COMMUTATORS AND NUMERICAL RANGES OF POWERS OF OPERATORS

Elias Sai Wan Shiu
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If 0 does not lie in the closure of the numerical range of any positive integral power of a Hilbert space operator $T$, then an odd power of $T$ is normal. If, in addition, $T$ is convexoid, then $T$ itself is normal; in fact, $T$ is the direct sum of at most three rotated positive operators. A version of these results is given in terms of commutators.

1. Introduction. In [8] C. R. Johnson proved: For an $m \times m$ complex matrix $A$, if $A^n$ is not normal for any positive integer $n$, then there exist a positive integer $n_0$ and a nonzero vector $x \in C^m$ such that $(A^{n_0}x, x) = 0$. Later he and M. Neuman [9] obtained a number theoretic result which strengthens the above theorem. We generalize these theorems to the Hilbert space operator case in this paper.

Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators on a Hilbert space $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, $\overline{W}(T)$ denotes the closure of the numerical range of $T$. Our main results are: If $0 \notin \overline{W}(T^n)$, $n = 1, 2, 3, \ldots$, then an odd power of $T$ is normal; in fact, $T$ is similar to the direct sum of at most three rotated positive operators. Moreover, under the above hypothesis, $T$ is normal if and only if $T$ is convexoid.

These results can be applied to the theory of commutators: Let $\mathcal{H}$ denote a separable infinite dimensional Hilbert space. For $T \in \mathcal{B}(\mathcal{H})$, $T^n \in \{SX - XS: S, X \in \mathcal{B}(\mathcal{H}), S$ positive}, $n = 1, 2, 3, \ldots$, then there are an odd integer $k$ and a compact operator $K$ such that $T^k + K$ is normal; furthermore, $T$ is a compact perturbation of a normal operator if and only if the essential numerical range of $T$ is a polygon (possibly degenerate).

2. Preliminaries. Let $C$ denote the set of complex numbers and $R^+$ the set of strictly positive numbers. For $\Omega \subset C$, $\text{Co}(\Omega)$ denotes its convex hull; $\Omega^n = \{z^n: z \in \Omega\}$, $n$ a positive integer. We write $\Omega > r$, $r$ a real number, if $\Omega$ is a real subset and each number in $\Omega$ is greater than $r$. Let $\alpha, \beta \in C$ and $\varepsilon \in (0, 1]$, $\theta(\alpha, \beta; \varepsilon)$ denotes the closed elliptical disc with eccentricity $\varepsilon$ and foci at $\alpha$ and $\beta$,

$$\theta(\alpha, \beta; \varepsilon) = \{z \in C: |z - \alpha| + |z - \beta| \leq |\alpha - \beta|/\varepsilon\}.$$
Note that $\Theta(a, \beta; 1)$ is the line segment joining $a$ and $\beta$.

**Lemma 1.** Let $a, \beta$ be two distinct nonzero complex numbers. For $\varepsilon \in (0, 1]$, if $|\text{Arg}(a/\beta)| \geq \arccos (-\varepsilon^2)$, then $0 \in \Theta(a, \beta; \varepsilon)$.

For $T \in \mathcal{B} (\mathcal{H})$, $\sigma(T)$ denotes the spectrum and $W(T)$ the numerical range of $T$, $W(T) = \{(Tx, x) : \|x\| = 1\}$. We say $T$ is positive and write $T > 0$ if $W(T) > 0$. $T$ is called convexoid if $\text{Co}(\sigma(T)) = W(T)$ [6, p. 114].

The following result describes the numerical range of a $2 \times 2$ matrix with distinct eigenvalues ([12], [10]).

**Lemma 2.** If $\alpha \neq \beta$, then $W\left(\begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}\right) = \Theta(\alpha, \beta; (1 + |\gamma/(\alpha - \beta)|^2)^{-1/2})$.

Let $\mathcal{H} \oplus \mathcal{H}$ denote the direct sum of two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}$; an operator on $\mathcal{H} \oplus \mathcal{H}$ may be expressed as a $2 \times 2$ matrix whose entries are operators. See [6, Chapter 7].

**Lemma 3.** Let $T \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then

$$W(T) = \bigcup \left\{ W\left(\begin{pmatrix} Ax, x \\ Cy, y \end{pmatrix} : x \in \mathcal{H}, y \in \mathcal{H}, \|x\| = \|y\| = 1\right) : (By, x) \right\}.$$  

Let $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) = \sigma_1 \cup \sigma_2$, where $\sigma_1$ and $\sigma_2$ are disjoint, nonempty and closed. Let $E$ be the spectral projection associated with $\sigma_1$ [18, § 5.7]; then $E^2 = E$, $ET = TE$, $\sigma(T|_{E^2}) = \sigma_1$ and $\sigma(T|_{E^2}) = \sigma_2$. We note that $E$ may not be Hermitian.

**Lemma 4** (cf. [13, § 0.4]). Let $T$ and $E$ be as above and let $P$ be the orthogonal projection on $E\mathcal{H}$. Then, with respect to the decomposition $E\mathcal{H} \oplus (E\mathcal{H})^1$, the operator matrix corresponding to $T$ has the form $\begin{pmatrix} T_1 & T_1A - AT_2 \\ 0 & T_2 \end{pmatrix}$, where $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = E - P$ and

$$\sigma(T_i) = \sigma_i, \ i = 1, 2.$$  

Furthermore, $T_1A - AT_2 = 0$ if and only of $A = 0$.

The following result is proved in ([14], [15]).

**Lemma 5.** For $T \in \mathcal{B}(\mathcal{H})$ and $\sigma(T) \cap R^+ \neq \emptyset$, if $\{z \in \mathbb{C} : |z| \leq \gamma^2\} \not\subset W(T^n)$ for infinitely many positive integers $n$, then $T > 0$.

3. Main results. The following generalizes [8, Theorem 1].

**Theorem 1.** Let $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) \cap R^+ \neq \emptyset$. Suppose
0 \in \overline{W}(T^n), n = 1, 2, 3, \ldots, then either (i) there is a positive odd integer m such that $T^m > 0$ or (ii) there exist a proper closed subspace $\mathcal{H}_1$ of $\mathcal{H}$ and positive operators $T_1$ and $T_2$ on $\mathcal{H}_1$ and $\mathcal{H}_1 \perp$ respectively such that $T = T_1 \oplus e^{i\theta}T_2$, $\theta$ being irrational modulo $2\pi$.

**Proof.** Since $0 \in \overline{W}(T^n) \supset \text{Co}(\sigma(T^n)) = \text{Co}(\sigma(T)^n)$, $n = 1, 2, 3, \ldots$, either (i) there is an odd integer $m$ such that $\sigma(T)^m \subset \mathbb{R}^+$ or (ii) $\sigma(T) \subset \mathbb{R}^+ \cup e^{i\theta} \cdot \mathbb{R}^+$, $\theta$ being irrational modulo $2\pi$.

In case (i), $\sigma(T^m) > 0$. Thus we have $T^m > 0$ by Lemma 5.

In case (ii) we apply Lemma 4 with $\sigma_x = \sigma(T)_{\mathbb{R}}^{-\frac{1}{2}}$. Then

$$T = \begin{pmatrix} T_1 T_A - e^{i\theta} T_2 & 0 \\ 0 & e^{i\theta} T_2 \end{pmatrix},$$

where $\sigma(T_1) > 0$ and $\sigma(T_2) > 0$. Since $T^n = \begin{pmatrix} T_1^n T_A - e^{i\theta} T_2^n & 0 \\ 0 & e^{i\theta} T_2^n \end{pmatrix}$, $W(T^n) \supset W(T_1^n)$ and $W(T^n) \supset W(e^{i\theta} T_2^n)$, we have $T_1 > 0$ and $T_2 > 0$ by Lemma 5.

To show that $T = T_1 \oplus e^{i\theta}T_2$, we have to show $A = 0$. Assume $A \neq 0$. For a positive integer $n$ and $y \in (E_{\mathcal{H}})^\perp$, with $\|y\| = 1$ and $Ay \neq 0$, let $\theta[n, y]$ denote the numerical range of the $2 \times 2$ matrix

$$\left(\begin{array}{cc} (T_1^n Ay, Ay)/\|Ay\|^2 & ((T_1^n Ay, Ay) - e^{i\theta}(AT_2^n y, Ay))/\|Ay\| \\ 0 & e^{i\theta}(T_2^n y, y) \end{array}\right).$$

By Lemma 3, $\theta[n, y] \subset W(T^n)$. By Lemma 2, $\theta[n, y] = \theta(\alpha, \beta; \varepsilon[n, y])$, where $\alpha \in \mathbb{R}^+$, $\beta \in e^{i\theta} \cdot \mathbb{R}^+$ and

$$\varepsilon[n, y] = \left(1 + \frac{||((T_1^n Ay, Ay) - e^{i\theta}(AT_2^n y, Ay))/\|Ay\|\|^2}{(T_1^n Ay, Ay)/\|Ay\|^2 - e^{i\theta}(T_2^n y, y)}\right)^{-1/2}.$$  

Let $y_m, m = 1, 2, 3, \ldots$ be a sequence in $(E_{\mathcal{H}})^\perp$ such that $\|y_m\| = 1$ and $\lim_{m \to \infty} \|Ay_m\| = \|A\|$. For each $n$,

$$\frac{((T_1^n Ay_m, Ay_m) - e^{i\theta}(T_2^n y_m, A^*Ay_m))/\|Ay_m\|^2}{(T_1^n Ay_m, Ay_m)/\|Ay_m\|^2 - e^{i\theta}(T_2^n y_m, y_m)} = 1 + \frac{e^{i\theta}(T_2^n y_m, (||Ay_m||^2 - A^*A)y_m)}{(T_1^n Ay_m, A_m)/\|Ay_m\|^2 - e^{i\theta}(T_2^n y_m, y_m)} \to 1$$

as $m \to \infty$.

Hence $\lim_{m \to \infty} \varepsilon[n, y_m] = (1 + \|A\|^2)^{-1/2}$. Thus for each integer $n$, there is an integer $m(n)$ such that

$$\varepsilon[n, y_{m(n)}] \leq (1 + \|A\|^2/2)^{-1/2} < 1.$$  

Since $\theta$ is irrational modulo $2\pi$, we can pick a positive integer $N$ for which $|\text{Arg} e^{iN\theta}| \geq \text{arccos} (-/(1 + \|A\|^2/2))$. Then $0 \in \theta[N, y_{m(N)}]$ by Lemma 1. However, $0 \in W(T^N)$ by hypothesis; $A = 0$ and $T = T_1 \oplus e^{i\theta}T_2$. 


We note that if $\mathcal{H}$ is finite dimensional, the proof of case (ii) can be greatly simplified: Let $\alpha, \beta \in \mathbb{C}$ and $\alpha^n \neq \beta^n$, $n = 1, 2, 3, \ldots$, then $W\left(\begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix}\right) = \Theta(\alpha^n, \beta^n; (1 + |\gamma/(\alpha - \beta)|^{1/2})$ by Lemma 2.

For $\mathcal{C} \subset \mathbb{C}\{0\}$, let $\#\text{Arg } \mathcal{C}$ denote the cardinality of the set $\{\lambda/|\lambda| : \lambda \in \mathcal{C}\}$. The result in [9] may be stated as follows: Let $\mathcal{C}$ be a compact set of nonzero complex numbers such that $\mathcal{C} \cap \mathbb{R}^+ \neq \emptyset$. If $0 \in \text{Co}(\mathcal{C}^n)$, $n = 1, 2, 3 \ldots$, and if $\#\text{Arg } \mathcal{C} \geq 3$, then $\#\text{Arg } \mathcal{C} = 3$ and $\mathcal{C}^n \subset \mathbb{R}^+$.

**Theorem 1'.** Let $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) \cap \mathbb{R}^+ \neq \emptyset$. Suppose $0 \notin \overline{W}(T^n)$, $n = 1, 2, 3, \ldots$. We have the following cases:

(i) $\#\text{Arg } \sigma(T) = 1$ then $T > 0$.

(ii) $\#\text{Arg } \sigma(T) \geq 3$, then $\#\text{Arg } \sigma(T) = 3$ and $T^\gamma > 0$.

(iii) $\#\text{Arg } \sigma(T) = 2$, then either there is a positive odd integer $m$ such that $T^m > 0$ or there exist a closed subspace $\mathcal{H}_1$ of $\mathcal{H}$ and positive operators $T_1$ and $T_2$ on $\mathcal{H}_1$ and $\mathcal{H}_1^\perp$ respectively such that $T = T_1 \oplus e^{i\theta}T_2$, $\theta$ being irrational modulo $2\pi$.

**Theorem 2.** Let $T \in \mathcal{B}(\mathcal{H})$. Suppose $0 \notin \overline{W}(T^n)$, $n = 1, 2, 3, \ldots$. Then $T$ is normal if $T$ is convexoid.

**Proof.** By Theorem 1', $\#\text{Arg } \sigma(T) \leq 3$. First, we consider the case $\#\text{Arg } \sigma(T) = 2$, i.e., there are two real numbers $\theta_1$ and $\theta_2$ such that $\sigma(T) \subset e^{i\theta_1}\cdot \mathbb{R}^+ \cup e^{i\theta_2}\cdot \mathbb{R}^+$. Let $E$ be the spectral projection associated with $\sigma(T) \cap e^{i\theta_1}\cdot \mathbb{R}^+$. With respect to $E \mathcal{H} \oplus (E \mathcal{H})^\perp$, put $E = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix}$, then $T = \begin{pmatrix} e^{i\theta_1}T_1 & e^{i\theta_1}T_1A - Ae^{i\theta_2}T_1 \\ 0 & e^{i\theta_2}T_2 \end{pmatrix}$, where $T_1 > 0$ and $T_2 > 0$. Assume $A \neq 0$; thus there is a two-dimensional compression of $T$ whose numerical range consists of an elliptical disc with foci on each of the two half-rays $e^{i\theta_j}\cdot \mathbb{R}^+$, $j = 1, 2$, and eccentricity strictly less than unity. However, $T$ is a convexoid by hypothesis and $\text{Co}(\sigma(T))$ is a quadrilateral, a triangle or a line segment with all of its vertices lying on the two half-rays $e^{i\theta_j}\cdot \mathbb{R}^+$, $j = 1, 2$. Therefore, $A = 0$ and $T = e^{i\theta_1}T_1 \oplus e^{i\theta_2}T_2$.

The case that $\#\text{Arg } \sigma(T) = 3$ is treated in a similar fashion. Nevertheless, we note that the above geometric argument fails if $\#\text{Arg } \sigma(T) \geq 4$. Fortunately this case cannot arise.

By the term polygon, we mean the rectilinear figure together with its interior domain; moreover, we do not exclude the degenerate cases of singletons and line segments. For $T \in \mathcal{B}(\mathcal{H})$, if $\overline{W}(T)$ is a polygon, then $T$ is convexoid [7, Satz 1]. Thus we have
COROLLARY 1. Let $T \in \mathcal{B}(\mathcal{H})$. Suppose $0 \in \overline{W}(T^n)$, $n = 1, 2, 3, \ldots$. Then $T$ is normal if and only if $\overline{W}(T)$ is a polygon.

We note that the polygon mentioned in Corollary 1 may have at most six sides.

4. Commutators. There are interesting applications of the above results to the theory of commutators. Let $\mathcal{S}$ be a separable infinite dimensional Hilbert space, $\mathcal{H}(\mathcal{S})$ the set of all compact operators on $\mathcal{S}$ and $\Pi$ the canonical homomorphism from $\mathcal{B}(\mathcal{S})$ onto the Calkin algebra, $\mathcal{B}(\mathcal{S})/\mathcal{K}(\mathcal{S})$. There exists an isometric *-isomorphism $\tau$ of the Calkin algebra onto a closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a suitably chosen Hilbert space [16, Theorem 12.41]. For $T \in \mathcal{B}(\mathcal{S})$, the Weyl spectrum $\sigma_w(T)$ is the largest subset of $\sigma(T)$ which is invariant under compact perturbations, $\sigma_w(T) = \cap \{\sigma(T + K) : K \in \mathcal{K}(\mathcal{S})\}$. In [5] it is shown that $\sigma_w(T)$ consists of $\sigma(\tau(\Pi(T)))$ together with some of the bounded components of the complement of $\sigma(\tau(\Pi(T)))$. Consequently if $\sigma_w(T)$ lies on a simple arc, $\sigma_w(T) = \sigma(\tau(\Pi(T)))$.

LEMMA 6 ([11], [4, p. 62]). Let $T \in \mathcal{B}(\mathcal{S})$. Suppose $\tau(\Pi(T))$ is normal and $\sigma(\tau(\Pi(T)))$ lies on a simple arc. Then, there exists a compact operator $K$ such that $T + K$ is normal and $\sigma(T + K) = \sigma(\tau(\Pi(T)))$.

The essential numerical range of $T \in \mathcal{B}(\mathcal{S})$ is the set $W_e(T) = \cap \{\overline{W}(T + K) : K \in \mathcal{H}(\mathcal{S})\}$. By [17, Theorem 9] and [2, Theorem 3], $W_e(T) = \overline{W}(\tau(\Pi(T)))$. Let $\mathcal{R}$ denote $\{SX - XS : S, X \in \mathcal{B}(\mathcal{S}), S > 0\}$. In [1], J. H. Anderson proved the following deep result: $\mathcal{R} = \{T \in \mathcal{B}(\mathcal{S}) : 0 \in W_e(T)\}$; also see [3, §34]. Corresponding to Theorem 1', we have

THEOREM 3. Let $T \in \mathcal{B}(\mathcal{S})$. Suppose $T^* \in \mathcal{R}$, $n = 1, 2, 3, \ldots$.

Then we have the following cases:

(i) $\# \arg \sigma_w(T) = 1$, then there exist $\theta \in [0, 2\pi)$ and a compact operator $K$ such that $(e^{i\theta}T + K) > 0$.

(ii) $\# \arg \sigma_w(T) \geq 3$, then $\# \arg \sigma_w(T) = 3$ and there exist $\theta \in [0, 2\pi)$ and a compact operator $K$ such that $(e^{i\theta}T^* + K) > 0$.

(iii) $\# \arg \sigma_w(T) = 2$, then either there exist a positive odd integer $m$, $\theta \in [0, 2\pi)$ and a compact operator $K$ such that $(e^{i\theta}T^m + K) > 0$, or there exist a closed subspace $\mathcal{S}_1$ of $\mathcal{S}$ and positive operators $T_1$ and $T_2$ on $\mathcal{S}_1$ and $\mathcal{S}_1^*$ respectively such that $(T - e^{i\theta_1}T_1 \oplus e^{i\theta_2}T_2)$ is compact, where $(\theta_1 - \theta_2)$ is a number irrational modulo $2\pi$.

Proof. We only need to prove the second half of case (iii).
We know $\tau(\Pi(T)) = e^{i\theta_1}V_1 \mathbb{1} e^{i\theta_2}V_2$ on $\mathcal{H}_1 \mathbb{1} \mathcal{H}_1 \mathbb{1} = \mathcal{H}$, where $V_1 > 0$ and $V_2 > 0$. Thus $\tau(\Pi(T))$ is normal and $\sigma(\tau \circ \Pi(T))$ lies on a simple arc. By Lemma 6, there is a compact operator $K$ such that $T + K$ is normal and $\sigma(T + K) = \sigma(\tau(\Pi(T)))$. Consequently, there exist a closed subspace $\mathfrak{S}_1$ of $\mathfrak{S}$ and positive operators $T_1$ and $T_2$ on $\mathfrak{S}_1$ and $\mathfrak{S}_1 \mathbb{1}$ respectively such that $(T - e^{i\theta_1}T_1 \mathbb{1} e^{i\theta_2}T_2)$ is compact.

**Theorem 4.** Let $T \in \mathcal{B}(\mathfrak{S})$. Suppose $T^n \notin \mathcal{R}$, $n = 1, 2, 3, \ldots$. Then $T$ is a compact perturbation of a normal operator if and only if $W_n(T)$ is a polygon.

**Proof.** Apply Corollary 1.

**References**

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