DELOOPING THE CONTINUOUS $K$-THEORY OF A VALUATION RING

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OF A VALUATION RING

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In this note the continuous algebraic $K$-theory groups of a complete discrete valuation ring are described as the inverse limit of the ordinary algebraic $K$-theory of its finite quotient rings.

In [4] we defined continuous algebraic $K$-theory groups $K^\text{top}_i$, $i \geq 2$, both for a complete discrete valuation ring $\mathcal{O}$ with finite residue field of positive characteristic $p$ and for its fraction field and proved that $K^\text{top}_i$ agrees with the fundamental group of the special linear group as defined in [2] by means of universal topological central extensions. The definition of $K^\text{top}_i$ in [4] is in terms of $BN$-pairs and is similar to the theory $K^B_N$ of [5] which is known [6] to deloop to ordinary algebraic $K$-theory. The purpose of this note is to deloop $K^\text{top}_i(\mathcal{O})$ in the sense of the following result: Let $\mathcal{P} \cap \mathcal{O}$ be the maximal ideal and let $K_i$ be the algebraic $K$-theory groups of Quillen [3].

**Theorem.** For $i \geq 2$ there is a natural isomorphism

$$K^\text{top}_i(\mathcal{O}) \cong \lim_{\longleftarrow n} K_i(\mathcal{O}/\mathcal{P}^n).$$

In a forthcoming paper of the author and R. J. Milgram, this equation allows us to use the continuous cohomology of $\text{SL}(l, \mathcal{O})$ to compute the rank of the free part of $K^\text{top}_i(\mathcal{O})$ as a module over the $p$-adic completion of the rational integers.

In §2 a step in the proof of this theorem is used to describe the homotopy fiber of $\text{BE}(A)^+ \to \text{BE}(A/J)^+$ where $J$ is an ideal in a commutative ring $A$ such that $1 + J \subseteq A^*$. At least, we construct a space $\text{B}(U_f(A, J))^+$ whose homotopy groups fit into the appropriate exact sequence.

Actually, in this paper we shall let

$$K^\text{top}_i(\mathcal{O}) = \lim_{\longleftarrow m|n} \lim_{\longleftarrow l} \pi_{i-1} \text{SL}_n^\text{top}(l, \mathcal{O})$$

whereas in [4] the order of the inverse and direct limits is reversed. The above definition is perhaps better as it still gives the main results of [4]. To see the two are the same one would have to prove that

$$\to \pi_{i-1} \text{SL}_n^\text{top}(l, \mathcal{O}) \to \pi_{i-1} \text{SL}_n^\text{top}(l + 1, \mathcal{O}) \to \cdots$$
eventually stabilizes to an isomorphism.

The theorem makes it clear that the natural map $K_t(\mathcal{O}) \to K_t^{\text{top}}(\mathcal{O})$ comes from the ring maps $\mathcal{O} \to \mathcal{O}/\mathcal{P}^n$.

1. Delooping. Let $n$ and $l$ be fixed. The main step is to prove

**PROPOSITION 1.1.** There is a natural homotopy equivalence

$$\text{SL}^\text{ab}(l, \mathcal{O}/\mathcal{P}^n) \cong \text{SL}_n^{\text{top}}(l, \mathcal{O})$$

such that if $m|n$ there is a homotopy commutative diagram

$$\begin{array}{ccc}
\text{SL}^\text{ab}(l, \mathcal{O}/\mathcal{P}^n) & \cong & \text{SL}_n^{\text{top}}(l, \mathcal{O}) \\
\downarrow & & \downarrow \\
\text{SL}^\text{ab}(l, \mathcal{O}/\mathcal{P}^m) & \cong & \text{SL}_m^{\text{top}}(l, \mathcal{O}).
\end{array}$$

See [4] for notation. From this result and [6] we see that for $i \geq 2$

$$\lim_{l \to i} \pi_{i-1} \text{SL}_n^{\text{top}}(l, \mathcal{O}) = \lim_{l \to i} \pi_{i-1} \text{SL}^\text{ab} l, \mathcal{O}/\mathcal{P}^n) = \pi_{i-1} \text{SL}_n^\text{ab}(\mathcal{O}/\mathcal{P}^n) = K_t(\mathcal{O}/\mathcal{P}^n).$$

Here $\text{SL}^\text{ab}(A)$ of [4] is the same as $E^{\text{BN}}(A)$ of [5]. The main theorem now follows from commutativity of (*).

For simplicity of notation let $S_i = \text{SL}^\text{ab}(l, \mathcal{O}/\mathcal{P}^n)$ and $T_i = \text{SL}_n^{\text{top}}(l, \mathcal{O})$. Let $P^i$ (resp. $Q^i$) be the complex whose $k$-simplices are $(k + 1)$-tuples $(F_0 < F_1 < \cdots < F_i)$ where $F_i$ is a linear (resp. affine) facette or $R^i$. $P^i \subset S_i$ by the imbedding $F \to U_F$ and $Q^i \subset T_i$ via $F \to U_F$. Let $\text{st}_i(\Delta) < Q^i$ be the star of $\Delta$ consisting of all affine facettes $F$ such that $\Delta < F$. Let $K_i < T_i$ be the subcomplex whose $k$-simplices $(\alpha_0 \cdot U_{F_0}^* < \cdots < \alpha_k \cdot U_{F_k}^*)$ have $F_i \in \text{st}_i(\Delta)$.

Now for each affine facette $F \in \text{st}_i(\Delta)$ there is a unique linear facette $F'$ which contains $F$ such that $F < G$ implies $F' < G'$. The map $\text{st}_i(\Delta) \to P^i$ sending $F$ to $F'$ is an isomorphism of partially ordered sets. Let $\pi: \text{SL}(l, \mathcal{O}) \to \text{SL}(l, \mathcal{O}/\mathcal{P}^n)$ be reduction modulo $\mathcal{P}^n$. We claim that

$$\pi(U_F^*) = U_F^*$$

for $F \in \text{st}_i(\Delta)$. This is clear for the fundamental chamber $C = \{x_i + 1 > x_i > \cdots > x_i\}$ and also for any $F < C$. For an arbitrary $F \in \text{st}_i(\Delta)$ choose an element $w$ of the linear Weyl group $W_0$ so that $w \cdot F < C$. Thus by [4, Lemma 3]
$$\pi(U_F^*) = \pi(w^{-1}(wU_{w,F}^*w^{-1})w)$$
$$= w^{-1} \cdot \pi(U_{w,F}^*) \cdot w$$
$$= w^{-1} \cdot U_{w,F}^* \cdot w$$
$$= U_F^*.$$

Moreover for each $F \in \text{st}_i(\mathcal{A})$ we have

$$(1.2') \quad \pi^{-1}(U_F^n) = U_F^n$$

These two equations imply the correspondence

$$\alpha \cdot U_F^n \longrightarrow \pi(\alpha) \cdot U_F^n,$$

preserves order and defines a simplicial isomorphism $K_1 \to S_i$. Hence to prove (1.1) it suffices to show $K_1$ is a deformation retract to $T_i$.

Let $f; g: Q^I \to Q^I$ be two simplicial maps arising from order preserving maps of vertices.

**Lemma 1.3.** There is a triangulation $(Q^i \times I)'$ of $Q^i \times I$ as a partially ordered set which refines the standard triangulation leaving $Q^i \times 0$ and $Q^i \times 1$ fixed and there is a simplicial map $w: (Q^i \times I)' \to Q^i$ such that

(a) $w|Q^i \times 0 = f$ and $w|Q^i \times 1 = g$

(b) if $\sigma = (v_0 < \cdots < v_n)$ is a simplex of the standard triangulation $Q \times I$, $v \in \sigma$ is a vertex in the new triangulation, and $e_{ij}(\lambda)$ is in $U_{w(v_i)}^n$ for $0 \leq s \leq k$, then

$$e_{ij}(\lambda) \in U_{w(v_i)}^n.$$

This is the affine analogue of Lemma 3.3 of [6] and the proof is similar. For (b) compare (B) of Lemma 4 of [4].

Now let $r: Q^i \to \text{st}_i(\mathcal{A}) \subset Q^i$ be defined by

$$r(F) = \begin{cases} 
\text{the unique affine facette of } \text{st}_i(\mathcal{A}) \text{ which is} \\
\text{contained in the same linear facette as } F.
\end{cases}$$

This is an order preserving map which is the identity on $\text{st}_i(\mathcal{A})$.

**Lemma 1.4.** For each affine facette $F$ we have $U_{r(F)}^n \subset U_{r(F)}^n$.

*Proof.* If $w \in W_0$, then $w \cdot r(F) = r(w \cdot F)$ and $w \cdot F^* \cdot w^{-1} = U_{w,F}^*$, so by choosing a $w$ such that $w \cdot F$ is contained in the fundamental linear chamber $C_0 = \{x_1 > \cdots > x_i\}$ of $\mathcal{C}_0$ we can assume $F \subset \tilde{C}_0$. In this case $r(F) = C$. When $i > j$, $e_i - e_j \geq 0$ on $F$; so for the generator $e_{ij}(\lambda)$ of $U_F^n$ the element $\lambda \in \mathcal{C}$ can be arbitrary and $e_{ij}(\lambda) \in U_F^n$. When $i < j$, $e_i - e_j \leq 0$ on $F$ so $k(F, e_i - e_j)_n = n = k(C, e_i - e_j)_n$; hence any generator $e_{ij}(\lambda)$ of $U_F^n$ also belongs to $U_F^n$. 


We can now complete the proof of (1.1). Apply Lemma 1.3 in the case $f = \text{id}$ and $g = r$ to get $w: (Q^i \times I') \rightarrow Q^i$ satisfying (a) and (b). The map $\rho: T_i \rightarrow Q^i$ taking $\alpha \cdot U_F^n$ to $F'$ is nondegenerate on simplices and so is $\rho \times 1: T_i \times I \rightarrow Q^i \times I$. Therefore the triangulation $(Q^i \times I')$ induces a subdivision $(T_i \times I')$ of $T_i \times I$. Let $\sigma = (\alpha_0 \cdot U_{F_0}^n < \cdots < \alpha_k \cdot U_{F_k}^n)$ be a simplex of $T_i$ and let $v$ be a vertex of $\sigma \times I$. Let $u = (\rho \times 1)(v)$. By (1.4) we have $U_{F_0}^n \subset U_{w(v)}^n$. Hence by (b) of (1.3) we still have

$$U_{F_0}^n \subset U_{w(v)}^n$$

if $v$ is any vertex of $(\sigma \times I')$.

Let $R: T_i \rightarrow T_i$ be defined by $R(\alpha \cdot U_F^n) = \alpha \cdot U_{r(F)}^n$. This retracts $T_i$ onto $K_i$. Define a homotopy $H: (T_i \times I') \rightarrow T_i$ from the identity to $R$ as follows: Let $v$ be a vertex $(\sigma \times I')$ and let $u = (\rho \times 1)(v)$. Let

$$H(v) = \alpha_0 \cdot U_{w(v)}^n .$$

Then (1.5) shows this is independent of the choice $\alpha_0 \in U_{F_0}^n$ so we get a well defined map.

2. A fibration in $K$-theory. Let $A$ be a commutative ring and $J \subset A$ be an ideal such that $1 + J \subset A^*$. Then $K_i(A) \rightarrow K_i(A/J)$ is surjective for $i = 1, 2$. In this section we build a space $B(U_F(A, J))^+$ such that for $i \geq 2$ there is a natural exact sequence

$$\cdots \rightarrow K_{i+1}(A/J) \rightarrow \pi_i B(U_F(A, J))^+ \rightarrow K_i(A) \rightarrow K_i(A/J) \rightarrow \cdots .$$

Let $P^i$ denote the set of linear facettes in $R^i$ and identify $P^i$ as a subset of $P^{i+1}$ by the map

$$(x_1, \ldots, x_i) \mapsto (x_1, \ldots, x_i, x_{i+1}).$$

Let $P^\infty = \bigcup_i P^i$. If $F \in P^\infty$ define the subgroup $U_F(A, J)$ of the group $E(A)$ of elementary matrices to be the one generated by

(a) $e_{ij}(\lambda)$ where $\lambda \in A$ for $e_i - e_j > 0$ on $F$

(b) $e_{ii}(\lambda)$ where $\lambda \in J$ for $e_i - e_j < 0$ on $F$

(c) diagonal matrices diag $\{1 + \lambda_1, \ldots, 1 + \lambda_r\}$ of determinant one where $\lambda_i \in J$.

If $F < G$, then $U_F(A, J) < U_G(A, J)$. When $J = 0$, we just get the groups $U_F$ of [4] and [5]. In this case we write $U_F(A, J) = U_F(A)$. Let $\pi: E(A) \rightarrow E(A/J)$ be reduction mod $J$. Then as in (1.2) and (1.2)' we have

$$\pi[U_F(A, J)] = U_F(A/J) \quad \text{and} \quad \pi^{-1}[U_F(A/J)] = U_F(A, J) .$$
Let $B\{ U_F(A, J)\}$ be the realization of the simplicial space which in dimension $k \geq 0$ is the disjoint union of the spaces

$$(F_0 < \cdots < F_k) \times BU_{F_0}(A, J)$$

where $F_i \in P^\infty$. Let $E\{ \alpha \cdot U_F(A, J)\}$ be defined as the pullback

$$E\{ \alpha \cdot U_F(A, J)\} \rightarrow EG$$

$$B\{ U_F(A, J)\} \rightarrow BG$$

where $G = E(A)$. When $J = 0$ we recover $E\{ \alpha \cdot U_F\}$ as in [1]. Moreover just as in [1] the space $E\{ \alpha \cdot U_F(A, J)\}$ has the homotopy type of the space $E^{BN}(A, J)$ whose $k$-simplices are $(k + 1)$-tuples

$$\sigma_0 \cdot U_{F_0}(A, J) \prec \cdots \prec \alpha_k \cdot U_{F_k}(A, J)$$

where $\alpha \cdot U_{F}(A, J) \prec \beta \cdot U_{0}(A, J)$ iff $F \prec G$ and $\alpha \cdot U_{F}(A, J) \prec \beta \cdot U_{0}(A, J)$. As in [1] we have a homotopy fibration

$$E\{ \alpha \cdot U_F(A, J)\} \rightarrow B\{ U_F(A, J)\} \rightarrow BE(A) \cdot$$

Suppose for the moment we have

**Lemma 2.3.** $\pi_1 B\{ U_F(A, J)\}$ is perfect.

Then essentially the same argument as in [1] shows that

$$E\{ \alpha \cdot U_F(A, J)\} \rightarrow B\{ U_F(A, J)\}^+ \rightarrow BE(A)^+$$

is also a homotopy fibration. It follows from (2.2) that the map

$$E^{BN}(A, J) \rightarrow E^{BN}(A/J)$$

given by $\alpha \cdot U_F(A, J) \rightarrow \pi(\alpha) \cdot U_F(A/J)$ is an isomorphism. By [6] we therefore have $\pi_{i-1} E^{BN}(A, J) = K_i(A/J)$ and the homotopy sequence of the fibration (***) gives (2.1).

To prove the lemma, it is enough to show the generators are products of commutators and the formula $w \cdot U_F \cdot w^{-1} = U_{w \cdot F}$ reduces the argument to the case where $F' = C_0 = \{ x_i > x_{i+1} > \cdots > x_l \}$ considered as lying in $P^l$. Here $l \geq 3$. For generators $e_{ij}(\lambda)$ of $\pi_i(BU_{C_0})$ the third Steinberg relation $e_{ij}(\lambda \beta) = [e_{ij}(\lambda), e_{jk}(\beta)]$ shows $e_{ij}(\lambda)$ is a commutator: for example, if $\lambda \in J$ we have $e_{21}(\lambda) = [e_{23}(1), e_{31}(\lambda)]$. Now consider the generators $e_{ij}(\lambda)$, $\lambda \in 1 + J$, where $\lambda$ is in the $i$th row and $j$th column and $\lambda^{-1}$ is in the $j$th row and $i$th column. For simplicity take $i = 1$ and $j = 2$. Recall that if $M, N \in U_F$ are considered as generators of $\pi_1 BU_F$ their composition as loops is homotopic to
Let $\lambda = 1 + \sigma$ and $\lambda^{-1} = 1 + \tau$ where $\tau, \sigma \in J$. We have the following matrix identity valid in $E(A)$:

$$
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix} =
\begin{pmatrix}
1 & \lambda \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\tau & 1
\end{pmatrix}
\begin{pmatrix}
1 & \sigma \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
0 & 1
\end{pmatrix}.
$$

Thus modulo the commutator subgroup

$$
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & \sigma \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix} =
\begin{pmatrix}
1+\sigma & \sigma \\
-\sigma & 1-\sigma
\end{pmatrix}.
$$

Now let $D = \{x_t = x_2 > \cdots > x_1\}$ and $C_0' = \{x_2 > x_1 > \cdots > x_1\}$. We have $U_{c_0} \supset U_d \subset U_{c_0}$ and the matrix $\begin{pmatrix}
1+\sigma & \sigma \\
-\sigma & 1-\sigma
\end{pmatrix}$ lies in $U_d$. Each of $\begin{pmatrix}
1 & 0 \\
-1 & 1
\end{pmatrix}$, $\begin{pmatrix}
1 & \sigma \\
0 & 1
\end{pmatrix}$, and $\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}$ belong to $U_{c_0}$ and therefore by the above argument lie in the commutator subgroup. Therefore so does $\begin{pmatrix}
1+\sigma & \sigma \\
-\sigma & 1-\sigma
\end{pmatrix}$, and we conclude that the loop $\begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix}$ lies in the commutator subgroup.

It is probably true that

$$
B(U_F(A, J))^+ \longrightarrow BE(A)^+ \longrightarrow BE(A/J)^+
$$

is a homotopy fibration.

References


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