Pacific Journal of Mathematics

DELOOPING THE CONTINUOUS K-THEORY OF A VALUATION RING

JOHN BASON WAGONER

Vol. 65, No. 2 October 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 65, No. 2, 1976

DELOOPING THE CONTINUOUS K-THEORY OF A VALUATION RING

J. B. WAGONER

In this note the continuous algebraic K-theory groups of a complete discrete valuation ring are described as the inverse limit of the ordinary algebraic K-theory of its finite quotient rings.

In [4] we defined continuous algebraic K-theory groups K_i^{top}, $i \geqq 2$, both for a complete discrete valuation ring $\mathscr{O}$ with finite residue field of positive characteristic p and for its fraction field and proved that K_2^{top} agrees with the fundamental group of the special linear group as defined in [2] by means of universal topological central extensions. The definition of K_i^{top} in [4] is in terms of BN-pairs and is similar to the theory K_i^{BN} of [5] which is known [6] to deloop to ordinary algebraic K-theory. The purpose of this note is to deloop $K_i^{\text{top}}(\mathscr{O})$ in the sense of the following result: Let $\mathscr{P} \cap \mathscr{O}$ be the maximal ideal and let K_i be the algebraic K-theory groups of Quillen [3].

THEOREM. *For $i \geqq 2$ there is a natural isomorphism*

$$K_i^{\text{top}}(\mathscr{O}) \cong \lim_{\underset{n}{\longleftarrow}} K_i(\mathscr{O}/\mathscr{P}^n) .$$

In a forthcoming paper of the author and R. J. Milgram, this equaation allows us to use the continuous cohomology of $\mathrm{SL}(l, \mathscr{O})$ to compute the rank of the free part of $K_i^{\text{top}}(\mathscr{O})$ as a module over the p-adic completion of the rational integers.

In § 2 a step in the proof of this theorem is used to describe the homotopy fiber of $BE(A)^+ \to BE(A/J)^+$ where J is an ideal in a commutative ring A such that $1 + J \subset A^*$. At least, we construct a space $B\{U_F(A, J)\}^+$ whose homotopy groups fit into the appropriate exact sequence.

Actually, in this paper we shall let

$$K_i^{\text{top}}(\mathscr{O}) = \lim_{\underset{m|n}{\longleftarrow}} [\lim_{\underset{l}{\longrightarrow}} \pi_{i-1} \mathrm{SL}_n^{\text{top}}(l, \mathscr{O})]$$

whereas in [4] the order of the inverse and direct limits is reversed. The above definition is perhaps better as it still gives the main results of [4]. To see the two are the same one would have to prove that

$$\longrightarrow \pi_{i-1} \mathrm{SL}_n^{\text{top}} (l, \mathscr{O}) \longrightarrow \pi_{i-1} \mathrm{SL}_n^{\text{top}} (l + 1, \mathscr{O}) \longrightarrow \cdots$$

eventually stabilizes to an isomorphism.

The theorem makes it clear that the natural map $K_i(\mathscr{O}) \to K_i^{\text{top}}(\mathscr{O})$ comes from the ring maps $\mathscr{O} \to \mathscr{O}/\mathscr{P}^n$.

1. **Delooping.** Let n and l be fixed. The main step is to prove

PROPOSITION 1.1. *There is a natural homotopy equivalence*

$$\mathrm{SL}^{ab}(l, \mathscr{O}/\mathscr{P}^n) \cong \mathrm{SL}_n^{\text{top}}(l, \mathscr{O})$$

such that if $m|n$ there is a homotopy commutative diagram

$$(*) \qquad \begin{array}{ccc} \mathrm{SL}^{ab}(l, \mathscr{O}/\mathscr{P}^n) & \cong & \mathrm{SL}_n^{\text{top}}(l, \mathscr{O}) \\ \downarrow & & \downarrow \\ \mathrm{SL}^{ab}(l, \mathscr{O}/\mathscr{P}^m) & \cong & \mathrm{SL}_m^{\text{top}}(l, \mathscr{O}) . \end{array}$$

See [4] for notation. From this result and [6] we see that for $i \geqq 2$

$$\begin{aligned} \varinjlim_l \pi_{i-1} \mathrm{SL}_n^{\text{top}}(l, \mathscr{O}) &= \varinjlim_l \pi_{i-1} \mathrm{SL}^{ab}\, l, \mathscr{O}/\mathscr{P}^n) \\ &= \pi_{i-1} \mathrm{SL}^{ab}(\mathscr{O}/\mathscr{P}^n) \\ &= K_i(\mathscr{O}/\mathscr{P}^n) . \end{aligned}$$

Here $\mathrm{SL}^{ab}(A)$ of [4] is the same as $E^{BN}(A)$ of [5]. The main theorem now follows from commutativity of (*).

For simplicity of notation let $S_l = \mathrm{SL}^{ab}(l, \mathscr{O}/\mathscr{P}^n)$ and $T_l = \mathrm{SL}_n^{\text{top}}(l, \mathscr{O})$. Let P^l(resp. Q^l) be the complex whose k-simplices are $(k+1)$-tuples $(F_0 < F_1 < \cdots < F_k)$ where F_i is a linear (resp. affine) facette or R^l. $P^l \subset S_l$ by the imbedding $F \to U_F$ and $Q^l \subset T_l$ via $F \to U_F^n$. Let $\mathrm{st}_l(\Delta) < Q^l$ be the star of Δ consisting of all affine facettes F such that $\Delta < F$. Let $K_l < T_l$ be the subcomplex whose k-simplices $(\alpha_0 \cdot U_{F_0}^n < \cdots < \alpha_k \cdot U_{F_k}^n)$ have $F_i \in \mathrm{st}_l(\Delta)$.

Now for each affine facette $F \in \mathrm{st}_l(\Delta)$ there is a unique linear facette F' which contains F such that $F < G$ implies $F' < G'$. The map $\mathrm{st}_l(\Delta) \to P^l$ sending F to F' is an isomorphism of partially ordered sets. Let π: $\mathrm{SL}(l, \mathscr{O}) \to \mathrm{SL}(l, \mathscr{O}/\mathscr{P}^n)$ be reduction modulo $\mathscr{P}^n$. We claim that

$$(1.2) \qquad \pi(U_F^n) = U_{F'}$$

for $F \in \mathrm{st}_l(\Delta)$. This is clear for the fundamental chamber $C = \{x_l + 1 > x_1 > \cdots > x_l\}$ and also for any $F < C$. For an arbitrary $F \in \mathrm{st}_l(\Delta)$ choose an element w of the linear Weyl group W_0 so that $w \cdot F < C$. Thus by [4, Lemma 3]

$$\begin{aligned}\pi(U_F^n) &= \pi(w^{-1}(wU_F^n w^{-1})w)\\ &= w^{-1}\cdot\pi(U_{w\cdot F}^n)\cdot w\\ &= w^{-1}\cdot U_{w\cdot F'}\cdot w\\ &= U_{F'}\,.\end{aligned}$$

Moreover for each $F\in \mathrm{st}_l(\varDelta)$ we have

$$\pi^{-1}(U_{F'}) = U_F^n \tag{1.2'}$$

These two equations imply the correspondence

$$\alpha\cdot U_F^n \longrightarrow \pi(\alpha)\cdot U_{F'}$$

preserves order and defines a simplicial isomorphism $K_l\to S_l$. Hence to prove (1.1) it suffices to show K_l is a deformation retract to T_l.

Let $f; g$: $Q^l\to Q^l$ be two simplicial maps arising from order preserving maps of vertices.

LEMMA 1.3. *There is a triangulation $(Q_l\times I)'$ of $Q_l\times I$ as a partially ordered set which refines the standard triangulation leaving $Q^l\times 0$ and $Q^l\times 1$ fixed and there is a simplicial map w: $(Q^l\times I)'\to Q^l$ such that*

(a) *$w|Q^l\times 0=f$ and $w|Q^l\times 1=g$*

(b) *if $\sigma=(v_0<\cdots<v_n)$ is a simplex of the standard triangulation $Q\times I$, $v\in\sigma$ is a vertex in the new triangulation, and $e_{ij}(\lambda)$ is in $U_{w(v_s)}^n$ for $0\leqq s\leqq k$, then*

$$e_{ij}(\lambda)\in U_{w(v)}^n\,.$$

This is the affine analogue of Lemma 3.3 of [6] and the proof is similar. For (b) compare (B) of Lemma 4 of [4].

Now let r: $Q^l\to \mathrm{st}_l(\varDelta)\subset Q^l$ be defined by

$$r(F)=\begin{cases}\text{the unique affine facette of } \mathrm{st}_l(\varDelta) \text{ which is}\\ \text{contained in the same linear facette as } F.\end{cases}$$

This is an order preserving map which is the identity on $\mathrm{st}_l(\varDelta)$.

LEMMA 1.4. *For each affine facette F we have $U_F^n\subset U_{r(F)}^n$.*

Proof. If $w\in W_0$, then $w\cdot r(F)=r(w\cdot F)$ and $w\cdot F_F^n\cdot w^{-1}=U_{w\cdot F}^n$; so by choosing a w such that $w\cdot F$ is contained in the closure $\bar{C}_0$ of the fundamental linear chamber $C_0=\{x_1>\cdots>x_l\}$ we can assume $F\subset\bar{C}_0$. In this case $r(F)=C$. When $i>j$, $e_i-e_j\geqq 0$ on F; so for the generator $e_{ij}(\lambda)$ of U_F^n the element $\lambda\in\mathscr{O}$ can be arbitrary and $e_{ij}(\lambda)\in U_C^n$. When $i<j$, $e_i-e_j\leqq 0$ on F so $k(F, e_i-e_j)_n\geqq n=k(C, e_i-e_j)_n$; hence any generator $e_{ij}(\lambda)$ of U_F^n also belongs to U_C^n.

We can now complete the proof of (1.1). Apply Lemma 1.3 in the case $f = \mathrm{id}$ and $g = r$ to get $w: (Q^l \times I)' \to Q^l$ satisfying (a) and (b). The map $\rho: T_l \to Q^l$ taking $\alpha \cdot U_F^n$ to F is nondegenerate on simplices and so is $\rho \times 1: T_l \times I \to Q^l \times I$. Therefore the triangulation $(Q^l \times I)'$ induces a subdivision $(T_l \times I)'$ of $T_l \times I$. Let $\sigma = (\alpha_0 \cdot U_{F_0}^n < \cdots < \alpha_k \cdot U_{F_k}^n)$ be a simplex of T_l and let v be a vertex of $\sigma \times I$. Let $u = (\rho \times 1)(v)$. By (1.4) we have $U_{F_0}^n \subset U_{w(v)}^n$. Hence by (b) of (1.3) we still have

$$(1.5) \qquad U_{F_0}^n \subset U_{w(v)}^n$$

if v is any vertex of $(\sigma \times I)'$.

Let $R: T_l \to T_l$ be defined by $R(\alpha \cdot U_F^n) = \alpha \cdot U_{r(F)}^n$. This retracts T_l onto K_l. Define a homotopy $H: (T_l \times I)' \to T_l$ from the identity to R as follows: Let v be a vertex $(\sigma \times I)'$ and let $u = (\rho \times 1)(v)$. Let

$$H(v) = \alpha_0 \cdot U_{w(v)}^n .$$

Then (1.5) shows this is independent of the choice $\alpha_0 \in U_{F_0}^n$ so we get a well defined map.

2. **A fibration in K-theory.** Let A be a commutative ring and $J \subset A$ be an ideal such that $1 + J \subset A^*$. Then $K_i(A) \to K_i(A/J)$ is surjective for $i = 1, 2$. In this section we build a space $B\{U_F(A, J)\}^+$ such that for $i \geqq 2$ there is a natural exact sequence

$$(2.1) \qquad \cdots \longrightarrow K_{i+1}(A/J) \longrightarrow \pi_i B\{U_F(A, J)\}^+ \longrightarrow K_i(A) \longrightarrow K_i(A/J) \longrightarrow \cdots .$$

Let P^l denote the set of linear facettes in R^l and identify P^l as a subset of P^{l+1} by the map

$$(x_1, \cdots, x_l) \longrightarrow (x_1, \cdots, x_l, x_l) .$$

Let $P^\infty = \cup_l P^l$. If $F \in P^\infty$ define the subgroup $U_F(A, J)$ of the group $E(A)$ of elementary matrices to be the one generated by

(a) $e_{ij}(\lambda)$ where $\lambda \in A$ for $e_i - e_j > 0$ on F

(b) $e_{ij}(\lambda)$ where $\lambda \in J$ for $e_i - e_j < 0$ on F

(c) diagonal matrices diag $\{1 + \lambda_1, \cdots, 1 + \lambda_r\}$ of determinant one where $\lambda_i \in J$.

If $F < G$, then $U_F(A, J) < U_G(A, J)$. When $J = 0$, we just get the groups U_F of [4] and]5]. In this case we write $U_F(A, J) = U_F(A)$. Let $\pi: E(A) \to E(A/J)$ be reduction mod J. Then as in (1.2) and (1.2)′ we have

$$(2.2) \qquad \pi[U_F(A, J)] = U_F(A/J) \quad \text{and} \quad \pi^{-1}[U_F(A/J)] = U_F(A, J) .$$

Let $B\{U_F(A, J)\}$ be the realization of the simplicial space which in dimension $k \geqq 0$ is the disjoint union of the spaces

$$(F_0 < \cdots < F_k) \times BU_{F_0}(A, J)$$

where $F_i \in P^\infty$. Let $E\{\alpha \cdot U_F(A, J)\}$ be defined as the pullback

$$\begin{array}{ccc} E\{\alpha \cdot U_F(A, J)\} & \longrightarrow & EG \\ \downarrow & & \downarrow \\ B\{U_F(A, J)\} & \longrightarrow & BG \end{array}$$

where $G = E(A)$. When $J = 0$ we recover $E\{\alpha \cdot U_F\}$ as in [1]. Moreover just as in [1] the space $E\{\alpha \cdot U_F(A, J)\}$ has the homotopy type of the space $E^{BN}(A, J)$ whose k-simplices are $(k+1)$-tuples

$$\sigma_0 \cdot U_{F_0}(A, J) < \cdots < \alpha_k \cdot U_{F_k}(A, J)$$

where $\alpha \cdot U_F(A, J) < \beta \cdot U_G(A, J)$ iff $F < G$ and $\alpha \cdot U_F(A, J) \subset \beta \cdot U_G(A, J)$. As in [1] we have a homotopy fibration

$$E\{\alpha \cdot U_F(A, J)\} \longrightarrow B\{U_F(A, J)\} \longrightarrow BE(A) .$$

Suppose for the moment we have

LEMMA 2.3. *$\pi_1 B\{U_F(A, J)\}$ is perfect.*

Then essentially the same argument as in [1] shows that

$$(**) \qquad E\{\alpha \cdot U_F(A, J)\} \longrightarrow B\{U_F(A, J)\}^+ \longrightarrow BE(A)^+$$

is also a homotopy fibration. It follows from (2.2) that the map

$$E^{BN}(A, J) \longrightarrow E^{BN}(A/J)$$

given by $\alpha \cdot U_F(A, J) \rightarrow \pi(\alpha) \cdot U_F(A/J)$ is an isomorphism. By [6] we therefore have $\pi_{i-1} E^{BN}(A, J) = K_i(A/J)$ and the homotopy sequence of the fibration (**) gives (2.1).

To prove the lemma, it is enough to show the generators are products of commutators and the formula $w \cdot U_F \cdot w^{-1} = U_{u \cdot F}$ reduces the argument to the case where $F = C_0 = \{x_1 > x_2 > \cdots > x_l\}$ considered as lying in P^l. Here $l \geqq 3$. For generators $e_{ij}(\lambda)$ of $\pi_1(BU_{C_0})$ the third Steinberg relation $e_{ij}(\alpha\beta) = [e_{ij}(\alpha), e_{jk}(\beta)]$ shows $e_{ij}(\lambda)$ is a commutator: for example, if $\lambda \in J$ we have $e_{21}(\lambda) = [e_{23}(1), e_{31}(\lambda)]$. Now consider the generators $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \in 1 + J$, where λ is in the ith row and ith column and λ^{-1} is in the jth row and jth column. For simplicity take $i = 1$ and $j = 2$. Recall that if $M, N \in U_F$ are considered as generators of $\pi_1 BU_F$ their composition as loops is homotopic to

MN. Let $\lambda = 1 + \sigma$ and $\lambda^{-1} = 1 + \tau$ where $\tau, \sigma \in J$. We have the following matrix identity valid in $E(A)$:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\tau & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Thus modulo the commutator subgroup

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+\sigma & \sigma \\ -\sigma & 1-\sigma \end{pmatrix}.$$

Now let $D = \{x_1 = x_2 > \cdots > x_l\}$ and $C_0' = \{x_2 > x_1 > \cdots > x_l\}$. We have $U_{C_0} \supset U_D \subset U_{C_0'}$ and the matrix $\begin{pmatrix} 1+\sigma & \sigma \\ -\sigma & 1-\sigma \end{pmatrix}$ lies in U_D. Each of $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ belong to $U_{C_0'}$ and therefore by the above argument lie in the commutator subgroup. Therefore so does $\begin{pmatrix} 1+\sigma & \sigma \\ -\sigma & 1-\sigma \end{pmatrix}$, and we conclude that the loop $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ lies in the commutator subgroup.

It is probably true that

$$B\{U_F(A, J)\}^+ \longrightarrow BE(A)^+ \longrightarrow BE(A/J)^+$$

is a homotopy fibration.

References

1. D. Anderson, M. Karoubi and J. Wagoner, *Relations between Higher Algebraic K-theories*, Springer-Verlag Lecture Notes in Mathematics, No. 341, pp. 73–81. See also: *Higher algebraic K-theories*, preprint, University of California, Berkeley.
2. C. Moore, *Group extensions of $\mathscr{P}$-adic and adelic linear groups*, I.H.E.S. Pub. Math., No. 34, 1968.
3. D. Quillen, *Higher K-theory I*, *Algebraic K-theory I*, Springer-Verlag Lecture Notes in Mathematics, No. 341, pp. 85–147.
4. J. Wagoner, *Homotopy theory for the $\mathscr{P}$-adic special linear group*, Comm. Math. Helv., **50** (1975), 535–559.
5. ———, *Buildings, stratifications, and Higher K-theory*, *Algebraic K-theory I*, Springer-Verlag Lecture Notes in Mathematics, No. 341, pp. 148–165.
6. ———, *Equivalence of algebraic K-theories*, preprint, University of California, Berkeley.

Received January 5, 1976. Partially supported by NSF Grant GP-43843X.

UNIVERSITY OF CALIFORNIA-BERKLEY

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 65, No. 2 October, 1976