $\mathfrak{m}$  satisfies the condition  $(\text{ad } H)\mathfrak{m} \subset \mathfrak{m}$ . It has been shown that the  $G$ -invariant connections on  $G/H$  correspond to certain non-associative algebras  $(\mathfrak{m}, \alpha)$  and that these algebras, in turn, correspond to certain local analytic multiplications on  $G/H$ . These correspondences generalize many of the results of Lie theory; it has been shown, for example, that there is a change of coordinates at  $\bar{e} = eH$  which makes the algebras associated with a local multiplication anti-commutative. However, if  $G/H$  has pseudo-Riemannian structures and we require that the change of coordinate maps be local isometries, then the existence of a change of coordinates which gives an anti-commutative algebra is no longer guaranteed. Thus it is natural to ask when an algebra  $(\mathfrak{m}, \alpha)$  inducing a pseudo-Riemannian connection is anti-commutative and it is shown in this paper that a necessary and sufficient condition is basically that  $(\mathfrak{m}, \alpha)$  be power-associative.

1. Basics. Let  $G$  be a connected Lie group with Lie algebra  $\mathfrak{g}$  and let  $H$  be a closed (Lie) subgroup with Lie algebra  $\mathfrak{h}$ . Then the pair  $(G, H)$  or  $(\mathfrak{g}, \mathfrak{h})$  is called a *reductive pair* if there exists a subspace  $\mathfrak{m}$  of  $\mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$  (subspace direct sum) and  $(\text{ad } H)\mathfrak{m} \subset \mathfrak{m}$ . The corresponding analytic manifold  $M = G/H$  is called a *reductive homogeneous space* and  $\mathfrak{m}$  is identified with the tangent space  $M_{\bar{e}}$ . For a reductive space with a fixed Lie algebra decomposition  $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$  it is shown in [2], [6] that there is a 1-1 correspondence between  $G$ -invariant connections  $\nabla$  and nonassociative algebras  $(\mathfrak{m}, \alpha)$  with  $\text{ad } H \subset \text{Aut}(\mathfrak{m}, \alpha)$ . ( $\alpha$  is the bilinear algebra multiplication on  $\mathfrak{m}$  and  $\text{Aut}(\mathfrak{m}, \alpha)$  is the automorphism group of the algebra  $(\mathfrak{m}, \alpha)$ .)

A  $G$ -invariant pseudo-Riemannian connection on a reductive homogeneous space  $G/H$  corresponds to an algebra  $(\mathfrak{m}, \alpha)$  with a nondegenerate symmetric bilinear form  $C$  such that for all  $X, Y, Z \in \mathfrak{m}$  and  $U \in \mathfrak{h}$

$$(1) \quad C((\text{ad } U)X, Y) + C(X, (\text{ad } U)Y) = 0 \quad \text{and}$$

$$(2) \quad C(\alpha(Z, X), Y) + C(X, \alpha(Z, Y)) = 0.$$

We denote such algebras by  $(\mathfrak{m}, \alpha, C)$  and they are discussed in

[4], [6], [7]. In particular since the torsion tensor is zero we have from [2] that for  $X, Y \in m$

$$(3) \quad \alpha(X, Y) - \alpha(Y, X) = XY$$

where we use the notation  $XY = [X, Y]_m$  (resp.  $h(X, Y)$ ) for the projection of  $[X, Y]$  in  $g$  onto  $m$  (resp.  $h$ ). Thus the algebra  $(m, \alpha, C)$  is reductive Lie admissible [5] and in particular for  $h = \{0\}$  the algebra  $(g, \alpha, C)$  is Lie admissible [1].

As an example let  $\pi: G \rightarrow G/H$  be the canonical projection of  $G$  onto the reductive space  $G/H$ . For any  $X \in m$  the curves  $\gamma(t) = \pi \exp tX$  are geodesics relative to the  $G$ -invariant pseudo-Riemannian connection  $\nabla$  given by  $(m, \alpha, C)$  if and only if  $\alpha(X, Y) = (1/2)XY$ . This connection is called the *pseudo-Riemannian connection of the first kind* [2], [4] and we use the notation  $(m, (1/2)XY, B)$  for the corresponding algebra where  $B$  now denotes the nondegenerate form. In particular, let  $g$  and  $h$  be semi-simple and let Kill denote the Killing form of  $g$ . Since  $\text{Kill}|_{h \times h}$  is nondegenerate we can write  $g = m + h$  with  $m = h^\perp$  relative to the Killing form. Thus  $(g, h)$  is a reductive pair. The form  $B = \text{Kill}|_{m \times m}$  and the multiplication  $\alpha(X, Y) = (1/2)XY$  give an algebra  $(m, (1/2)XY, B)$  which satisfies conditions (1) and (2) and therefore induces a pseudo-Riemannian connection of the first kind. (One, of course, considers  $B = -\text{Kill}|_{m \times m}$  in case  $\text{Kill}|_{m \times m}$  is negative definite as is the case for  $G = SO(n)$  and  $H = SO(k)$ .)

Now let the reductive space  $G/H$  have a pseudo-Riemannian connection of the first kind given by the algebra  $(m, (1/2)XY, B)$  and suppose  $G/H$  has another pseudo-Riemannian connection given by the algebra  $(m, \alpha, C)$ . Then the nondegeneracy of  $B$  and  $C$  implies the existence of an  $S \in GL(m)$  such that

$$C(X, Y) = B(SX, Y)$$

for all  $X, Y \in m$ . Also by the symmetry and equation (1) we obtain

$$(*) \quad S^b = S \text{ and } [\text{ad } U, S] = 0$$

for all  $U \in h$ , where  $b$  denotes the adjoint relative to  $B$ . In [3], [4], [6] it is noted that the set,  $J$ , of endomorphisms of  $m$  satisfying (\*) forms a Jordan algebra relative to the usual multiplication  $S_1 \cdot S_2 = (1/2)(S_1 S_2 + S_2 S_1)$ . Also the formula for  $\alpha$  is given by

$$2\alpha(X, Y) = XY + S^{-1}[X(SY) - (SX)Y]$$

where  $XY = [X, Y]_m$  is the multiplication in the algebra  $(m, (1/2)XY, B)$ . Many examples of the algebras  $(m, \alpha, C)$  determined by the Jordan algebra  $J$  are given in [4]. In the next section we discuss some of

the algebraic identities which these algebras may satisfy. These identities for the algebras  $(m, \alpha, C)$  are related to isometric coordinate changes and  $H$ -spaces  $(G/H, \mu)$  as discussed in [7].

2. Power-associative algebras. An algebra  $A$  over a field  $F$  is power-associative if every element  $X \in A$  generates an associative subalgebra  $F[X]$ ; see [9]. We now assume the algebra  $(m, \alpha, C)$  discussed in §1 is power-associative and use the notation  $X^n = \alpha(X, \dots, \alpha(X, X) \dots)$  where  $X$  occurs  $n$  times; this notation is used only for the algebra  $(m, \alpha, C)$  and is not to be confused with the product  $XY$  in  $(m, (1/2)XY, B)$ . The following result indicates that an algebra  $(m, \alpha, C)$  which defines an invariant Riemannian connection on a reductive space  $G/H$  does not satisfy the "usual" identities unless the algebra is anti-commutative; that is, unless the connection is of the first kind.

**THEOREM 1.** *Let  $(G, H)$  be a reductive pair with a corresponding Lie algebra decomposition  $g = m + h$ .*

(a) *If the algebra  $(m, \alpha, C)$  defines an invariant Riemannian connection on  $G/H$ , then  $\alpha(X^2, X) = \alpha(X, X^2)$  if and only if  $\alpha(X, Y) = (1/2)XY$  for all  $X, Y \in m$ .*

(b) *Let  $G/H$  have an invariant Riemannian connection of the first kind which is determined by the algebra  $(m, (1/2)XY, B)$ . If the algebra  $(m, \alpha, C)$  defines an invariant pseudo-Riemannian connection on  $G/H$ , then the algebra  $(m, \alpha, C)$  is power associative if and only if  $\alpha(X, Y) = (1/2)XY$  for all  $X, Y \in m$ .*

*Proof.* Since an anti-commutative algebra is power-associative, we need only prove the converses of the above statements.

(a) From formula (2) the positive definite form  $C$  must satisfy  $C(V, \alpha(U, V)) = 0$  for all  $U, V \in m$ . Now using this and formula (2) we see that for any  $X \in m$

$$\begin{aligned} C(\alpha(X, X), \alpha(X, X)) &= -C(X, \alpha(X, \alpha(X, X))) \\ &= -C(X, \alpha(\alpha(X, X), X)) \\ &= 0. \end{aligned}$$

where the identity  $\alpha(X, X^2) = \alpha(X^2, X)$  is used for the second equality. Thus  $\alpha(X, X) = 0$ . Using (3), we obtain  $\alpha(X, Y) = (1/2)XY$ .

(b) If we are given an algebra  $(m, (1/2)XY, B)$  which induces a Riemannian connection of the first kind and a second algebra  $(m, \alpha, C)$  which induces another pseudo-Riemannian connection, then, as remarked in §1, we can write  $C(X, Y) = B(SX, Y)$  and  $2\alpha(X, Y) = XY + S^{-1}[X(SY) - (SX)Y]$  for some  $S \in GL(m)$ . Using the fact

that the positive definite form  $B$  satisfies  $B(ZX, Y) + B(X, ZY) = 0$ , we now show that the algebra  $(m, \alpha, C)$  has no nonzero idempotent elements. For suppose  $E = \alpha(E, E)$ ; then from the above formula  $E = S^{-1}[E(SE)]$  so that  $SE = E(SE)$ . From this  $SE = E(E(SE))$  and therefore

$$\begin{aligned} B(SE, SE) &= B(SE, E(E(SE))) \\ &= -B(E(SE), E(SE)) \\ &= -B(SE, SE) \end{aligned}$$

so that  $B(SE, SE) = 0$  and  $SE = 0$ . As  $S$  is nonsingular,  $E = 0$ .

Since the power-associative algebra  $(m, \alpha, C)$  contains no idempotents, the associative subalgebra  $F[X]$  generated by any  $X \in m$  is nil [9; Prop. 3.3]; that is, for each  $X \in m$ , there exists a positive integer  $p$  such that  $X^p = 0$  in the algebra  $(m, \alpha, C)$ . By power-associativity if  $X^{r+t} = 0$  for positive integers  $r$  and  $t$ , then

$$0 = X^{r+t} = \alpha(X^r, X^t) = \frac{1}{2} X^r X^t + \frac{S^{-1}}{2} [X^r(SX^t) - (SX^r)X^t].$$

Thus using  $\alpha(X, Y) - \alpha(Y, X) = XY$  we also see  $X^r X^t = \alpha(X^r, X^t) - \alpha(X^t, X^r) = X^{r+t} - X^{r+t} = 0$  which implies

$$(4) \quad X^r(SX^t) = (SX^r)X^t$$

whenever  $X^{r+t} = 0$ .

We now show  $X^3 = 0$  implies  $X^2 = 0$ . For suppose  $X^3 = 0$ ; then from formula (4) we obtain

$$X(SX^2) = (SX)X^2.$$

Using the formula for  $\alpha(X, Y)$  we note  $SX^2 = X(SX)$  and have

$$\begin{aligned} B(SX^2, SX^2) &= B(X(SX), SX^2) \\ &= -B(SX, X(SX^2)) \\ &= -B(SX, (SX)X^2) \\ &= -B((SX)(SX), X^2) \\ &= 0 \end{aligned}$$

using the anti-commutativity  $ZZ = 0$  in  $(m, (1/2)XY, B)$ . Thus  $SX^2 = 0$  which implies  $X^2 = 0$ .

Next we show  $X^{n+1} = 0$  implies  $X^n = 0$  for  $n \geq 3$  and consequently by induction  $X^{n+1} = 0$  implies  $X^2 = 0$ . For suppose  $X^{n+1} = 0$ ; then  $X^{2n-1} = 0$  and from formula (4) we obtain

$$X(SX^n) = (SX)X^n \text{ and } X^{n-1}(SX^n) = (SX^{n-1})X^n.$$

Using these we see

$$\begin{aligned} B(X(SX^{n-1}), SX^n) &= -B(SX^{n-1}, X(SX^n)) \\ &= -B(SX^{n-1}, (SX)X^n) \\ &= B((SX^{n-1})X^n, SX) \end{aligned}$$

and

$$\begin{aligned} B((SX)X^{n-1}, SX^n) &= B(SX, X^{n-1}SX^n) \\ &= B(SX, (SX^{n-1})X^n) . \end{aligned}$$

Thus using  $X^{n-1}X = \alpha(X^{n-1}, X) - \alpha(X, X^{n-1}) = X^n - X^n = 0$ , we obtain  $2SX^n = X(SX^{n-1}) - (SX)X^{n-1}$  and

$$\begin{aligned} 2B(SX^n, SX^n) &= B(X(SX^{n-1}) - (SX)X^{n-1}, SX^n) \\ &= B(X(SX^{n-1}), SX^n) - B((SX)X^{n-1}, SX^n) \\ &= 0 \end{aligned}$$

and therefore  $X^n = 0$ . Since the algebra  $(m, \alpha, C)$  is nil, we have for every  $X \in m$  that  $X^p = 0$  for some integer  $p$ . Thus by the above  $0 = X^2 = \alpha(X, X)$ . Using (3), we obtain  $\alpha(X, Y) = (1/2)XY$ .

REMARKS. The conclusion of Theorem 1 that  $\alpha(X, Y) = (1/2)XY$  need not imply the forms  $B$  and  $C$  are equal. However, let us consider the algebra  $(m, (1/2)XY, B)$  as given where we can assume  $B$  is just nondegenerate. Then the endomorphism  $S$  which determines  $C$  for another algebra  $(m, \alpha, C)$  with  $\alpha(X, Y) = (1/2)XY$  is in the multiplication centralizer of  $(m, (1/2)XY, B)$ . To see this first recall that the multiplication centralizer,  $\Gamma$ , of the algebra  $(m, (1/2)XY, B)$  consists of those endomorphisms  $T$  of  $m$  satisfying  $L(X)T = TL(X)$  for all  $X \in m$ , where  $L(X): m \rightarrow m: Y \rightarrow XY$ . In [9; p. 15] the multiplication centralizer is discussed in general. It is proven that  $\Gamma$  is a subalgebra of the algebra of all endomorphisms of  $m$  and if the algebra  $(m, (1/2)XY, B)$  is simple,  $\Gamma$  is a field. Now, to see that  $S$  is in  $\Gamma$  we use formula (2) and  $\alpha(X, Y) = (1/2)XY$  and note that

$$\begin{aligned} B(S(XY), Z) &= C(XY, Z) \\ &= 2C(\alpha(X, Y), Z) \\ &= -2C(Y, \alpha(X, Z)) \\ &= -C(Y, XZ) \\ &= -B(SY, XZ) \\ &= B(X(SY), Z) . \end{aligned}$$

Since  $B$  is nondegenerate,  $S(XY) = X(SY)$ ; that is,  $SL(X) = L(X)S$  which implies  $S \in \Gamma$ . Conversely, a nonsingular endomorphism  $S$  in  $\Gamma \cap J$  determines an algebra  $(m, \alpha, C)$  with  $\alpha(X, Y) = (1/2)XY$ . In

particular, if  $S$  is chosen so that  $C$  is positive definite, then the corresponding connection is Riemannian.

As an example, let the pseudo-Riemannian connection determined by the nonzero algebra  $(m, (1/2)XY, B)$  be holonomy irreducible. Then as discussed in [3], [4], [6], the algebra  $(m, (1/2)XY, B)$  is simple. If we require that the algebra  $(m, (1/2)XY, C)$  be such that  $C$  is positive definite, then the following computations prove  $S$  is symmetric relative to  $C$ . For  $X, Y \in m$ ,

$$\begin{aligned} C(X, SY) &= B(SX, SY) \\ &= B(SY, SX) \\ &= C(Y, SX) \\ &= C(SX, Y) \end{aligned}$$

so that  $S^c = S$ , where  $c$  denotes the adjoint relative to  $C$ . Therefore,  $S$  has a nonzero real characteristic root  $\lambda$  and the characteristic root space  $n = \{X \in m: SY = \lambda Y\}$  is a nonzero ideal of  $(m, (1/2)XY, B)$ ; this uses  $L(X)S = SL(X)$  for all  $X \in m$ . Since  $(m, (1/2)XY, B)$  is simple, we see  $n = m$  and consequently  $S = \lambda I$ ; thus the original form  $B$  must be definite in this case. More generally, if  $(m, (1/2)XY, B)$  is semi-simple (that is, a direct sum of simple ideals), then the corresponding  $S$  is diagonalizable. These semi-simple algebras often occur when  $g$  and  $h$  are semi-simple Lie algebras as discussed in [4], [8].

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