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**COMMUTATORS AND NUMERICAL RANGES OF POWERS OF
OPERATORS**

ELIAS SAI WAN SHIU

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If 0 does not lie in the closure of the numerical range of any positive integral power of a Hilbert space operator T , then an odd power of T is normal. If, in addition, T is convexoid, then T itself is normal; in fact, T is the direct sum of at most three rotated positive operators. A version of these results is given in terms of commutators.

1. Introduction. In [8] C. R. Johnson proved: For an $m \times m$ complex matrix A , if A^n is not normal for any positive integer n , then there exist a positive integer n_0 and a nonzero vector $x \in \mathbb{C}^m$ such that $(A^{n_0}x, x) = 0$. Later he and M. Neuman [9] obtained a number theoretic result which strengthens the above theorem. We generalize these theorems to the Hilbert space operator case in this paper.

Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators on a Hilbert space \mathcal{H} . For $T \in \mathcal{B}(\mathcal{H})$, $\overline{W}(T)$ denotes the closure of the numerical range of T . Our main results are: If $0 \notin \overline{W}(T^n)$, $n = 1, 2, 3, \dots$, then an odd power of T is normal; in fact, T is similar to the direct sum of at most three rotated positive operators. Moreover, under the above hypothesis, T is normal if and only if T is convexoid.

These results can be applied to the theory of commutators: Let \mathfrak{H} denote a separable infinite dimensional Hilbert space. For $T \in \mathcal{B}(\mathfrak{H})$, if $T^n \notin \{SX - XS: S, X \in \mathcal{B}(\mathfrak{H}), S \text{ positive}\}$, $n = 1, 2, 3, \dots$, then there are an odd integer k and a compact operator K such that $T^k + K$ is normal; furthermore, T is a compact perturbation of a normal operator if and only if the essential numerical range of T is a polygon (possibly degenerate).

2. Preliminaries. Let \mathbb{C} denote the set of complex numbers and \mathbb{R}^+ the set of strictly positive numbers. For $\Omega \subset \mathbb{C}$, $\text{Co}(\Omega)$ denotes its convex hull; $\Omega^n = \{z^n: z \in \Omega\}$, n a positive integer. We write $\Omega > r$, r a real number, if Ω is a real subset and each number in Ω is greater than r . Let $\alpha, \beta \in \mathbb{C}$ and $\varepsilon \in (0, 1]$, $\theta(\alpha, \beta; \varepsilon)$ denotes the closed elliptical disc with eccentricity ε and foci at α and β ,

$$\theta(\alpha, \beta; \varepsilon) = \{z \in \mathbb{C}: |z - \alpha| + |z - \beta| \leq |\alpha - \beta|/\varepsilon\}.$$

Note that $\Theta(\alpha, \beta; 1)$ is the line segment joining α and β .

LEMMA 1. *Let α, β be two distinct nonzero complex numbers. For $\varepsilon \in (0, 1]$, if $|\operatorname{Arg}(\alpha/\beta)| \geq \arccos(-\varepsilon^2)$, then $0 \in \Theta(\alpha, \beta; \varepsilon)$.*

For $T \in \mathcal{B}(\mathcal{H})$, $\sigma(T)$ denotes the spectrum and $W(T)$ the numerical range of T , $W(T) = \{(Tx, x) : \|x\| = 1\}$. We say T is positive and write $T > 0$ if $\bar{W}(T) > 0$. T is called convexoid if $\operatorname{Co}(\sigma(T)) = \bar{W}(T)$ [6, p. 114].

The following result describes the numerical range of a 2×2 matrix with distinct eigenvalues ([12], [10]).

LEMMA 2. *If $\alpha \neq \beta$, then $W\left(\begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}\right) = \Theta(\alpha, \beta; (1 + |\gamma/(\alpha - \beta)|^2)^{-1/2})$.*

Let $\mathcal{H} \oplus \mathcal{K}$ denote the direct sum of two Hilbert spaces \mathcal{H} and \mathcal{K} ; an operator on $\mathcal{H} \oplus \mathcal{K}$ may be expressed as a 2×2 matrix whose entries are operators. See [6, Chapter 7].

LEMMA 3. *Let $T \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$, $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then*

$$W(T) = \cup \left\{ W\left(\begin{pmatrix} (Ax, x) & (By, x) \\ (Cx, y) & (Dy, y) \end{pmatrix}\right) : x \in \mathcal{H}, y \in \mathcal{K}, \|x\| = \|y\| = 1 \right\}.$$

Let $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are disjoint, nonempty and closed. Let E be the spectral projection associated with σ_1 [18, §5.7]; then $E^2 = E$, $ET = TE$, $\sigma(T|_{E\mathcal{H}}) = \sigma_1$ and $\sigma(T|_{(I-E)\mathcal{H}}) = \sigma_2$. We note that E may not be Hermitian.

LEMMA 4 (cf. [13, §0.4]). *Let T and E be as above and let P be the orthogonal projection on $E\mathcal{H}$. Then, with respect to the decomposition $E\mathcal{H} \oplus (E\mathcal{H})^\perp$, the operator matrix corresponding to T has the form $\begin{pmatrix} T_1 & T_1A - AT_2 \\ 0 & T_2 \end{pmatrix}$, where $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = E - P$ and*

$$\sigma(T_i) = \sigma_i, \quad i = 1, 2.$$

Furthermore, $T_1A - AT_2 = 0$ if and only if $A = 0$.

The following result is proved in ([14], [15]).

LEMMA 5. *For $T \in \mathcal{B}(\mathcal{H})$ and $\sigma(T) > \gamma > 0$, if $\{z \in \mathbf{C} : |z| \leq \gamma^n\} \not\subset W(T^n)$ for infinitely many positive integers n , then $T > 0$.*

3. Main results. The following generalizes [8, Theorem 1].

THEOREM 1. *Let $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) \cap \mathbf{R}^+ \neq \emptyset$. Suppose*

$0 \notin \bar{W}(T^n)$, $n = 1, 2, 3, \dots$, then either (i) there is a positive odd integer m such that $T^m > 0$ or (ii) there exist a proper closed subspace \mathcal{H}_1 of \mathcal{H} and positive operators T_1 and T_2 on \mathcal{H}_1 and \mathcal{H}_1^\perp respectively such that $T = T_1 \oplus e^{i\theta} T_2$, θ being irrational modulo 2π .

Proof. Since $0 \notin \bar{W}(T^n) \supset \text{Co}(\sigma(T^n)) = \text{Co}(\sigma(T)^n)$, $n = 1, 2, 3, \dots$, either (i) there is an odd integer m such that $\sigma(T)^m \subset \mathbf{R}^+$ or (ii) $\sigma(T) \subset \mathbf{R}^+ \cup e^{i\theta} \cdot \mathbf{R}^+$, θ being irrational modulo 2π .

In case (i), $\sigma(T^m) > 0$. Thus we have $T^m > 0$ by Lemma 5.

In case (ii) we apply Lemma 4 with $\sigma_1 = \sigma(T) \cap \mathbf{R}^+$. Then

$$T = \begin{pmatrix} T_1 & T_1 A - e^{i\theta} A T_2 \\ 0 & e^{i\theta} T_2 \end{pmatrix},$$

where $\sigma(T_1) > 0$ and $\sigma(T_2) > 0$. Since $T^n = \begin{pmatrix} T_1^n & T_1^n A - e^{in\theta} A T_2^n \\ 0 & e^{in\theta} T_2^n \end{pmatrix}$, $W(T^n) \supset W(T_1^n)$ and $W(T^n) \supset W(e^{in\theta} T_2^n)$, we have $T_1 > 0$ and $T_2 > 0$ by Lemma 5.

To show that $T = T_1 \oplus e^{i\theta} T_2$, we have to show $A = 0$. Assume $A \neq 0$. For a positive integer n and $y \in (E\mathcal{H})^\perp$, with $\|y\| = 1$ and $Ay \neq 0$, let $\theta[n, y]$ denote the numerical range of the 2×2 matrix

$$\begin{pmatrix} ((T_1^n Ay, Ay) / \|Ay\|^2 & ((T_1^n Ay, Ay) - e^{in\theta}(AT_2^n y, Ay)) / \|Ay\| \\ 0 & e^{in\theta}(T_2^n y, y) \end{pmatrix}.$$

By Lemma 3, $\theta[n, y] \subset W(T^n)$. By Lemma 2, $\theta[n, y] = \theta(\alpha, \beta; \varepsilon[n, y])$, where $\alpha \in \mathbf{R}^+$, $\beta \in e^{in\theta} \mathbf{R}^+$ and

$$\varepsilon[n, y] = \left(1 + \left| \frac{((T_1^n Ay, Ay) - e^{in\theta}(AT_2^n y, Ay)) / \|Ay\|}{(T_1^n Ay, Ay) / \|Ay\|^2 - e^{in\theta}(T_2^n y, y)} \right|^2 \right)^{-1/2}.$$

Let y_m , $m = 1, 2, 3, \dots$ be a sequence in $(E\mathcal{H})^\perp$ such that $\|y_m\| = 1$ and $\lim_{m \rightarrow \infty} \|Ay_m\| = \|A\|$. For each n ,

$$\begin{aligned} & \frac{((T_1^n Ay_m, Ay_m) - e^{in\theta}(T_2^n y_m, A^* Ay_m)) / \|Ay_m\|^2}{(T_1^n Ay_m, Ay_m) / \|Ay_m\|^2 - e^{in\theta}(T_2^n y_m, y_m)} \\ &= 1 + \frac{e^{in\theta}(T_2^n y_m, (\|Ay_m\|^2 - A^* A)y_m)}{(T_1^n Ay_m A, y_m) / \|Ay_m\|^2 - e^{in\theta}(T_2^n y_m, y_m)} \longrightarrow 1 \text{ as } m \longrightarrow \infty. \end{aligned}$$

Hence $\lim_{m \rightarrow \infty} \varepsilon[n, y_m] = (1 + \|A\|^2)^{-1/2}$. Thus for each integer n , there is an integer $m(n)$ such that

$$\varepsilon[n, y_{m(n)}] \leq (1 + \|A\|^2/2)^{-1/2} < 1.$$

Since θ is irrational modulo 2π , we can pick a positive integer N for which $|\text{Arg } e^{iN\theta}| \geq \arccos(-/(1 + \|A\|^2/2))$. Then $0 \in \theta[N, y_{m(N)}]$ by Lemma 1. However, $0 \notin W(T^N)$ by hypothesis; $A = 0$ and $T = T_1 \oplus e^{i\theta} T_2$.

We note that if \mathcal{H} is finite dimensional, the proof of case (ii) can be greatly simplified: Let $\alpha, \beta \in \mathbf{C}$ and $\alpha^n \neq \beta^n$, $n = 1, 2, 3, \dots$, then $W\left(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}^n\right) = \theta(\alpha^n, \beta^n; (1 + |\gamma/(\alpha - \beta)|^2)^{-1/2})$ by Lemma 2.

For $\mathcal{E} \subset \mathbf{C} \setminus \{0\}$, let $\#\text{Arg } \mathcal{E}$ denote the cardinality of the set $\{\lambda/|\lambda|: \lambda \in \mathcal{E}\}$. The result in [9] may be stated as follows: Let \mathcal{E} be a compact set of nonzero complex numbers such that $\mathcal{E} \cap \mathbf{R}^+ \neq \emptyset$. If $0 \notin \text{Co}(\mathcal{E}^n)$, $n = 1, 2, 3, \dots$, and if $\#\text{Arg } \mathcal{E} \geq 3$, then $\#\text{Arg } \mathcal{E} = 3$ and $\mathcal{E}^\tau \subset \mathbf{R}^+$.

THEOREM 1'. *Let $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) \cap \mathbf{R}^+ \neq \emptyset$. Suppose $0 \notin \bar{W}(T^n)$, $n = 1, 2, 3, \dots$. We have the following cases:*

- (i) $\#\text{Arg } \sigma(T) = 1$ then $T > 0$.
- (ii) $\#\text{Arg } \sigma(T) \geq 3$, then $\#\text{Arg } \sigma(T) = 3$ and $T^\tau > 0$.
- (iii) $\#\text{Arg } \sigma(T) = 2$, then either there is a positive odd integer m such that $T^m > 0$ or there exist a closed subspace \mathcal{H}_1 of \mathcal{H} and positive operators T_1 and T_2 on \mathcal{H}_1 and \mathcal{H}_1^\perp respectively such that $T = T_1 \oplus e^{i\theta} T_2$, θ being irrational modulo 2π .

THEOREM 2. *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose $0 \notin \bar{W}(T^n)$, $n = 1, 2, 3, \dots$. Then T is normal if T is convexoid.*

Proof. By Theorem 1', $\#\text{Arg } \sigma(T) \leq 3$. First, we consider the case $\#\text{Arg } \sigma(T) = 2$, i.e., there are two real numbers θ_1 and θ_2 such that $\sigma(T) \subset e^{i\theta_1} \cdot \mathbf{R}^+ \cup e^{i\theta_2} \cdot \mathbf{R}^+$. Let E be the spectral projection associated with $\sigma(T) \cap e^{i\theta_1} \cdot \mathbf{R}^+$. With respect to $E\mathcal{H} \oplus (E\mathcal{H})^\perp$, put $E = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix}$, then $T = \begin{pmatrix} e^{i\theta_1} T_1 & e^{i\theta_1} T_1 A - A e^{i\theta_2} T_1 \\ 0 & e^{i\theta_2} T_2 \end{pmatrix}$, where $T_1 > 0$ and $T_2 > 0$. Assume $A \neq 0$; thus there is a two-dimensional compression of T whose numerical range consists of an elliptical disc with foci on each of the two half-rays $e^{i\theta_j} \cdot \mathbf{R}^+$, $j = 1, 2$, and eccentricity strictly less than unity. However, T is a convexoid by hypothesis and $\text{Co}(\sigma(T))$ is a quadrilateral, a triangle or a line segment with all of its vertices lying on the two half-rays $e^{i\theta_j} \cdot \mathbf{R}^+$, $j = 1, 2$. Therefore, $A = 0$ and $T = e^{i\theta_1} T_1 \oplus e^{i\theta_2} T_2$.

The case that $\#\text{Arg } \sigma(T) = 3$ is treated in a similar fashion. Nevertheless, we note that the above geometric argument fails if $\#\text{Arg } \sigma(T) \geq 4$. Fortunately this case cannot arise.

By the term polygon, we mean the rectilinear figure together with its interior domain; moreover, we do not exclude the degenerate cases of singletons and line segments. For $T \in \mathcal{B}(\mathcal{H})$, if $\bar{W}(T)$ is a polygon, then T is convexoid [7, Satz 1]. Thus we have

COROLLARY 1. *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose $0 \notin \bar{W}(T^n)$, $n = 1, 2, 3, \dots$. Then T is normal if and only if $\bar{W}(T)$ is a polygon.*

We note that the polygon mentioned in Corollary 1 may have at most six sides.

4. Commutators. There are interesting applications of the above results to the theory of commutators. Let \mathfrak{H} be a separable infinite dimensional Hilbert space, $\mathcal{K}(\mathfrak{H})$ the set of all compact operators on \mathfrak{H} and Π the canonical homomorphism from $\mathcal{B}(\mathfrak{H})$ onto the Calkin algebra, $\mathcal{B}(\mathfrak{H})/\mathcal{K}(\mathfrak{H})$. There exists an isometric *-isomorphism τ of the Calkin algebra onto a closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, where \mathcal{H} is a suitably chosen Hilbert space [16, Theorem 12.41]. For $T \in \mathcal{B}(\mathfrak{H})$, the Weyl spectrum $\sigma_w(T)$ is the largest subset of $\sigma(T)$ which is invariant under compact perturbations, $\sigma_w(T) = \bigcap \{\sigma(T + K) : K \in \mathcal{K}(\mathfrak{H})\}$. In [5] it is shown that $\sigma_w(T)$ consists of $\sigma(\tau(\Pi(T)))$ together with some of the bounded components of the complement of $\sigma(\tau(\Pi(T)))$. Consequently if $\sigma_w(T)$ lies on a simple arc, $\sigma_w(T) = \sigma(\tau(\Pi(T)))$.

LEMMA 6 ([11], [4, p. 62]). *Let $T \in \mathcal{B}(\mathfrak{H})$. Suppose $\tau(\Pi(T))$ is normal and $\sigma(\tau(\Pi(T)))$ lies on a simple arc. Then, there exists a compact operator K such that $T + K$ is normal and $\sigma(T + K) = \sigma(\tau(\Pi(T)))$.*

The essential numerical range of $T \in \mathcal{B}(\mathfrak{H})$ is the set $W_e(T) = \bigcap \{\bar{W}(T + K) : K \in \mathcal{K}(\mathfrak{H})\}$. By [17, Theorem 9] and [2, Theorem 3], $W_e(T) = \bar{W}(\tau(\Pi(T)))$. Let \mathcal{R} denote $\{SX - XS : S, X \in \mathcal{B}(\mathfrak{H}), S > 0\}$. In [1], J. H. Anderson proved the following deep result: $\mathcal{R} = \{T \in \mathcal{B}(\mathfrak{H}) : 0 \in W_e(T)\}$; also see [3, §34]. Corresponding to Theorem 1', we have

THEOREM 3. *Let $T \in \mathcal{B}(\mathfrak{H})$. Suppose $T^n \in \mathcal{R}$, $n = 1, 2, 3, \dots$. Then we have the following cases:*

- (i) $\#\text{Arg } \sigma_w(T) = 1$, then there exist $\theta \in [0, 2\pi)$ and a compact operator K such that $(e^{i\theta}T + K) > 0$.
- (ii) $\#\text{Arg } \sigma_w(T) \geq 3$, then $\#\text{Arg } \sigma_w(T) = 3$ and there exist $\theta \in [0, 2\pi)$ and a compact operator K such that $(e^{i\theta}T^3 + K) > 0$.
- (iii) $\#\text{Arg } \sigma_w(T) = 2$, then either there exist a positive odd integer m , $\theta \in [0, 2\pi)$ and a compact operator K such that $(e^{i\theta}T^m + K) > 0$, or there exist a closed subspace \mathfrak{H}_1 of \mathfrak{H} and positive operators T_1 and T_2 on \mathfrak{H}_1 and \mathfrak{H}_1^\perp respectively such that $(T - e^{i\theta_1}T_1 \oplus e^{i\theta_2}T_2)$ is compact, where $(\theta_1 - \theta_2)$ is a number irrational modulo 2π .

Proof. We only need to prove the second half of case (iii).

We know $\tau(\Pi(T)) = e^{i\theta_1}V_1 \oplus e^{i\theta_2}V_2$ on $\mathcal{H}_1 \oplus \mathcal{H}_1^1 = \mathcal{H}$, where $V_1 > 0$ and $V_2 > 0$. Thus $\tau(\Pi(T))$ is normal and $\sigma(\tau \circ \Pi(T))$ lies on a simple arc. By Lemma 6, there is a compact operator K such that $T + K$ is normal and $\sigma(T + K) = \sigma(\tau(\Pi(T)))$. Consequently, there exist a closed subspace \mathfrak{S}_1 of \mathfrak{S} and positive operators T_1 and T_2 on \mathfrak{S}_1 and \mathfrak{S}_1^\perp respectively such that $(T - e^{i\theta_1}T_1 \oplus e^{i\theta_2}T_2)$ is compact.

THEOREM 4. *Let $T \in \mathcal{B}(\mathfrak{S})$. Suppose $T^n \in \mathcal{R}$, $n = 1, 2, 3, \dots$. Then T is a compact perturbation of a normal operator if and only if $W_e(T)$ is a polygon.*

Proof. Apply Corollary 1.

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