STANDARD REGULAR SEMIGROUPS

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We give a structure theorem for a class of regular semigroups. Let $S$ be a regular semigroup, let $T$ denote the union of the maximal subgroups of $S$, and let $E(T)$ denote the set of idempotents of $T$. Assume $T$ is a semigroup (equivalently, $T$ is a semilattice $Y$ of completely simple semigroups $(T_y; y \in Y)$). If $Y$ has a greatest element and $e, f, g \in E(T)$, $e \geq f$, and $e \geq g$ imply $fg = gf$, we term $S$ a standard regular semigroup. The structure of $S$ is given modulo right groups and an inverse semigroup $V$ in which every subgroup is a single element by means of an explicit multiplication. We specialize the structure theorem to orthodox, $\mathcal{D}$-unipotent, and inverse semigroups, and to a class of semigroups with $Y$ an $\omega Y$-semilattice.

Finally, we show that $S$ is a regular extension of $T$ by $V$ in the sense of Yamada [19].

Let us first state the structure theorem. Let $V$ be a semilattice with greatest element. Let $V$ be an inverse semigroup with semilattice of idempotents $Y$ such that each subgroup of $V$ consists of a single element. Let $(L, \circ)$ be a standard regular semilattice $Y$ of left zero semigroups $(L_y; y \in Y)$. Let $(J, *)$ be a standard regular semilattice $Y$ of right groups $(J_y; y \in Y)$. Suppose $I_y \cap J_y = \{e_y\}$, a single idempotent element, and $e_y e_z = e_y e_y = e_y z$ for all $y, z \in Y$. Let $H_y$ denote the maximal subgroup of $J_y$ containing $e_y$. Let $i \to B_i$ be a homomorphism of $(L, \circ)$ into $P(J)$, the semigroup of right translations of $(J, *)$; let $b \to \beta_b$ be a mapping of $V$ into $\text{End}(J, *)$, the semigroup of endomorphism of $(J, *)$, and let $g$ be a mapping of $V \times V$ into $H = \bigcup (H_y; y \in Y)$, a semilattice $Y$ of groups $(H_y; y \in Y)$ (with respect to the multiplication $*$ in $J$) such that $1(a) jB_i \in H_{yz}$ for $i \in I_y$ and $j \in J_z$, (b) $J_y \beta_b \subseteq H_{b^{-1}zd}$, (c) $g(c, d) \in H_{(cd)^{-1}zd}$. 2(a) $hB_{e_y} = h \beta_y = h e_y$ for $h \in J$ and $y \in Y$. (b) if $j \in H_z$ and $i \in I_y$, $jB_i = j$ (c) $g(y, z) = e_{yz}$ for $y, z \in Y$. 3(a)

$$\beta_d \beta_d = \beta_{ed} C_{g(e, d)}(x C_z = z^{-1} x z \text{ for } x, z \in H)$$

(b) $g(a, b) * g(b, c) = g(ab, c) * (g(a, b) \beta_c)$. Let $(Y, I, J, V, B, \beta, g)$ denote the $(i, a, j) \in V$, $i \in I_{aa^{-1}}$, $j \in J_{a^{-1}a}$ under the multiplication (4)

$$(i, a, j)(w, b, v) = (i \circ e_{(ab)(ab)^{-1}}, ab, g(a, b) * jB_{w} \beta_b * v).$$

We show (Theorem 3.14) that $(Y, I, J, V, B, \beta, g)$ is a standard regular semigroup, and, conversely, every standard regular semigroup is isomorphic to some $(Y, I, J, V, B, \beta, g)$. 

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If $X$ is a semigroup, $E(X)$ will denote the set of idempotents of $X$. Let $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, and $\mathcal{D}$ denote Green’s relations (notation of [1]).

Using [1, Theorem 2.3], $\mathcal{H}$ is the identity congruence on $V$. Hence, using a result of Munn [5, Theorem 2.3; see also 6], $V$ is isomorphic to a subsemigroup $U$ of the semigroup $X$ of isomorphisms between principal ideals of $E(V)$ with $E(U) = E(X)$.

In special cases, explicit multiplications for $V$ have been given (see for example [8] and [4]). Probably, the most familiar example of $V$ is the bicyclic semigroup.

The multiplication for $J$ is described by means of “connecting homomorphisms” between the $J_y$ (i.e. if $a \in J_y$, $b \in J_z$, $a \cdot b = a\zeta_{y,z} b\zeta_{y,z}$ where $\zeta_{y,w}(y \geq w)$ is a homomorphism of $J_y$ into $J_w$). The multiplication for $I$ is similarly characterized (see Remarks 1.7, 1.8, and 3.15).

A regular semigroup $X$ is termed locally inverse if $e, f, g \in E(X)$, $e \geq f$, and $e \geq g$ imply $fg = gf$. (Let $X$ be a regular semigroup and let $e \in E(X)$. Hence, if $a \in eXe$, there exists $y \in X$ such that $a = aya = (ae)y(ea) = a(aye)a$. Thus, the semigroup $eXe$ is also regular. Hence, using [1, Theorem 1.17], a regular semigroup $X$ is locally inverse if and only if $eXe$ is an inverse semigroup for all $e \in E(X)$.) Thus, a standard regular semigroup is a locally inverse semigroup such that $T$ is a semigroup and $Y$ has a greatest element.

Following Hall [3], a regular semigroup $X$ is termed orthodox if $E(X)$ is a semigroup. In general, a standard regular semigroup is not orthodox.

Yamada [18, Theorem 2] described the structure of locally inverse orthodox semigroups in terms of inverse semigroups (locally inverse orthodox = generalized inverse in the sense of Yamada [18, Theorem 1]).

A Cliffordian semigroup is a semigroup which is a union of its subgroups. A semigroup $S$ is Cliffordian if and only if $S$ is a semilattice $Y$ of completely simple semigroups ($S_y: y \in Y$) (Clifford, [1, Theorem 4.6]).

In §1, we show that the multiplication of a locally inverse Cliffordian semigroup $S$ is described by means of connecting homomorphisms between the $S_y$ (Theorem 1.6) and give some consequences of this theorem. The results of this section are applied repeatedly in the sequel.

In §§2 and 3, we prove the converse and direct parts, respectively, of our structure theorem (Theorem 3.14).

Let $N$ denote the nonnegative integers and let $Y$ be a semilattice with greatest element. If $W = N \times Y$ with $(k, \alpha)A(s, \lambda) = (k, \alpha)$, $(s, \lambda)$, or $(k, \alpha)A\lambda$ according to whether $k > s, s > k$, or $s = k$, we term $W$ an $\omega Y$-semilattice. A regular semigroup $S$ is termed $\omega Y$-unipotent if $E(S)$ is an $\omega Y$-semilattice of right zero semigroups ($E_{(s, \delta)}: (n, \delta) \in$
If $f_{(n, \delta)}(f_{(m, \lambda)}) \in E((T_{(n, \delta)}; f_{(m, \lambda)})$ if and only if $\delta = \lambda$.

If $E((T_{(n, \delta)})$ is a single element for each $(n, \delta) \in N \times Y$, we term $S$ an $\omega Y$-inverse semigroup.

Munn [5, Theorem 3.3] described the structure of simple $\omega Y$-inverse semigroups. In [10, Theorem 4], Warne described the structure of simple $\omega Y$-unipotent semigroups such that $e \in E((T_{(n, \delta)}; f_{(m, \lambda)})$ and $(n, \delta) < (m, \lambda)$ implies $e < f$. When specialized to inverse semigroups, this result yields a theorem [10, Corollary 5] equivalent to Munn’s theorem (see also [15, Lemma 2.1]). In [11, theorem and corollary], we show “simple” may be omitted. In [15, Theorem 6.1], we give a structure theorem for $\omega Y$-unipotent semigroups.

Let $S$ be a standard regular semigroup such that $T$ is an $\omega Y$-semilattice of completely simple semigroups $(T_{(n, \delta)}; (n, \delta) \in N \times Y)$. If $f_{(n, \delta)}(f_{(m, \lambda)}) \in E((T_{(n, \delta)}; f_{(m, \lambda)})$ if and only if $\delta = \lambda$, we term $S$ a standard regular semigroup of type $\omega Y$.

In §4, we specialize Theorem 3.14 to obtain the structure of standard regular semigroups of type $\omega Y$ (Theorem 4.2). In Theorem 4.2, the factor terms “$g(c, d)$” are omitted and $V$ is an $\omega Y$-inverse semigroup with each subgroup a single element. Hence, an explicit multiplication for $V$ is given by [15, Theorem 2.3]. Further specialization yields the structure of simple and bisimple standard regular semigroups of type $\omega$ ($T$ is an $\omega$-chain of completely simple semigroups—no condition of the $\mathbb{D}$-classes).

In §5, we describe the structure of standard orthodox, standard $\mathbb{L}$-unipotent, and standard inverse semigroups (Theorems 5.1, 5.3, and 5.5 respectively). A standard regular semigroup is termed standard orthodox ($\mathbb{L}$-unipotent)(inverse) if $T$ is a semilattice of rectangular groups (right groups)(groups). The structure theorems are obtained by specializing Theorem 3.14. In each of the theorems the term “$B$”, is omitted. In Theorem 5.3 and 5.5, $I_y = \{e_y\}$ for each $y \in Y$. In Theorem 5.5, $J_y = H_y$ for each $y \in Y$.

Warne [9, page 206, paragraph 3] and Munn [5, page 66, paragraph 3] have exhibited inverse semigroups with identity on which $\mathcal{H}$, Green’s relation, is not a congruence. Using Lemma 2.13, these semigroups are not standard.

Let $S$ be a standard regular semigroup. If $a \in S$, let $\mathcal{J}(a)$ denote the collection of inverses of $a$. Let $t = \{(a, b) \in S^2; aa', bb' \in E(T_a)$ and $a'a, b'b \in E(T_b)$ for some $a' \in \mathcal{J}(a), b' \in \mathcal{J}(b),$ and $y, z \in Y\}.$

In §5, we show $t$ is a congruence on $S, S/t \cong V,$ ker $t = T,$ and $S$ is a regular extension of $T$ by $S/t$ in the sense of Yamada [19].

We use the definitions of Clifford and Preston [1] unless otherwise specified. In particular, $R, L, \mathcal{H},$ and $\mathcal{D}$ will denote Green’s relations on a semigroup $S$, i.e., $(a, b) \in R$ if $a \cup aS = b \cup bS; (a, b) \in L$ if $a \cup Sa = b \cup Sb; \mathcal{H} = R \cap L; \mathcal{D} = R \circ L (a, b) \in \mathcal{D}$ if there
exists \( x \in S \) such that \((a, x)\in R_a \) and \((x, b)\in L_b \)). \( R_a \) will denote the \( R \)-class containing \( a \in S \). A semigroup consisting of a single \( R \)-class is termed a bisimple semigroup. A semigroup \( S \) which is a union of a collection of pairwise disjoint subsemigroups \((S_y: y \in Y)\) where \( Y \) is a semilattice and \( S_y S_z \subseteq S_y \Lambda S_z \) for all \( y, z \in Y \) is termed a semilattice \( Y \) of the semigroups \((S_y: y \in Y)\). If \( Y = N \) with \( n \Lambda m = \max(n, m) \), \( S \) is termed an \( \omega \)-chain of the semigroups \((S_n: n \in N)\). A semigroup \( S \) is termed regular if \( a \in aSa \) for all \( a \in S \). If \( S \) is a regular semigroup, for each \( a \in S \), there exists \( \# \in S \) such that \( a\#a = a \) and \( \#a\# = a \) (for example, if \( a = axa \), let \( y = xax \) [1, Lemma 1.14]). The element \( y \) is termed an inverse of \( a \). A regular semigroup \( S \) is termed an inverse semigroup if each \( a \in S \) has precisely one inverse. A rectangular band is the algebraic direct product of a left zero semigroup \( U \) (\( x, y \in U \) implies \( xy = x \)) and a right zero semigroup. A rectangular group is the algebraic direct product of a group and a rectangular band. A right group is a semigroup \( X \) such that \( a, b \in X \) implies there exists a unique \( x \in X \) such that \( ax = b \).

### 1. Locally inverse Cliffordian semigroups.

In this section, we give a characterization of locally inverse Cliffordian semigroups (Theorem 1.6) and related results to be used in the sequel.

In the remainder of this section, \( S \) will denote a locally inverse Cliffordian semigroup, i.e. \( S \) is a locally inverse semilattice \( Y \) of completely simple semigroups \((S_y: y \in Y)\).

**Lemma 1.1.** If \( E \in E(S_y) \) and \( y \geq z \), there exists precisely one \( e \in E(S_z) \) such that \( E \geq e \). Furthermore, \( S_z L_e \subseteq L_z \) and \( R_z S_z \subseteq R_z \).

**Proof.** If \( y = z \), take \( e = E \). Suppose \( y > z \). Using the proof of [7, Theorem], there exists \( e \in E(S_z) \) such that \( E > e \). Let \( g, h \in E(S_y) \) with \( g \leq E \) and \( h \leq E \). Hence, since \( S \) is locally inverse, \( gh = hg \). Thus, \((hg)(hg) = hhgg = hg \), and, hence, \( hg \in E(S_y) \). Furthermore, \( g(hg) = hg = (hg)g \). Thus, \( hg \leq g \). Thus, since \( S_z \) is completely simple, \( hg = g \). Similarly, \( hg = h \) and, thus, \( g = h \). The proof of the second sentence of the lemma is contained in the proof of [7, Theorem] for \( y > z \). If \( y = z \), apply the Rees theorem [1,
Theorem 3.5].

Let \( A \in S_y \) and suppose that \( A \in R_E \cap L_F \) where \( E, F \in E(S_y) \). Let \( e \) and \( f \) denote the unique idempotents of \( S_y(y \geq z) \) under \( E \) and \( F \) respectively. We define \( A \zeta_{y,z} = eAf \). It is shown in the proof of [7, Theorem] that \( \zeta_{y,z} \) is well defined (i.e. \( \zeta_{y,z} \) does not depend on the selection of \( E \) and \( F \)).

**Lemma 1.2.** For \( y > z, \zeta_{y,z} \) is a homomorphism of \( S_y \) into \( S_z \). Let \( A \in S_y \) and \( B \in S_z \). If \( y > z, AB = A \zeta_{y,z}B \). If \( z > y, AB = A (B \zeta_{z,y} ) \).

**Proof.** The proof of Lemma 1.2 is contained in the proof of [7, Theorem].

**Lemma 1.3.** If \( y \in Y, \zeta_{y,y} \) is the identity mapping of \( S_y \).

**Proof.** Let \( A \in S_y \). Hence, \( A \in R_E \cap L_F \) for some \( E, F \in E(S_y) \). Let \( e \) and \( f \) denote the unique idempotents of \( S_y \) under \( E \) and \( F \) respectively. Hence, since \( S_y \) is completely simple, \( e = E \) and \( f = F \), thus, \( A \zeta_{y,y} = eAf = EAF = A \).

Let \( yz = y \wedge z \) in the semilattice \( Y \).

**Lemma 1.4.** If \( A \in S_y \) and \( B \in S_z \), \( AB = A \zeta_{y,z} B \zeta_{z,y} \).

**Proof.** Let \( A \in L_F \) and \( F \in E(S_y) \). Thus, utilizing Lemma 1.2 or 1.3, \( AB = A(FB) = A \zeta_{y,z} (FB) = A \zeta_{y,z} z F \zeta_{y,z} y B = (AF) \zeta_{y,z} y B = A \zeta_{y,z} z B = A \zeta_{z,y} z B \zeta_{z,y} y \).

**Lemma 1.5.** For \( x \geq y \geq z, \zeta_{z,y} \zeta_{y,z} = \zeta_{z,z} \).

**Proof.** Let \( A \in S_z \) and suppose that \( A \in R_E \subseteq L_F \) for some \( E, F \in E(S_y) \). Let \( e \) and \( f \) denote the idempotents of \( S_y \) under \( E \) and \( F \) respectively. Hence, \( A \zeta_{z,y} = eAf \). By Lemma 1.1, \( eA \in S_y L_F \subseteq L_f \) and \( Af \in R_E S_y \subseteq R_y \). Let \( e' \) and \( f' \) denote the unique idempotents of \( S_z \) under \( e \) and \( f \) respectively. Hence, \( A \zeta_{z,y} \zeta_{y,z} = e' eAf f' = e'A f' \). However, \( E \geq e \geq e' \) and \( F \geq f \geq f' \). Hence, by Lemma 1.1, \( A \zeta_{z,z} = e'A f' = A \zeta_{z,y} \zeta_{y,z} \).

**Theorem 1.6.** Let \( \{ S_y : y \in Y \} \) be a collection of pairwise disjoint completely simple semigroups indexed by the semilattice \( Y \). For each \( y, z \in Y \) with \( y \geq z \), let \( \zeta_{y,z} \) be a homomorphism of \( S_y \) into \( S_z \) such that

1. \( \zeta_{y,y} \) is the identity automorphism of \( S_y \).
\[
(2) \quad \zeta_{x,y,z} = \zeta_{z,z} \text{ for } x \geq y \geq z \text{ in } Y.
\]

Let \( S = \bigcup (S_y: y \in Y) \) and define a product on \( S \) by the rule
\[
(3) \quad A \circ B = A_{xy}B_{yz}
\]
where the right hand product is taken in \( S_{yz} \). Then, \( (S, \circ) \) is a locally inverse Cliffordian semigroup.

Conversely, let \( (S, \circ) \) be a locally inverse Cliffordian semigroup. Then, \( S \) is the union of a collection of pairwise disjoint completely simple semigroups \( (S_y: y \in Y) \) indexed by a semilattice \( Y \). For each \( y, z \in Y \) with \( y \geq z \), there exists a homomorphism \( \zeta_{y,z} \) of \( S_y \) into \( S_z \) such that (1) and (2) are valid and the multiplication is given by (3).

**Proof.** The converse is a consequence of [1, Theorem 4.6] and Lemmas 1.2-1.5. Let us now establish the direct part. Let \( xy = x'y \). Let \( A \in S_x, B \in S_y, \) and \( C \in S_z \). Hence, using (3), the fact \( \zeta_{xyz} \) is a homomorphism, and (2).

\[
A \circ (B \circ C) = A \circ (B_{x'y}C_{yz}) = A_{x'y}((B_{x'y}C_{yz})_{x'y})
\]

Similarly, \( (A \circ B) \circ C = (A_{x'y}B_{x'y})C_{yz} \). Hence \( (A \circ B) \circ C = A \circ (B \circ C) \) by associativity in \( S_{xyz} \). By (1) and (3), \( S_y \) is a completely simple subsemigroup of \( S \) for all \( y \in Y \). Thus, \( S_y \) is a Cliffordian semigroup for each \( y \in Y \) by [1, Theorem 2.52]. Hence, \( S \) is Cliffordian. Clearly, \( S \) is a regular semigroup. Finally, let \( E \in E(S_x), f \in E(S_y), \) and \( g \in E(S_z) \) and suppose that \( E \geq f \) and \( E \geq g \). Hence, \( x \geq y \) and \( x \geq z \). Thus, using (3) and (1), \( f = E \circ f = E_{x'y}f = f \circ E = f(E_{x'y}) \). Hence, \( E_{x'y} \geq f(E_{x'y}) \). Thus, \( f \in E_{x'y} \) since \( S_y \) is a completely simple semigroup. Similarly, \( g \in E_{x'y} \). Hence, \( f \circ g = f_{x'y}g_{x'y} = E_{x'y}f_{x'y}g_{x'y} = E_{x'y}g_{x'y}f_{x'y} = E_{x'y}g_{x'y}f_{x'y} = E_{x'y}g_{x'y}f_{x'y} = E_{x'y}g_{x'y}f_{x'y} \). Similarly, \( g \circ f = E_{x'y}g_{x'y}f_{x'y} \). Thus, \( g \circ f = f \circ g \), and hence, \( S \) is locally inverse.

**Remark.** In Theorem 1.6, we term \( \{\zeta_{y,z}: y, z \in Y\} \) the collection of structure homomorphisms of \( S \).

**Remark 1.7.** In the statement of Theorem 1.6, we may replace "completely simple semigroup" by "left zero semigroup" and "Cliffordian semigroup" by "semilattice of left zero semigroups". Using Theorem 1.6, a band \( E \) is left normal [17] if and only if \( E \) is a locally inverse semilattice of left zero semigroups. Hence, we have obtained the Yamada-Kimura characterization of left normal bands [17, Theorem 1].

**Remark 1.8.** In the statement of Theorem 1.6, we may replace "completely simple semigroup" by "right group" and "Cliffordian
semigroup” by “semilattice of right groups”.

The following result will be used in the sequel.

**Proposition 1.9.** Let $S$ be a locally inverse Cliffordian semigroup. Then, $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$ are congruence relations on $S$.

**Proof.** Let $S$ be a locally inverse Cliffordian semigroup. We first show that $\mathcal{L}$ is a congruence relation on $S$. We will apply Theorem 1.6 and its notation. Let $(x, y) \in \mathcal{L}$ in $S_u$ for some $u \in Y$. Let $z \in S_u$. Since $\zeta_{u, uv}$ is a homomorphism of $S_u$ into $S_{uv}$, $x\zeta_{u, uv} \mathcal{L} y\zeta_{u, uv}$ in $S_{uv}$. Using the Rees theorem [1, Theorem 3.5], $\mathcal{L}$ is a congruence relation on $S_{uv}$. Hence,

$$z \circ x = z\zeta_{u, uv} \circ x \zeta_{u, uv} = z \circ y .$$

Thus, since $\mathcal{L}$ is a right congruence on any semigroup, $\mathcal{L}$ is a congruence on $S$. Similarly, $\mathcal{R}$ is a congruence relation on $S$.

2. Structure theorem for standard regular semigroups (proof of converse). In this section, we will use a sequence of twenty-one lemmas to establish the converse of our structure theorem for standard regular semigroups (Theorem 2.22).

Let $S$ be a standard regular semigroup and let $T$ denote the union of the maximal subgroups of $S$. Hence, $T$ is a semilattice $Y$ of completely simple semigroups ($T_y : y \in Y$) where $Y$ has a greatest element $y_0$. Let $\{\zeta_y : y, z \in Y\}$ denote the set of structure homomorphisms of $T$. Let $E_y = E(T_y)$. Select and fix $e_{y_0} \in E_{y_0}$. For each $y \in Y$, define $e_y = e_{y_0} \zeta_{y_0, y}$. Let $S_0 = e_{y_0} S e_{y_0}$.

**Lemma 2.1.** $E(S_0) = \{e_y : y \in Y\}$.

**Proof.** Since $e_{y_0} e_y = e_y \zeta_{y_0, y} e_y = e_y$, and, similarly, $e_y e_{y_0} = e_y$, $e_y \in E(S_0)$ for all $y \in Y$. Suppose $f \in E(S_0)$ and $f \in E_{y_0}$, say. Hence, $f \leq e_{y_0}$ and $e_y \leq e_{y_0}$ implies $f e_y = e_y f$. Thus, since $T_y$ is completely simple, $f = e_y$.

**Lemma 2.2.** $y \mapsto e_y$ defines an isomorphism of $Y$ onto $E(S_0)$.

**Proof.** Let $y, z \in Y$. Hence,

$$e_y e_z = e_y \zeta_{y, y} e_z \zeta_{y, y} = (e_{y_0} \zeta_{y_0, y} \zeta_{y, y})(e_{y_0} \zeta_{y_0, y} \zeta_{y, y}) = e_{y_z} .$$

**Lemma 2.3.** $S_0 = \bigcup (R_y \cap L_z : y, z \in Y)$.

**Proof.** Let $x \in R_y \cap L_z$ where $y, z \in Y$. Using Lemma 2.2, $e_{y_0} x =$
\[ e_v e_s x = e_s x = x \text{ and, similarly, } x e_v s = x. \] Conversely, if \( x \in S_o \), let \( x^{-1} \) denote the unique inverse of \( x \) in \( S_o \). Thus, using Lemma 2.1, \( x x^{-1} = e_v \) and \( x^{-1} x = e_s \) for some \( y, z \in Y \). Hence, \( x \in R_e \cap L_e \).

By the Rees theorem [1, Theorem 3.5], for \( y \in Y \), \( T_y = G_y \times M_y \times N_y \), where \( G_y \) is a group and \( M_y \) and \( N_y \) are sets under the multiplication \((g, i, j)(h, p, q) = (gf_s(j, i)h, i, q)\) where \((j, p) \rightarrow f_s(j, p)\) is a mapping of \( N_y \times M_y \) into \( G_y \). We note \( e_y = (f_s(j_y, i_y)^{-1}, i_y, j_y) \), say, where \( f_s(j_y, i_y)^{-1} \) is the inverse of \( f_s(j_y, i_y) \) in the group \( G_y \). Let \( I_y \) denote the set of idempotents of the \( \mathcal{L}_p \)-class of \( T_y \) containing \( e_y \) and let \( J_y \) denote the \( \mathcal{R}_p \)-class of \( T_y \) containing \( e_y \). Hence, \( I_y = \{(f_s(j_y, i_y)^{-1}, i, j_y); i \in M_y\} \) and \( J_y = \{(g, i_y, j); g \in G \text{ and } j \in N_y\} \). Let \( I = \bigcup (I_y; y \in Y) \) and \( J = \bigcup (J_y; y \in Y) \).

**Lemma 2.4.** \( I \) is a standard regular semilattice \( Y \) of left zero semigroups (\( I_y; y \in Y \)).

**Proof.** Let \( a \in I_y \) and \( b \in I_z \). Thus, \( a \mathcal{L} e_y \) and \( b \mathcal{R} e_z \). Using Proposition 1.9 and Lemma 2.2, \( ab \mathcal{L} e_y \). Furthermore, \( abab = abe_y ab = abe_y e_z b = abe_y b = abb = ab \). Thus, \( ab \in I_y \).

**Lemma 2.5.** \( J \) is a standard regular semilattice \( Y \) of right groups (\( J_y; y \in Y \)).

**Proof.** Apply Proposition 1.9 and Lemma 2.2.

The next two lemmas are special cases of left-right duals of [14, Lemmas 1.3 and 1.4]. Note our arbitrary representations of the "\( e_y \)" requires a slight modification in the proof of [14, Lemma 1.3].

**Lemma 2.6.** Every element of \( T \) may be uniquely expressed in the form \( x = ij \) where \( i \in I_y \) and \( j \in J_y \) for some \( y \in Y \).

If \( X \) is a set, \( T_X \) will denote the semigroup (iteration) of mappings of \( X \) into \( X \).

**Lemma 2.7.** There exists a mapping \( j \rightarrow A_j \) of \( J \) into \( T \) and a mapping \( p \rightarrow B_p \) of \( I \) into \( T \) such that \( I_y A_j \subseteq I_y \) for \( j \in J_y \) and \( J_y B_p \subseteq J_y \) for \( p \in I_y \). If \( j \in J \) and \( p \in I \), \( jp = pA_j jB_p \). Furthermore, \( jA_j p \mathcal{R} pA_j (eT) \) and \( jB_p \mathcal{L} jB_p (eT) \).

**Lemma 2.8.** \( iA_j = e_s \) for \( i \in I_y \) and \( j \in J_y \).

**Proof.** First, we show that \( A_j = A_s \) for \( j \in J_y \). Since \( \mathcal{R} \) is a congruence relation on \( T \), \((j, e_s) \in \mathcal{R} \) implies \((j, e_s) \in \mathcal{R} \) for all
Hence, using Lemma 2.7, \((iA_1f_iA_e) \in E^T\) for all \(i \in I\). Thus, \(iA_j = iA_{e_j}\) for \(i \in I\). Let \(i \in I\), say. Thus, since \(e_i \in I_{e_i}\), we utilize Lemma 2.7 to obtain \((e_i f_iA_e) = e_i = iA_{e_i} e_i B_i\). Thus, by Lemmas 2.7 and 2.6, \(iA_{e_i} = e_i\). Since \(iA_{e_i} \in I_{e_i}\), we utilize Lemma 2.7 to obtain \((e_i f_iA_e) = e_i = iA_{e_i} e_i B_i\). Thus, by Lemmas 2.7 and 2.6, \(iA_{e_i} = e_i\). Since \(i^1 \in I_{e_i}\), \(i^1 \in I_{e_i}\). Hence, using Theorem 1.6, \(e_i = e_i (iA_{e_i} e_i) = e_i\). Thus, \(iA_j = iA_{e_j} = e_i\).

**Definition [1, p. 10]**. A transformation \(\rho\) of a semigroup \(S\) is a right translation of \(S\) if \((\alpha \delta)|0 \leftarrow \alpha(\beta \theta)\) for all \(a, b \in S\).

**Lemma 2.9**. For each \(i \in I\), \(B_i\) is a right translation of \(I\).

**Proof**. Let \(r \in J_u\), \(s \in J_v\) and \(x \in I_p\), say. Hence, utilizing Lemmas 2.5 and 2.7, \((rs)x = xA_r (rs) = (rs)A_x\) while \(r(sx) = r(xA_s B_x) = xA_r (rB_s A_x)\). However, using Lemma 2.8, \(rB_x A_s = rB_{x^p}\). Using Lemma 2.5, \(rB_x A_s = rB_{x^p}\). Hence, \(e_{x^p} (rB_x A_s) = e_x = rB_{x^p} A_s rB_{x^p}\). Hence, using Lemmas 2.7 and 2.6, \((rs)B_x = rB_{x^p} sB_x = r(sB_x)\).

**Lemma 2.10**. \(i \rightarrow B_i\) is a homomorphism of \(I\) into \(P(J)\), the semigroup of right translations of \(J\).

**Proof**. Let \(r, s \in I\) and \(x \in J\). Thus, proceeding as in the proof of Lemma 2.9, \((xs) = (rs)A_x xB_r s\) and

\[
(xr)s = rA_x (xB_r s) = (rA_x xA_x)(xB_r s).
\]

Thus \(xB_{rs} = xB_r B_s\).

For each \(y \in Y\), let \(H_y\) denote the maximal subgroup of \(S\) containing \(e_y\).

**Lemma 2.11**. If \(i \in I_y\) and \(j \in J_y\), \(jB_i \in H_y\). If \(j \in H_y\), \(jB_i = j\).

**Proof**. Let \(i \in I_y\) and \(j \in J_y\). Since \(j \in P(e_y) \leftarrow e_y \in J_y\) and \(j \in P(e_y) \leftarrow e_y\), we utilize Lemma 2.7 to obtain \((i \in P(e_y) \leftarrow e_y = iA_{e_i} e_i B_i\). Thus, \(iA_{e_i} jB_i = e_y jB_i = jB_i\). Thus, \(jB_i \in H_y\). If \(j \in H_y\), \(jB_i = j = je_j = \epsilon_y = j\).

**Lemma 2.12**. If \(j \in J\), \(jB_{e_j} = je_j\).

**Proof**. Utilize the proof of Lemma 2.11.

**Lemma 2.13**. Let \(X\) be an inverse semigroup such that the union
of the maximal subgroups of $X$ is a subsemigroup. Then, $\mathcal{H}$ is a congruence relation on $X$.

Proof. Let $H = \bigcup (H_e : e \in E(X))$. Using [1, Theorem 1.7], $H$ is an inverse semigroup which is a union of groups. Hence, $E(H)$ is contained in the center of $H$ by [1, Lemma 4.8]. Let $(a, b) \in \mathcal{H}$. Hence, $aa^{-1} = bb^{-1}$ and $a^{-1}a = b^{-1}b$. Let $c \in S$. Thus,

$$(bc)(bc)^{-1} = bcc^{-1}b^{-1} = ba^{-1}acc^{-1}ab^{-1} = ba^{-1}(acc^{-1}a^{-1})ab^{-1}$$

while $(bc)^{-1}bc = c^{-1}b^{-1}bc = c^{-1}a^{-1}ac = (ac)^{-1}ac$. Hence, $(ac, bc) \in \mathcal{H}$. Similarly, $(ca, cb) \in \mathcal{H}$.

**Lemma 2.14.** $\mathcal{H}$ is a congruence relation on $S_0$.

Proof. Using Lemma 2.3, $\{H_y : y \in Y\}$ is the collection of maximal subgroups of $S_0$. Let $a \in H_x$ and $b \in H_x$, say. Since $a \mathcal{H} e_y$, $a \mathcal{H} e_y$ and, thus, $a \mathcal{H} e_y \in H_y$. Similarly, $b \mathcal{H} e_y \in H_y$. Hence, using Lemma 2.13, $\mathcal{H}$ is a congruence relation on $S_0$.

**Lemma 2.15.** There exists a homomorphism $\phi$ of $S_0$ onto an inverse semigroup $V$ where $E(V) = Y$ and each $\mathcal{H}$-class of $V$ consists of a single element. Furthermore, $(a, b) \in \mathcal{H}(S_0)$ if and only if $a\phi = b\phi$. Thus, if $h_c = c\phi^{-1}$, $\{h_c : c \in V\}$ is the collection of $\mathcal{H}$-classes of $S_0$.

Proof. Using Theorem 2.14 and [1, Theorem 7.36], $S_{0/\mathcal{H}}$ is an inverse semigroup. Let $a \rightarrow \bar{a}$ denote the natural homomorphism of $S_0$ onto $S_{0/\mathcal{H}}$. Suppose $(\bar{a}, \bar{b}) \in \mathcal{H}(\varepsilon S_0)$. Hence, $\bar{a}\bar{a}^{-1} = \bar{b}\bar{b}^{-1}$ and $\bar{a}^{-1}\bar{a} = \bar{b}^{-1}\bar{b}$. Thus, $\bar{a}\bar{a}^{-1} = \bar{b}\bar{b}^{-1}$ and $\bar{a}^{-1}\bar{a} = \bar{b}^{-1}\bar{b}$. Hence, $(a\bar{a}^{-1}, b\bar{b}^{-1}) \in \mathcal{H}(\varepsilon S_0)$ and $(a^{-1}a, b^{-1}b) \in \mathcal{H}(\varepsilon S_0)$. Thus, $aa^{-1} = bb^{-1}$ and $a^{-1}a = b^{-1}b$. Hence, $(a, b) \in \mathcal{H}$, and, thus, $\bar{a} = \bar{b}$. Thus, each $\mathcal{H}$-class of $S_{0/\mathcal{H}}$ consists of a single element. Using Lemma 2.2 and [1, Lemma 7.34], $\varepsilon_y \rightarrow y$ defines an isomorphism of $E(S_{0/\mathcal{H}})$ onto the semilattice $Y$. Hence, we may extend this isomorphism to an isomorphism $\lambda$ of $S_{0/\mathcal{H}}$ onto a semigroup $V$ with $E(V) = Y$. For $a \in S_0$, define $a\phi = \bar{a}\lambda$.

For each $c \in V$, select a representative element $v_c \in h_c$. For $y \in Y$, let $v_y = e_y$. Hence, using Lemma 2.15 and its proof, $v_cv_c^{-1} = e_{c^{-1}}$ and $v_cv_c = e_{c^{-1}}$ for $c \in V$.

**Lemma 2.16.** Every element of $S$ may be uniquely expressed in the form $iv_j$ where $i \in I_{\alpha^{-1}}$ and $j \in J_{\beta^{-1}}$. 


Proof. Let \( x \in S \). Hence, \( x \in R_0 \cap L_f \) where \( e \in E_y \) and \( f \in E_z \) for some \( y, z \in Y \). Thus, \( (e, i) \in R \) for some \( i \in I_y \) and hence \( (x, i) \in R \). We note that \( e_y \in E_y \) for some \( y \in Y \). Hence, \( R_y \cap L_e \neq \emptyset \). Thus, using Lemmas 2.3 and 2.15, \( R_y \cap L_e = h_z \) for some \( c \in V \). Thus, \( v_xv_e = e_y \) and \( v_xv_e = e_z \). Hence, \( x = ix = ie_xx = iv_xv_e(xv_e)'x \). We will show that \( xv_e'x \in T_{e-1} \). Using the proof of [1, Theorem 2.18], there exists an inverse \( x' \) of \( x \) such that \( x' \in R_f \cap L_s \), \( xx' = e \), and \( x'x = f \). Let \( xx' = (f_y(s, r)^{-1}, r, s) \) for some \( r \in M_y \) and \( s \in N_y \) and let \( z = f_s(j_y, r)^{-1} \). Using the Rees theorem, \( e_yx = e_y \). Hence, \( e_z = v_xv_e = v_xv_e = v_xv_e = v_xv_e \). Thus, \( v_xv_e'x \in e_xS \). Again, using the Rees theorem, \( ze_yx = xx' \). Hence, \( x = xe_yx = zv_xv_e'x \). Thus, \( f = x'x = x'zv_xv_e'x \in S v_e'x \). However, \( v_xv_e'x \in S_x = Sf \). Hence, \( v_xv_e'x \in R_y \cap L_f \subseteq T_y \). Since \( e_yv_xv_e'x = v_xv_e'x, v_xv_e'x \in J_z \). Since \( e_z = v_xv_e, z = e_z = (e_yv_xv_e')^{-1}v_xv_e = e_z \). Thus, \( v_xv_e'x \in J_z \). Hence, \( x = iv_xj \) where \( i \in I_{e-1} \) and \( j \in J_{e-1} \). Suppose \( iv_xj = rv_yd \) where \( r \in I_{e-1} \) and \( s \in J_{e-1} \). Since \( j_{k-e-1}r \), \( j_{k-e-1}r \) and \( v_yd \). Hence, \( c'c = c \). Furthermore, \( v_yd \in H_y \). Therefore, \( e_x = iv_xj = e_x = e_x = e_x \). Hence, \( v_xj = v_yd \). Thus, \( j = e_xe_x = e_xe_x = e_xe_x = e_xe_x = s \).

**Lemma 2.17.** If \( c, d \in V \), \( v_xv_y = v_xg(c, d) \) where \( g \) is a function of \( V \times V \) into \( H = \bigcup \{H_y: y \in Y\} \) such that \( g(c, d) \in H_{(cd)^{-1}cd} \). If \( y, z \in Y \), \( g(y, z) = e_y \).

**Proof.** Using Lemma 2.16, \( v_xv_y = iv_xj \) where \( i \in I_y \) and \( j \in J_{e-1} \). We first show \( x = cd \). By the proof of Lemma 2.16, \( iv_xj \in R_y \cap L_y \). However, \( (v_xv_y(v_xv_y)^{-1})^2 = (cd)(cd)^{-1} = e_{(cd)^{-1}cd} \). Hence, using Lemma 2.16, \( v_xv_yv_y = e_{(cd)^{-1}cd} \). Similarly, \( (v_xv_y)^{-1}v_xv_y = e_{(cd)^{-1}cd} \). Thus, \( xx = (cd)(cd)^{-1} \) and \( x^{-1}x = (cd)^{-1}cd \). Thus, \( (x, c) \in H(c) \). Hence, using Lemma 2.16, \( x = cd \). Thus, \( v_xv_y = iv_xj \). Let \( j = g(c, d) \). Hence, using Lemma 2.16, \( g \) is a function of \( V \times V \) into \( J \) and \( g(c, d) \in J_{(cd)^{-1}cd} \). Furthermore, \( v_xv_y = e_{(cd)^{-1}cd}v_yv_x = e_{(cd)^{-1}cd}iv_xg(c, d) = v_yd \). We note that \( g(c, d) = e_{(cd)^{-1}cd}g(c, d) = v_yd \). Thus, using Lemma 2.3, \( g(c, d) \in H_{(cd)^{-1}cd} \). To obtain the last statement of the theorem, utilize Lemma 2.2

If \( u \in J \) and \( s \in V \), define \( u_{\beta_s} = v_{(e_s')^{-1}u} \).

**Lemma 2.18.** For \( s \in V \), \( \beta_s \in \text{End} J \) and \( J_{r_s} \subseteq H_{e_s-1} \). Furthermore, if \( j \in J \) and \( s \in Y \), \( j_{\beta_s} = je_s \).

**Proof.** Let \( j, j_s \in J \). Suppose \( j \in J_s \) and \( j_s \in J_r \). Hence, using Lemma 2.5, \( j_{\beta_s}j_{\beta_s} = (v_x^{-1}j_{\beta_s}v_y)(v_x^{-1}j_{\beta_s}v_y) = v_x^{-1}j_{\beta_s}e_{s}^{-1}(j_{\beta_s}e_{s}^{-1})v_y = v_x^{-1}j_{\beta_s}v_y = (j, j_{\beta_s}j_{\beta_s} \beta_s \). Hence, \( \beta_s \) is a homomorphism of \( J \) into \( S \). Since is a \( J \)
union of groups, $\beta_s$ is a homomorphism of $J$ into $T$. Since $e_r\beta_s = v_r^{-1}e_rv_s = e_{s^{-1}r}$, $\beta_s$ implies $j\beta_s, \beta_s^{-1}$. Hence, $J, \beta_s \subseteq J_{s^{-1}r'}$. Thus, $\beta_s \in \text{End } J$. Let $j \in J_r$. Thus, $v_s^{-1}jv_s = v_s^{-1}jv_s, e_{s^{-1}r} = v_s^{-1}jv_s, e_{s^{-1}r} = v_s^{-1}jv_s$. Thus $J, \beta_s \subseteq J_{s^{-1}r}$. If $s \in Y$ and $j \in J_r$, $j\beta_s = e, \beta_s = j\beta_s$.

If $j \in H$ and $z \in H$, define $j\beta_z = z^{-1}jz$. (Using Proposition 1.9 and Lemma 2.2, $H$ is a semilattice $Y$ of the groups $(H, y \in Y)$.)

**Lemma 2.19.** $\beta_c \beta_d = \beta_c \beta_{c'd'}$.

**Proof.** Utilizing Lemma 2.17, $j\beta_s \beta_d = v_s^{-1}v_s^{-1}jv_s = (v_s^{-1}jv_s)^{-1} = (v_s^{-1}jv_s)^{-1}$.

**Lemma 2.20.** $g(a, bc)g(b, c) = g(ab, c)(g(a, b)\beta_c)$.

**Proof.** Using Lemmas 2.17 and 2.5,

\[
(v_s v_c)v_c = v_s g(a, b)v_c = v_s g(a, b) e_{c^{-1}} v_c = v_s e_{c^{-1}} g(a, b)v_c = v_{a b} e_{c^{-1}} g(a, b)v_c = g(a, b)\beta_c.
\]

Using Lemmas 2.17 and 2.18, $g(ab, c)(g(a, b)\beta_c) \in J_{(abc)^{-1}abc}$. However,

\[
v_a v_s v_c = g(ab, c)(g(a, b)\beta_c).
\]

We note that $g(a, bc)g(b, c) \in J_{(abc)^{-1}abc}$. Hence, using Lemma 2.16, $g(ab, bc)g(b, c) = g(ab, c)(g(a, b)\beta_c)$.

If $a, b \in J$, define $a^*b = ab$. If $a, b \in I$, define $a \circ b = ab$.

**Lemma 2.21.** $S \equiv \{(i, a, j) : a \in V, i \in I_{\text{sa}^{-1}}, \text{ and } j \in J_{\text{sa}^{-1}}\}$ under the multiplication $(i, a, j)(u, b, z) = (i \circ e_{(ab)(ab)^{-1}}) (ab, g(a, b)^* jB_u \beta_z)$.

**Proof.** Let $i \in I_{\text{sa}^{-1}}, j \in J_{\text{sa}^{-1}}, u \in I_{\text{ab}^{-1}},$ and $z \in J_{\text{sa}^{-1}}$. Using Lemmas 2.7, 2.8, and 2.2,

\[
(i v_s j)(u v_z) = i v_s u A_j jB_v v_z
\]

Using Lemmas 2.4, 2.5, 2.7, 2.17 and 2.18, $i \circ e_{(ab)(ab)^{-1}} \in I_{(ab)(ab)^{-1}}$ and $g(a, b)^* jB_u \beta_z \in J_{(ab)^{-1}ab}$. Hence, using Lemma 2.16, $(i v_s j) = (i, a, j)$ defines an isomorphism of $S$ onto the groupoid given in the statement.
of the lemma.

**Theorem 2.22.** Let $S$ be a standard regular semigroup. Then, $S$ is isomorphic to $(Y, I, J, V, B, \beta, g)$ for some $Y, I, J, V, B, \beta, g$.

**Proof.** Utilize the remark before the proof of Lemma 2.21 and Lemmas 2.2, 2.4, 2.5, 2.10–2.12, 2.15, and 2.17–2.21.

**Remark 2.23.** A semilattice $Y$ is said to be directed from above if $y, z \in Y$ implies there exists $w \in Y$ such that $w \geq y$ and $w \geq z$. Theorem 2.22 is valid if we replace the condition "$Y$ has a greatest element" by "$Y$ is directed from above" and "for each $y \in Y$, there exists $e_y \in E_y$ such that $e_y e_z = e_z$ for all $y, z \in Y$". Just replace $S$ by the inverse semigroup $\bigcup (e_y Se_y; y \in Y) = \bigcup (R_{e_y} \cap L_{e_z}; y, z \in Y)$ and note that for $z \leq y$, $e_{yv_yz} = e_z$.

3. Structure theorem for standard regular semigroups (proof of direct half). In this section, we show that $(Y, I, J, V, B, \beta, g)$ is a standard regular semigroup and establish other results to be used later in the sequel.

For brevity, let $S = (Y, I, J, V, B, \beta, g)$.

**Lemma 3.1.** $S$ is a semigroup.

**Proof.** Utilizing (4) and (1), closure is easily established. Let $(i, a, j)_s = i$ and $(i, a, j)_s = j$. Let $x = (i, a, j), y = (u, b, z)$, and $w = (p, c, q)$. Using the fact $e_r e_s = e_s$ for all $r, s \in Y$ and (4), $((xy)w)_z = (x(yw))_z$. Utilizing (4), the facts $\beta_s \in \text{End}(J, *)$ and $i \rightarrow B_i$ is a homomorphism of $(I, *)$ into $P(J)$, 3(b), 3(a), 1(c), the fact $e_r e_s = e_s$, 1(b), and 2(a),

$$((xy)w)_z = g(ab, c)^*(g(a, b)^* jB_u \beta_s z)B_x \beta_s^* q$$

$$= g(ab, c)^*(g(a, b)\beta_s)^*(jB_u \beta_s^* z)B_x \beta_s^* q$$

$$= g(ab, c)^*(g(a, b)\beta_s)^* jB_u \beta_s^* zB_x \beta_s^* q$$

$$= g(ab, c)^*(g(b, c))^* g(b, c)^* zB_x \beta_s^* q$$

$$= g(ab, c)^* e_{(bc)^{-1}}^* jB_u \beta_s^* g(b, c)^* zB_x \beta_s^* q$$

$$= g(ab, c)^* jB_u \beta_s^* g(b, c)^* zB_x \beta_s^* q$$

$$= g(ab, c)^* jB_u \beta_s^* e_{(bc)^{-1}}^* \beta_s^* zB_x \beta_s^* q$$

$$= g(ab, c)^* jB_u \beta_s^* g(b, c)^* zB_x \beta_s^* q$$

$$= g(ab, c)^* jB_u \beta_s^* g(b, c)^* zB_x \beta_s^* q$$

$$= (x(yw))_z.$$
Hence, \((xy)w = x(yw)\).

**Lemma 3.2.** \(E(S) = \{(i, a, j): a \in Y \text{ and } jB_i = e_a\}\).

**Proof.** Let \((i, a, j) \in E(S)\). Using 4, 2(a), 1(a), and 2(c), \(a \in Y\) and \(jB_i^*j = j\). Since \(J_a\) is a right group, there exists \(y \in J_a\) such that \(j^*y = e_a\). Hence, using 1(a), \(e_a = j^*y = jB_i^*j = jB_i e_a = jB_i\).

Conversely, if \(a \in Y\) and \(jB_i = e_a\), using (4), 2(c), and 2(a), \((i, a, j) \in E(S)\).

**Lemma 3.3.** Let \((i, a, j), (u, b, v) \in S\). Then, \((i, a, j) \not\sim (u, b, v)\) if and only if \(i = u\).

**Proof.** First suppose that \(i = u\). Hence, \(aa^{-1} = bb^{-1}\). Let \(x \in I_{(a^{-1}b)(a^{-1}b)^{-1}}\). Thus, using (1) and the fact \(J_{a^{-1}b}\) is a right group, there exists \(y \in J_{a^{-1}b}\) such that \((g(a, a^{-1}b)*jB_aB_{a^{-1}b})y = v\). Thus, using (4), \((i, a, j)(x, a^{-1}b, y) = (i, b, v)\). Similarly, there exists \(p \in I_{(b^{-1}a)(b^{-1}a)^{-1}}\) and \(q \in J_{b^{-1}a}\) such that \((i, b, v)(p, b^{-1}a, q) = (i, a, j)\). Thus, \((i, a, j) \not\sim (i, b, v)\). Conversely, suppose that \((i, a, j) \not\sim (u, b, v)\). Using Thus, (4), \(i \circ u = u\) and \(u \circ i = i\). Hence, \(i = u\).

**Lemma 3.4.** \(S\) is a regular semigroup.

**Proof.** Let \((i, a, j) \in S\). Using Lemma 3.3, \((i, aa^{-1}, e_{a^{-1}a}) \not\sim (i, a, j)\). By 2(b), \(e_{a^{-1}a}B_i = e_{a^{-1}a}\). Hence, using Lemma 3.2, \((i, a, j) \not\sim (i, a, i)\). Thus each \(\mathcal{H}\)-class of \(S\) contains an idempotent.

**Lemma 3.5.** Let \((i, a, j), (w, b, z) \in S\). Then, \((i, a, j) \not\sim (w, b, z)\) if and only if \(i = w, a = b, (j, z) \in \mathcal{H}(eJ_{a^{-1}b})\).

**Proof.** We first show that \((i, a, j) \not\sim (w, b, z)\) if \(a^{-1}a = b^{-1}b\) and \((j, z) \in \mathcal{H}(eJ_{a^{-1}b})\). Since \((j, z) \in \mathcal{H}(eJ_{a^{-1}b})\), there exists \(y \in H_{a^{-1}b}\) such that \(y^*j = z\). Since \((a^{-1}, a) \in H_{a^{-1}b}\) by 1(c), there exists \(x \in H_{a^{-1}a}\) such that \(x^*g(a^{-1}, a)^*j = z\). Using 1(b), 2(b), 3(a), 2(a), and 1(c), \(g(a^{-1}, a)*x\beta_a, B_a\beta_a^*j = x^*g(a^{-1}, a)^*j = z\). Thus, \((e_{a^{-1}a}, a^{-1}, x\beta_a, j) = (e_{a^{-1}a}, a^{-1}, a, z)\). However, \((w, b, (g(b, b^{-1}b))^{-1})(e_{a^{-1}b}, b^{-1}b, z) = (w, b, z)\). Hence, \(((w, b, (g(b, b^{-1}b))^{-1})(e_{a^{-1}b}, a^{-1}, x\beta_a, j))(i, a, j) = (w, b, z)\). Similarly, there exists \(y \in H_{a^{-1}b}\) such that \(((i, a, (g(a, a^{-1}a))^{-1})(e_{a^{-1}b}, b^{-1}b, p\beta_{a^{-1}b}))(w, b, z) = (i, a, j)\). Hence, \((i, a, j) \not\sim (w, b, z)\). Conversely, suppose that \((i, a, j) \not\sim (w, b, z)\). Using (4) and (1), \(a^{-1}a = b^{-1}b\) and \((j, z) \in \mathcal{H}(eJ_{a^{-1}b})\).

**Lemma 3.6.** Let \((i, a, j), (w, b, z) \in S\). Then, \((i, a, j) \not\sim (w, b, z)\) if and only if \(i = w, a = b, (j, z) \in \mathcal{H}(eJ_{a^{-1}b})\).
Proof. Just note that each $g$-class of $V$ consists of a single element, and combine Lemmas 3.3 and 3.5.

Remark. Lemmas 3.3 and 3.5 and their proofs were suggested by [16, Lemma 3.2] and its proof.

Lemma 3.7. Let $(i, a, j), (w, b, z) \in S$. Then, $(i, a, j)D(w, b, z)$ if and only if $a Db(\varepsilon V)$.

Proof. Suppose that $a Db(\varepsilon V)$. Hence, there exists $x \in V$ such that $aRx$ and $xDb$. Hence, $aa^{-1} = xx^{-1}$ and $b^{-1}b = x^{-1}x$. Thus, using Lemmas 3.3 and 3.5, $(i, a, j)R(i, x, z)L(w, b, z)$. Conversely, if $(i, a, j)D(w, b, z)$, using (4), $a Db(\varepsilon V)$.

Lemma 3.8. $S$ is a bisimple semigroup if and only if $V$ is a bisimple semigroup.

Proof. Apply Lemma 3.7.

Lemma 3.9 For each $y \in Y$, let $T_y = \{(i, y, j): i \in I_y$ and $j \in J_y\}$. Then, $T_y$ is a completely simple semigroup.

Proof. First, we show that $T_y$ is a simple semigroup. Let $(i, y, j), (u, y, v) \in T_y$. Hence, using 1(a), 2(a), and 2(c), $(i, y, j)(u, y, v) = (i, jB^*v) \in T_y$. Since $J_y$ is a right group, there exists $x \in J_y$ such that $j^*x = v$. Hence, using 2(b) and 2(a),

$$(u, y, e_y)(i, y, j)(e_y, y, x) = (u, y, v).$$

Next, we show $T_y$ is completely simple. Let $e, f \in E(T_y)$ and suppose that $e \leq f$. Hence, using Lemma 3.2, $e = (i, y, j)$ and $f = (w, y, z)$, say, where $jB_i = zB_w = e_y$. Thus, $(w, y, z)(i, y, j) = (i, y, j)$ implies $w = i$. Hence, $(i, y, j)(i, y, z) = (i, y, j)$. Thus, $z = j$ and hence, $(w, y, z) = (i, y, j)$.

Lemma 3.10. Let $T = \bigcup(T_y: y \in Y)$. Then, $T$ is a semilattice $Y$ of the completely simple semigroup $(T_y: y \in Y)$.

Proof. Apply (4), 2(c), 1(a), and 2(a).

Lemma 3.11. $T$ is the union of the maximal subgroups of $S$.

Proof. If $x \in T_y$, $x$ is contained in some subgroup of $S$ (each completely simple semigroup is a union of its subgroups by [1, Theorem 2.52]). Thus, $x \in H_e$ for some $e \in E(S)$. Hence, $T \subseteq X$, the
union of the maximal subgroups of $S$. If $c \in X$, $c \not\in e$ for some $e \in E(S)$. Hence, $e = (i, y, j)$ for some $y \in Y$ by Lemma 3.2. Thus, using Lemma 3.6, $c \in T_y$.

**Lemma 3.12.** $T$ is a locally inverse semigroup.

*Proof.* Let $(i, a, j), (u, b, v), (w, c, z) \in E(S)$ such that $(i, a, j) \geq (u, b, v)$ and $(i, a, j) \geq (w, c, z)$. Hence, $(i, a, j)(u, b, v) = (u, b, v)(i, a, j) = (u, b, v)$. Using Lemma 3.2, $a, b \in Y$, and hence, $a \geq b$. Since $i \circ e_b = u$, $i \circ u = i \circ (i \circ e_b) = i \circ e_b = u$ while $u \circ i = u \circ (u \circ i) = u$. Hence, $i \geq u$. Using (4), 2(c), 2(a) and 1(a), $jB_s^*v = vB_s^*j = v$. Using, the fact $jB_s \in H$, and $v \in J$, a right group, $jB_s = e_b$. Hence, using Lemma 3.2, $e_b = vB_s = (vB_s^*j)B_s = vB_s^*jB_s = vB_s^*e_b = vB_s$. Thus, $e_s^*j = v$. Let $\{\zeta_{p,q} : p, q \in Y\}$ denote the set of structure homomorphisms of $(J, \cdot)$.

Thus, $e_s^*j_{\zeta_{a,b}} = v$. Hence, $j_{\zeta_{a,b}}^* = v$. We have shown that $(i, a, j) \geq (u, b, v)$ implies $i \geq u, a \geq b$, and $v = j_{\zeta_{a,b}}^*$. Similarly, $(i, a, j) \geq (w, c, z)$ implies $i \geq w, a \geq c$, and $z = j_{\zeta_{a,c}}^*$. Since $(I, \cdot)$ is a locally inverse semigroup, $u \circ w = w \circ u$. Using 2(c), 2(a), and 1(a), $(u, b, v)(w, c, z) = (u \circ e_b, bc, vB_s^*z)$ while $(w, c, z)(u, b, v) = (w \circ e_b, bc, zB_s^*v)$. Let $\{\phi_{p,q} : p, q \in Y\}$ denote the set of structure homomorphisms of $(I, \circ)$. Hence, $w \circ u = w \circ e_b \circ e_b^* \circ \phi_{b, e_b} \circ \phi_{b, e_b} = w \circ e_b$. Similarly, $u \circ w = u \circ e_b$. Hence, $u \circ e_b = w \circ e_b$. Furthermore,

\[
vB_s^*z = vB_s^*e_b^*z = vB_sB_s^*e_b^*z = vB_sB_s^*e_b^*e_b^*z = vB_s^*e_b^*z = e_b^*z = \zeta_{e, e_b}^*z.
\]

Similarly, $zB_s^*v = v_{\zeta_{b, e_b}}^*$. Hence, $vB_s^*z = j_{\zeta_{a,c}}^* \zeta_{e, e_b}^* = j_{\zeta_{a,b}}^* \zeta_{e, e_b}^* = v_{\zeta_{b, e_b}}^* = zB_s^*v$. Thus, $(u, b, v)(w, c, z) = (w, c, z)(u, b, v)$.

**Theorem 3.13.** $(Y, I, J, V, B, \beta, g)$ is a standard regular semigroup.

*Proof.* Utilize Lemmas 3.1, 3.4, and 3.10–3.12.

**Theorem 3.14.** $(Y, I, J, V, B, \beta, g)$ is a standard regular semigroup and, conversely, every standard regular semigroup is isomorphic to some $(Y, I, J, V, B, \beta, g)$.

*Proof.* Combine Theorems 2.22 and 3.13.

**Remark 3.15.** Let $J$ and $H$ be as in the statement of Theorem 3.14. Using the proof of [1, Theorem 1.27], Theorem 1.6, and Pro-
position 1.9, \( J \cong \bigcup (H_y \times E(J_y) : y \in Y) \) where, if \( a \in H_y, c \in H_z, b \in E(J_y), \) and \( d \in E(J_z), (a, b)(c, d) = (ac, bd) \in H_y \times E(J_y). \) The multiplications in \( H \) and \( E(J) \) are given by the corresponding specializations of Theorem 1.6. These specializations yield theorems of Clifford [1, Theorem 4.11] and Yamada and Kimura [17, Theorem 1] respectively.

REMARK 3.16. Let \( e_y = (e_y, y, e_y). \) Hence, using (4), 2(c), 2(a), \( e_y e_z = e_y. \) Thus, if we replace "\( Y \) has a greatest element" by "\( Y \) is directed from above" in the definition of \((Y, I, J, V, B, \beta, g)\), we obtain the semigroup of Remark 2.23. Hence, Theorem 3.14 with appropriate modifications characterizes these semigroups.

4. Standard regular semigroups of type \( \omega Y. \) Let \( S \) be a regular semigroup such that \( T \) is a locally inverse \( \omega Y \)-semilattice \( A \) of completely simple semigroups \( \{T_n : (n, \delta) \in A\}. \) If \( f_{(n, \delta)} \mathcal{D} f_{(m, \lambda)} (f_{(n, \delta)} \in E_{(n, \delta)} f_{(m, \lambda)} \in E_{(m, \lambda)}) \) if and only if \( \delta = \lambda, \) we term \( S \) a standard regular semigroup of type \( \omega Y. \) If \( \delta_0 \) is the greatest element of \( Y, (\epsilon, \delta) \) is the greatest element of \( A. \) We give a characterization of standard regular semigroups of type \( \omega Y \) (Theorem 4.2). A regular semigroup \( S \) such that \( T \) is a locally inverse \( \omega \)-chain of completely simple semigroups \( (T_n : n \in N) \) (no further condition) is termed a standard regular semigroup of type \( \omega. \) We show that \( S \) is a simple (bisimple) standard regular semigroup of type \( \omega \) if and only if \( S \) is a standard regular semigroup of type \( \omega Y \) with \( Y \) a finite chain (a single element) (Theorem 4.3) (Theorem 4.4). Hence, the structure of these semigroups is given by specializing Theorem 4.2.

To establish Theorem 4.2, we use a more general result on "split" extensions (Theorem 4.1).

Let \( S \) be a standard regular semigroup. In the notation of §2, let \( \{h_x : c \in V\} \) denote the collection of \( \mathcal{H} \)-classes of \( S. \) For each \( c \in V, \) select \( v_c \in h_c. \) If \( v_cv_d = v_{cd} \) for all \( c, d \in V, \) we term \( S \) a split extension of \( T \) by \( V. \)

Let \( Y, V, I, J, H, \) and \( \{\alpha_i\} \) be as in the definition of \((I, J, V, B, \beta, g). \) Let \( i \mapsto B_i \) be a homomorphism of \((I, \circ) \) into \( P(J), \) and let \( v \mapsto \beta_v \) be a homomorphism of \( V \) into \( \text{End} (J, \ast) \) such that (1) \( jB_i \in H_y, \) for \( j \in J_z \) and \( i \in I(b)J_z \beta_b \subseteq H_k \) (2) \( gB_\alpha = g\beta_\alpha = g^*e_\alpha \) for \( g \in J \) (b) \( jB_i = j \) for \( j \in H, \) and \( i \in I. \) Let \( (Y, I, J, V, B, \beta) \) denote \( \{(i, a, j) : a \in V, i \in I_{ab}, \) and \( j \in J_{a-1}\} \) under the multiplication (3) \( (i, a, j)(u, b, z) = (i \circ e_{ab}(ab)^{-1}, ab, jB_{a}\beta_{ab}z). \)

THEOREM 4.1. \((Y, I, J, V, B, \beta) \) is a split extension of \( T \) by \( V. \) Conversely, every such semigroup is isomorphic to some \((Y, I, J, V, B, \beta). \)
Proof. Let $S = (Y, I, J, V, B, \beta)$. If we let $g(c, d) = \beta(\beta d)^{-i}\beta d$ for all $c, d \in V$, it is easily verified that (1)-(4) of the definition of $(Y, I, J, V, B, \beta, g)$ are valid. Hence, $S$ is a standard regular semigroup by Theorem 3.14. Let $y_0$ denote the greatest element of $Y$. Using 2(a) and Lemmas 3.9 and 3.2, $(e_{yQ}, y_0, e_{yo}) \in E(T_{y_0})$. Using (4), 2(a) and the fact $(J, \ast)$ is locally inverse, $(e_{yo}, 2/0, e_{yo}) \in S(\{\beta^\ast a, a, j\}: a \in V, j \in H_{a^{-1}a})$. Thus, using Lemmas 3.6, $h_a = \{(e_{aa^{-1}}, a, j): j \in H_{a^{-1}a}\}$.

Let $v_a = (e_{aa^{-1}}, a, e_{a^{-1}a})$. Hence, $v_av_b = v_{ab}$. Thus, $S$ is a split extension of $T$ by $B$. Conversely, let $S$ be a standard regular semigroup which is a split extension of $T$ by $F$. Hence, using Lemma 2.17, $g(c, d) = \delta(d)^{i}\delta (d)$ for all $c, d \in V$. Thus, using Theorem 3.14, $S = (Y, I, J, V, B, \beta, g)$ with $g(c, d) = \delta(d)^{i}\delta (d)$. Hence, $S = U$. For brevity, let $(Y, I, J, V, B, \beta, g) = 17$.

Let $A$ be an $\omega Y$-semilattice. Let $V = \{(n, k)\delta; n, k \in N, \delta \in Y\}$ under the multiplication

$$(n, k)\delta(r, s) = (n + r - \min(k, r), k + s - \min(k, r))$$

where $f(k, r) = \delta, \gamma$, or $\delta\gamma$ according to whether $k > r, r > k$, or $r = k$. Let $Y(=A), I, J, H,$ and $(\delta(k, \delta))$ be as in the definition of $(Y, I, J, V, B, \beta)$. Let $(n, k)_\delta = \beta(n, k)_\delta$ be a homomorphism of $V$ into $\text{End}(J, \ast)$ and let $i \rightharpoonup B$, be a homomorphism of $(I, \circ)$ into $P(J)$ such that $1(a) \ jB_i \in H_{(n, \delta)(m, \delta)}$ for $j \in J_{(m, \delta)}$ and $i \in I(\delta(m, \delta))$. 2(a) $hB_{(n, \delta)} = h\beta(n, \delta) = h\epsilon(n, \delta)$ for $h \in J(b) jB_i = j$ if $j \in H_{(n, \delta)}$ and $i \in I(\delta(m, \delta))$. Let $I = \bigcup(I(n, \delta): n \in N)$ and let $J = \bigcup(J(n, \delta): n \in N)$. Let $(\omega Y, I, J, B, \beta)$ denote $\bigcup(I_\delta \times J_\delta; \delta \in Y)$ under the multiplication (3): if $i \in I(n, \delta), j \in J(k, \epsilon), u \in I(\epsilon, r)$, and $z \in J(r, \gamma)$, $(i, j)(u, z) = (i \circ e_{(n+\min(k, r), f(k, r))}, jB_{(r, \epsilon)}\beta^{r}(r, \gamma))z$.

Theorem 4.2. $(\omega Y, I, J, B, \beta)$ is a standard regular semigroup of type $\omega Y$, and conversely every such semigroup is isomorphic to some $(\omega Y, I, J, B, \beta)$.

Proof. Let $S$ be a standard regular semigroup of type $\omega Y$. Using Lemmas 2.1 and 2.2, $E(S) = \{e(n, \delta); n \in N, \delta \in Y\} \cong A$. Furthermore, $e((n, \delta)^\prime)(e(n, \delta)) \in (S)$ if and only if $\delta = \lambda$. Hence, use [15, Lemma 2.1, Theorem 2.3, and Corollaries 2.2 and 2.4] to show $S$ is a split extension of $T$ by $V$ (given in the definition of $(\omega Y, I, J, B, \beta)$). Hence, $S \cong (A, I, J, V, B, \beta)$ by Theorem 4.1. Using [15, Corollary
THEOREM 4.3. S is a simple standard regular semigroup of type \( \omega \) if and only if S is a standard regular semigroup of type \( \omega Y \) with \( Y \) a finite chain \( 0 > 1 > 2 > \cdots > d - 1 \) where \( d \) is a positive integer.

Proof. Let \( S \) be a simple standard regular semigroup of type \( \omega \). Hence, \( S \) is a simple semigroup. Thus, using Lemma 2.15 and [15, Lemma 7.5 and Theorem 2.3], \( V \) is the semigroup described in the definition of \( (\omega Y, I, J, B, \beta) \) with \( Y \) the finite chain \( 0 > 1 > 2 > \cdots > d - 1 \) where \( d \) is a positive integer. Hence, using Theorem 3.14 (and its proof), [15, Corollary 2.4], Lemmas 3.2, 3.7, and 3.9–3.12, \( S \) is a standard regular semigroup of type \( \omega Y \) with \( Y \) the finite chain \( 0 > 1 > 2 > \cdots > d - 1 \) where \( d \) is a positive integer. Conversely, let \( S \) be a standard regular semigroup of type \( \omega Y \) with \( Y \) a finite chain. It is easily seen that \( S \) is a standard regular semigroup of type \( \omega \).

We next show that any standard regular semigroup of type \( \omega Y \) is simple. Let \( S = (\omega Y, I, J, B, \beta) \). Let \( i \in I_{(a, b)} \), \( j \in J_{(k, \delta)} \), \( u \in I_{(r, s)} \) and \( v \in J_{(s, \eta)} \). Let \( q = e_{(r+u, v)} B_{(k+u, \delta)} \beta_{(r+u, v)} \). Using (1), \( q \in H_{(r+u, v)} \). Hence, since \( J_{(s, \eta)} \) is a right group, there exists \( b \in J_{(s, \eta)} \) such that \( q^* b = v \). Thus, (3), (2(a)), the fact \( c \to \beta_c \) is a homomorphism of \( V \) into \( \text{End}(J, \cdot) \), and (1), \( (u, e_{(r+u, v)})(i, j)(e_{(k+u, \delta)} b)) = (u, v) \).

If \( A \) is a finite set, \( |A| \) will denote the number of elements of \( A \).

THEOREM 4.4. \( S \) is a standard regular bisimple semigroup of type \( \omega \) if and only if \( S \) is a standard regular semigroup of type \( \omega Y \) with \( |Y| = 1 \).

Proof. Apply Theorem 4.3, Lemma 3.7, and [15, Corollary 2.4].
5. Some other classes of standard regular semigroups. A regular semigroup \( S \) is termed standard orthodox (\( \mathbb{L}^2 \)-unipotent) if \( T \) is a locally inverse semilattice \( Y \) of rectangular groups (right groups) \((T_y: y \in Y)(J_i: y \in Y)\) where \( Y \) has a greatest element. An inverse semigroup \( S \) is termed standard inverse if \( T \) is a semilattice \( Y \) of groups \((H_y: y \in Y)\) where \( Y \) has a greatest element. In this section, we give structure theorems for these classes of semigroups.

Let \( Y, V, I, J, \{ e_i \}, H, \beta, \) and \( g \) be as in the definition of \((Y, I, J, V, \beta, g)\). Furthermore, assume that 1(b), 1(c), 2(c), 3(a), and 3(b) of that definition are valid and that 2(a) \( g\beta_y = g^*e_y \) for \( y \in Y \) and \( g \in J \). Let \((Y, I, J, V, \beta, g)\) denote \( \{(i, a, j): a \in V, i \in I_{a^{-1}}, j \in J_{a^{-1}}\} \) under the multiplication (4)

\[(i, a, j)(u, b, z) = (i \circ e_{(ab)(ab)^{-1}}, ab, g(a, b)^*j\beta^*_y z) .\]

**Remark 5.1.** \((Y, I, J, V, \beta, g)\) is a standard orthodox semigroup and conversely every standard orthodox semigroup is isomorphic to some \((Y, I, J, V, \beta, g)\). Proof. Let \( S \) be a standard orthodox semigroup. It is easily seen that \( S \) is standard regular. We apply Theorem 3.14. Let \( j \in J \) and \( u \in I \). By the proof of Lemma 2.11, \( jB_u = ju \). Let \( \{ \gamma_y, z \in Y \} \) denote the set of structure homomorphisms of \( T \). Let \( j \in J_y \) and \( u \in I_z \). Hence, using the fact that \( T_{yz} \) is a rectangular group, \( ju = jz_yu \epsilon_z \epsilon_{z, y} = jz_yu \epsilon_z \epsilon_{z, y} = je_z \). Hence \( jB_u = j^*e_z \). Let \( j \in J, u \in I_{a^{-1}}, \) and \( z \in J_{a^{-1}} \). Hence, using the fact \( \beta \in \text{End} (J, *) \) and 1(b), \( jB_\beta z = (j^*e_{a^{-1}})\beta^*_y z = j^*e_{a^{-1}} \beta^*_y z = j^* \beta^*_y z \). Thus, (4) of Theorem 3.14 reduces to (4) in the definition of \((Y, I, J, V, \beta, g)\).

Conversely, we show \((Y, I, J, V, \beta, g)\) is a standard orthodox semigroup. First, we apply Theorem 3.14 to show that \((Y, I, J, V, \beta, g)\) is a standard regular semigroup. We define \( jB_i = j^*e_z \) for \( j \in J \) and \( i \in I_y \). Let \( u \in I_y, v \in I_i, \) and \( j \in J \). Hence \( jB_uB_v = j^*e_z \epsilon^*_y \epsilon^*_v = jB_{uv} \). Let \( j, h \in J \) and \( i \in I_y \). Hence, \( (j^*h)B_i = (j^*h)^*e_v = j^*(h^*e_v) = j^*(hB_i) \). Thus, \( i \to B_i \) is a homomorphism of \((I, o)\) into \( P(J, *) \). Let \( j \in J_y \) and \( i \in I_z \). Let \( \{ \gamma_{yz}: y, z \in Y \} \) denote the set of structure homomorphisms of \((J, *) \). Hence, for \( i \in I_z, jB_i = j^*e_z = j^*e_{y} \epsilon_z \epsilon_{y, z} \). However, \( e_z \epsilon_{y, z} \leq e_z \) and \( e_{y, z} \leq e_z \). Hence, using the fact that \((J, *)\) is locally inverse and \( J_{y, z} \) is a right group \( e_z \epsilon_{y, z} = e_{y, z} \). Thus, \( jB_i = j^*e_{y, z} e_{y, z} \in H_{yz} \). Hence, 1(a) of Theorem 3.14 is valid. By definition, \( jB_{uv} = j^*e_{uv} \). If \( j \in H_y \) and \( i \in I_y, jB_i = j^*e_y = j \). Hence, 2(b) of Theorem 3.14 is valid. Let \( j \in J, u \in I_{a^{-1}}, \) and \( z \in J_{a^{-1}} \). As above, \( j^*\beta^*_y z = jB_{uv} \beta^*_y z \). Thus, (4) of Theorem 3.14 and (4) of the definition of \((Y, I, J, V, \beta, g)\) are equivalent. Hence, using Theorem 3.14, \((Y, I, J, V, \beta, g)\) is a standard regular semigroup. Let \((i, a, j), (u, y, z) \in T_y \)
Hence, using 2(c) and 2(a), \((i, y, j)(u, y, z) = (i, y, j^*z)\). Thus, using [1, Theorem 1.27], \(T_s\) is a rectangular group. Hence, using Lemmas 3.9–3.11, \((Y, I, J, V, \beta, g)\) is a standard orthodox semigroup.

**Remark 5.2.** By a result of Preston, Yamada, and Clifford, [2, Proposition 1], if \(T\) is a semilattice of rectangular groups, \(T\) is an orthodox semigroup. Conversely, every Cliffordian semigroup which is orthodox is a semilattice of rectangular groups.

Let \(Y, V, J,\) and \(\beta\) be as in Theorem 5.1. For each \(y \in Y\), select \(e_y \in E(J_y)\) such that \(e_y^*e_y = e_{yz}\) for all \(y, z \in Y\). Let \(H_y\) denote the maximal subgroup of \(J_y\) containing \(e_y\). Let \(g\) be as in Theorem 5.1 and assume 1(b), 1(c), 2(a), 2(c), 3(a), and 3(b) of Theorem 5.1 are valid. Let \((Y, J, V, \beta, g)\) denote \(\{(a, j): a \in V, j \in J_a^{-1}\}\) under the multiplication (4) \((a, j)(b, z) = (ab, g(a, b)^*j\beta^*z)\).

**Theorem 5.3.** \((Y, J, V, \beta, g)\) is a standard \(S\)-unipotent semigroup and, conversely, every standard \(S\)-unipotent semigroup is isomorphic to some \((Y, J, V, \beta, g)\).

**Proof.** Let \(S\) be a standard \(S\)-unipotent semigroup. Hence, using [1, Theorem 1.27], \(S\) is standard orthodox. Using Theorem 5.1, \(S = (Y, I, J, V, \beta, g)\). Let \(i \in I_y\). Using Lemma 3.2 and the proof of Lemma 5.1, \((i, y, e_y), (e_y, y, e_y) \in E(S)\). Hence, using Lemma 3.5 and the fact each \(S\)-class of \(S\) contains precisely one idempotent, \(i = e_y\). Thus, \(I_y = \{e_y\}\) for each \(y \in Y\). Hence, \((e_y)\) defines an isomorphism of \((Y, I, J, V, \beta, g)\) onto \((Y, J, V, \beta, g)\). Conversely, we show that \(S = (Y, J, V, \beta, g)\) is a standard \(S\)-unipotent semigroup. Let \(I_y = \{e_y\}\). Define \(e_y \circ e_y = e_{yz}\) and let \(I = \bigcup (I_y: y \in Y)\). Hence, \((I, \circ)\) is a standard regular semilattice \(Y\) of left zero semigroups \((I_y: y \in Y)\). Then, \(\phi^{-1}\) is an isomorphism of \(S\) onto \((Y, I, J, V, \beta, g)\). Hence, \(S\) is a standard orthodox semigroup. Using Lemmas 3.2 and 3.5, each \(S\)-class of \(S\) contains precisely one idempotent. Hence, it easily follows that \(S\) is standard \(S\)-unipotent.

**Remark 5.4.** A semigroup \(S\) is termed \(S\)-unipotent if each \(S\)-class of \(S\) contains precisely one idempotent [13]. Hence, a standard regular semigroup is \(S\)-unipotent in the sense of [13] if and only if it is standard \(S\)-unipotent.

Let \(Y, V,\) and \(\beta\) be as in Theorem 5.3. Let \((H, ^*)\) be a semilattice \(Y\) of groups \((H_y: y \in Y)\) and let \(e_y\) denote the identity of \(H_y\) and let \(g\) be as in Theorem 5.3 and assume 1(b), 1(c), 2(a), 2(c), 3(a) and 3(b) of Theorem 5.3 are valid. Let \((Y, H, V, \beta, g)\) denote \(\{(a, j): a \in V, j \in H_a^{-1}\}\) under the multiplication (4)
(a, j)(b, z) = (ab, g(a, b) j β f z).

**Theorem 5.5.** \((Y, H, V, \beta, g)\) is a standard inverse semigroup, and conversely every standard inverse semigroup is isomorphic to some \((Y, H, V, \beta, g)\).

**Proof.** Let \(S\) be a standard inverse semigroup. Then, \(S\) is standard \(\mathcal{J}\)-unipotent. Hence, using Theorem 5.3, \(S \cong (Y, J, V, \beta, g)\). Using Lemma 3.2, \(E(Y, J, V, \beta, g) = \{(y, j); y \in Y, j \in E(J_y)\}\). Let \(j \in E(J_y)\). Hence, using 2(c) and 2(a), \((y, j)(y, e_y) = (y, e_y)\) and \((y, e_y)(y, j) = (y, j)\). Thus, \(j = e_y\). Hence, \(J_y = \mathcal{I}\) for all \(y \in Y\). Thus, \(S \cong (Y, H, V, \beta, g)\). Conversely, let \(S = (Y, H, V, \beta, g)\). Since \(e_y^* e_y = e_y^*\), \(S\) is standard \(\mathcal{J}\)-unipotent by Theorem 5.3. Using Lemma 3.2, \(E(S) = \{(y, e_y); y \in Y\}\). Hence, \(E(S)\) is a semilattice. Thus, it is easily seen that \(S\) is a standard inverse semigroup.

**Remark 5.6.** A characterization of standard orthodox semigroups may be obtained by combining a theorem of M. Yamada [18, Theorem 2] with Theorem 5.5.

6. The congruence \(t\). Let \(S\) be a standard regular semigroup. Let \(t = \{(a, b) \in S^2; a a^* b b' \in E(T_a) \text{ and } a' a, b b' \in E(T_b)\} \text{ for some } a' \in \mathcal{I}(a), b' \in \mathcal{I}(b), \text{ and } y, z \in Y\). We introduced \(t\) in a special case in [12] and used it in subsequent papers (see [15] and [16] for example). We show \(t\) is a congruence on \(S, S/t \cong V\), and \(\ker t\) (the collection of \(t\)-classes of \(S\) containing idempotents) = \(T\). We note that \(S\) is a regular extension of \(T\) by \(V\) in the sense of Yamada [19].

**Lemma 6.1.** \(\mathcal{J}(i, a, j) \cap (I_{a^{-1}a} \times \{a^{-1}\} \times J_{a a^{-1}}) \neq \emptyset\).

**Proof.** Let \(y = a a^{-1}\) and let \(u \in I_{a^{-1}a}\). Hence, using (1), \(g(a, a^{-1}) j B a s_{a^{-1}} \in H_{a a^{-1}}\) for \(j \in J_{a^{-1}a}\). There exists \(v \in H_{a a^{-1}}\) such that \(g(a, a^{-1}) j B a s_{a^{-1}v} = e_y\). Hence, using (4), \((i, a, j)(u, a^{-1}, v) = (i, y, e_y)\). However, using 2(b), Lemmas 3.2, 3.3, and 3.5,

\[(i, a, j) = (i, y, e_y)(i, a, j) = (i, a, j)(u, a^{-1}, v)(i, a, j)\]

while \((u, a^{-1}, v)(i, a, j)(u, a^{-1}, v) = (u, a^{-1}, v)(i, y, e_y) = (u, a^{-1}, v)\).

**Lemma 6.2.** \((i, a, j)t(u, b, v)\ if \ and \ only \ if \ a = b\).

**Proof.** Using Lemmas 6.1 and 3.9, there exists \((i, a, j)' \in \mathcal{J}(i, a, j)\) and \((u, b, v)' \in \mathcal{J}(u, b, v)\) such that \((i, a, j)'(i, a, j) \in T_{a^{-1}a}, (u, b, v')(u, b, v) \in T_{b^{-1}b}, (i, a, j)'(i, a, j) \in T_{a^{-1}a}, \text{ and } (u, b, v')(u, b, v) \in T_{b^{-1}b}\). Hence,
$(i, a, j)t(u, \delta, v)$ if and only if $aa^{-1} = bb^{-1}$ and $a^{-1}a = b^{-1}b$.

**Theorem 6.3.** $t$ is a congruence relation on $S$, $S/t \cong V$, and $\ker t = T$.

**Proof.** The first two assertions are easily seen. Using Lemmas 3.2, 3.9, and 3.10, $\ker t = T$.

**Remark 6.4.** Using Lemmas 3.9-3.11 and the fact $(i, a, j)\phi = a$ defines a homomorphism of $S$ onto $V$, $S$ is a regular extension of $T$ by $V$ in the sense of Yamada [19, page 4]. Thus, using Theorem 6.3, $S$ is a regular extension of $T$ by $S/t$.

**References**


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