THE CENTRALISER OF $E \otimes_\lambda F$

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If $E$ is a real Banach space then $\mathcal{B}(E)$ is the space of all bounded linear operators on $E$, and $\mathcal{K}(E)$ the subspace of $M$-bounded operators, i.e. the centraliser of $E$. Two Banach spaces $E$ and $F$ are considered as well as the tensor product $E \otimes F$. There is a natural mapping of the algebraic tensor product $\mathcal{K}(E) \odot \mathcal{K}(F)$ into $\mathcal{K}(E \otimes F)$. It is shown that $\mathcal{K}(E \otimes F)$ is precisely the strong operator closure, in $\mathcal{K}(E \otimes F)$, of its image.

1. Definitions and statement of results. A linear operator $T$ on a real Banach space $E$ is $M$-bounded if there is $\lambda > 0$ such that if $e \in E$ and $D$ is a closed ball in $E$ containing $\lambda e$ and $-\lambda e$, then $Te \in D$. The centraliser of $E$, $\mathcal{K}(E)$, is the commutative Banach algebra of all $M$-bounded linear operators on $E$. Let $K$ denote the unit ball of $E^*$, the Banach dual of $E$, equipped with the weak* topology. We denote the set of extreme points of a convex set $C$ by $\mathcal{E}(C)$. In [2], Theorem 4.8 it is shown that a bounded linear operator $T$ on $E$ is $M$-bounded if and only if each point of $\mathcal{E}(K)$ is an eigenvalue for $T^*$, the adjoint of $T$. Thus there is a real valued function $T \mapsto T$ on $\mathcal{E}(K)$ such that $T^*p = \bar{T}(p)p (p \in \mathcal{E}(K))$.

An $L$-ideal in a real Banach space is a subspace $I$ with a complementary direct summand $J$ such that $\|i\| + \|j\| = \|i + j\| (i \in I, j \in J)$. The sets $I \cap \mathcal{E}(K)$ for $I$ a weak*-closed $L$-ideal in $E^*$ form the closed sets of the structure topology on $\mathcal{E}(K)$. The map $T \mapsto \bar{T}$ is an isometric algebra isomorphism of $\mathcal{E}(E)$ onto the bounded structurally continuous real valued functions on $\mathcal{E}(K)$ with the supremum norm and pointwise multiplication ([2], Theorem 4.9).

We shall consider two Banach spaces $E$ and $F$, $K$ will retain its meaning and $M$ will denote the corresponding subset of $F^*$. We use $E \odot F$ to denote the algebraic tensor product of $E$ and $F$. We shall consider the norm

$$\left\| \sum_{i=1}^{n} e_i \otimes f_i \right\|_2 = \sup \left\{ \left| \sum_{i=1}^{n} k(e_i)m(f_i) \right| : k \in K, m \in M \right\}.$$ 

$E \otimes_2 F$ will denote $E \odot F$ with this norm, and $E \boxtimes F$ its completion.

We may identify $E \boxtimes F$ concretely in a number of ways. The formula $(k, m) \mapsto \sum_{i=1}^{n} k(e_i)m(f_i)$ defines a real valued function on $K \times M$. Such functions are continuous and affine in each variable. $\| \sum_{i=1}^{n} e_i \otimes f_i \|_2$ is the same as the supremum norm for such a function, so we may identify $E \otimes_2 F$ with a subspace $H$, the closure of
these functions, in \( C(K \times M) \), the continuous real valued functions on \( K \times M \). We shall have need to call upon:

**Lemma.** Every extreme point of the unit ball of \( H^* \) is of the form \( h \mapsto h(p, q)(p \in \mathcal{E}(K), q \in \mathcal{E}(M)) \).

Let \( R: C(K \times M)^* \to H^* \) be the restriction map, and let \( B \) be the unit ball of \( C(K \times M)^* \). If \( f \) is an extreme point of the unit ball of \( H^* \), then \( R^{-1} f \cap B \) is a weak* closed face of \( B \) which is nonempty by the Hahn-Banach theorem. By the Krein-Milman theorem, \( R^{-1} f \cap B \) has an extreme point, which must be extreme in the unit ball of \( C(K \times M)^* \), so is of the form \( h \mapsto \pm h(p, q) \) for \( p \in K, q \in M \). By replacing \( p \) by \(-p\), if necessary, we may ensure a positive sign. If \( p \) (say) is not extreme, then \( p = 1/2(p_1 + p_2), p_1, p_2 \in K, p_1 \neq p_2 \). \( h(p, q) = 1/2h(p_1, q) + 1/2h(p_2, q)(h \in H) \) as these functions are affine in each variable. As the functions of \( H \) separate the points of \( K \times M \), this contradicts the extremality.

**Corollary.**

\[
\left\| \sum_{i=1}^{n} e_i \otimes f_i \right\|_2 = \sup \left\{ \left\| \sum_{i=1}^{n} p(e_i)q(f_i) \right\| : p \in \mathcal{E}(K), q \in \mathcal{E}(M) \right\}.
\]

We consider the centraliser of \( E \otimes_{\gamma} F \). We have quite easily:

**Proposition.** If \( S_i \in \mathcal{S}(E), T_i \in \mathcal{S}(F)(1 \leq i \leq n) \) there is \( U \in \mathcal{S}(E \otimes_{\gamma} F) \) such that if \( e_j \in E, f_j \in F(1 \leq j \leq m) \) then \( U(\sum_{j=1}^{m} e_j \otimes f_j) = \sum_{j=1}^{m} \sum_{i=1}^{n} (S_i e_j) \otimes (T_i f_j) \).

To show that \( U \) exists (as a bounded linear operator) we need only show that the linear operator defined on \( E \otimes_{\gamma} F \) by this formula is bounded. This is so because,

\[
\left\| \sum_{i,j} (S_i e_j) \otimes (T_i f_j) \right\|_2 \\
= \sup \left\{ \left\| \sum_{i,j} p(S_i e_j) (T_i e_j) \right\| : p \in \mathcal{E}(K), q \in \mathcal{E}(M) \right\} \\
= \sup \left\{ \left\| \sum_{i,j} \tilde{S}_i(p) \tilde{T}_i(p) p(e_j)q(f_j) \right\| : p \in \mathcal{E}(K), q \in \mathcal{E}(M) \right\} \\
\leq \sup \left\{ \sum_i \left\| \tilde{S}_i(p) \right\| \left\| \tilde{T}_i(p) \right\| \left\| \sum_j p(e_j)q(f_j) \right\| : p \in \mathcal{E}(K), q \in \mathcal{E}(M) \right\} \\
\leq \sum_i \left\| S_i \right\| \left\| T_i \right\| \sup \left\{ \left\| \sum_j p(e_j)q(f_j) \right\| : p \in \mathcal{E}(K), q \in \mathcal{E}(M) \right\} \\
= \sum_i \left\| S_i \right\| \left\| T_i \right\| \left\| \sum_j e_j \otimes f_j \right\|_2.
\]
It remains to show that each extreme point of the unit ball of $(E \otimes_1 F)^*$ is an eigenvalue for $U^*$. If we denote by $p \otimes q$ the functional $\sum_{j} e_j \otimes f_j \mapsto \sum_{j} p(e_j)q(f_j)$ then we have

\[
U^*(p \otimes q)\left(\sum_{j} e_j \otimes f_j\right) = (p \otimes q)U\left(\sum_{j} e_j \otimes f_j\right) = (p \otimes q)\sum_{i,j} (S_ie_i) \otimes (T_if_i) = \sum_{i,j} p(S_ie_i)q(T_if_i) = \sum_{i,j} \tilde{S}_i(p)\tilde{T}_i(p)p(e_i)q(f_i) = \left[\sum_i \tilde{S}_i(p)\tilde{T}_i(p)\right](p \otimes q)\left(\sum_{j} e_j \otimes f_j\right).
\]

It is immediate that $U^*(p \otimes q) = \left[\sum_i \tilde{S}_i(p)\tilde{T}_i(p)\right](p \otimes q)$.

We thus have an embedding of $\mathcal{K}(E) \circ \mathcal{K}(F)$ in $\mathcal{K}(E \otimes_1 F)$ in an obvious way. The remainder of this paper is devoted to a proof of the following result.

**Theorem.** $\mathcal{K}(E \otimes_1 F)$ is the closure, for the strong operator topology, of the canonical copy of $\mathcal{K}(E) \circ \mathcal{K}(F)$ in $\mathcal{B}(E \otimes_1 F)$.

2. The proof. For this proof we shall identify the element $\sum_{i=1}^n e_i \otimes f_i \in E \circ F$ with the function $k \mapsto \sum_{i=1}^n k(e_i)f_i$ from $K$ into $F$. This is continuous affine function vanishing at 0. The set of all $F$-valued continuous affine functions of $K$ which vanish at 0 we shall denote by $A_0(K, F)$, and norm it by $||a|| = \sup \{||a(k)||: k \in K\}$, which corresponds to the norm on $E \otimes_1 F$. We may thus identify $E \otimes_1 F$ with the closure, $H$, in $A_0(K, F)$ of the functions with finite dimensional range.

If $\sum_{i=1}^n S_i \otimes T_i \in \mathcal{K}(E) \circ \mathcal{K}(F)$ then $\pi: p \mapsto \sum_{i=1}^n \tilde{S}_i(p)T_i$ is a function from $\mathcal{E}(K)$ into $\mathcal{E}(F)$ which is bounded and continuous for the structure topology on $\mathcal{E}(K)$ and the strong operator topology on $\mathcal{E}(F)$. If $U$ is the image of $\sum_{i=1}^n S_i \otimes T_i$ in $\mathcal{E}(H)$ (using the proposition and the identification of $H$ with $E \otimes_1 F$) then we have

\[
(Uh)(p) = \pi(p)h(p) \quad (h \in H, p \in \mathcal{E}(K)).
\]

This is because, if $\varepsilon > 0$, we may find $\sum_{i=1}^n e_i \otimes f_i \in E \circ F$ with $||h - \sum_{i=1}^n e_i \otimes f_i||_2 < \varepsilon$ and then

\[
\left\| (Uh)(p) - \pi(p)h(p) \right\| \leq \left\| (Uh)(p) - U\left(\sum_{j=1}^m e_j \otimes f_j\right)(p) \right\| + \left\| U\left(\sum_{j=1}^m e_j \otimes f_j\right)(p) - \pi(p)h(p) \right\|.
\]

But
\[ U\left( \sum_{j=1}^{m} e_j \otimes f_j \right)(p) = \sum_{i,j} (S_i e_j) \otimes (T_i e_j)(p) = \sum_{i} p(S_i e_j)(T_i e_j) = \left( \sum_{i} \tilde{S}_i(p) \right) \left( \sum_{j} p(e_j) f_j \right) = \pi(p) \left( \left( \sum_{j=1}^{m} e_j \otimes f_j \right)(p) \right). \]

Thus \( (Uh)(p) - \pi(p)h(p) \leq \|U\| \varepsilon + \|\pi(p)\| \|\sum_{j=1}^{m} e_j \otimes f_j - h(p)\| \leq (\|U\| + \|\pi(p)\|)\varepsilon \), which can be made as small as desired, so that \( (Uh)(p) = \pi(p)h(p) \).

Let \( V(K) \) denote the set of extreme points, \( p^* \), of \( K \) for which there is \( x \in E \) with \( p(x) = \|x\| \), then \( V(K) \) is weak* dense in \( \mathcal{E}(K) \). To show this it will suffice to prove that \( K = \overline{\text{co}}(V(K)) \), the weak* closed convex hull of \( V(K) \), for then \( \mathcal{E}(K) \subset \overline{V(K)} \) by Milman's theorem. If \( \overline{\text{co}}(V(K)) \neq K \) we may, by Hahn-Banach separation, find \( x \in E \) with \( k(x) \leq \alpha < k_0(x) \) for some real \( \alpha \), all \( k \in \overline{\text{co}}(V(K)) \) and some \( k_0 \in K \). Then \( \{k \in K : k(x) = \|x\|\} \) is a nonempty weak* closed face of \( K \). This possesses an extreme point, which cannot lie in \( \overline{\text{co}}(V(K)) \), yet which is in \( V(K) \) by its construction, a contradiction.

If \( p \in V(K), q \in V(M) \) then \( p \otimes q \) is extreme in the unit ball of \((E \otimes F)^*\). Fix \( e \in E, f \in F \) with \( \|e\| = e(p) = 1, \|f\| = f(p) = 1 \). Define injections \( P : E \rightarrow E \otimes F, Q : F \rightarrow E \otimes F \) by \( P(x) = x \otimes f, Q(y) = e \otimes y \). \( P, Q \) are isometric injections so the image of the unit ball of \((E \otimes F)^*\) under \( P^* \) (respectively \( Q^* \)) is \( K \) (respectively \( M \)). \( P^*, Q^* \) are continuous and affine, so \( P^{*\otimes}(p) \) and \( Q^{*\otimes}(q) \) intersect the unit ball of \((E \otimes F)^*\) in weak* closed faces, as must \( P^{*\otimes}(p) \cap Q^{*\otimes}(q) \). This intersection is nonempty, for \( P^*(p \otimes q) = p, Q^*(p \otimes q) = q \). This is because for \( x \in E, (P^*(p \otimes q))(x) = (p \otimes q)(Px) = (p \otimes q)(x \otimes f) = p(x)q(f) = p(x) \), with a similar proof for \( Q^* \). This face must have an extreme point which is extreme in the unit ball of \((E \otimes F)^*\), so is \( p' \otimes q' \) for \( p' \in \mathcal{E}(K), q' \in \mathcal{E}(M) \). But now \( p = P^*(p \otimes q) = P^*(p' \otimes q') = p' \) and also \( q = q' \), so that \( p \otimes q \) is itself extreme.

It follows that if \( U \in \mathcal{H}(H) \) then all points \( p \otimes q \) for \( p \in \mathcal{E}(K), q \in \mathcal{E}(M) \) are eigenvectors for \( U^* \). For let \( p_\tau \rightarrow p, q_\delta \rightarrow q \) be nets with \( p_\tau \in V(K), q_\delta \in V(M) \). The continuity of the map \( (k, m) \rightarrow k \otimes m \) from \( K \times M \) into \((E \otimes F)^* \) implies that \( p_\tau \otimes q_\delta \rightarrow p \otimes q \). But \( U^*(p_\tau \otimes q_\delta) = \tilde{U}(p_\tau \otimes q_\delta)(p_\tau \otimes q_\delta) \) are bounded (by \( \|U\| \)) so we may suppose (by choosing a subnet if necessary) that \( \tilde{U}(p_\tau \otimes q_\delta) \rightarrow \lambda \). Now \( U^*(p \otimes q) = \lim U^*(p_\tau \otimes q_\delta) = \lim \tilde{U}(p_\tau \otimes q_\delta) \lim (p_\tau \otimes q_\delta) = \lambda(p \otimes q) \).

Suppose \( U \in \mathcal{H}(H), p \in \mathcal{E}(K) \) and \( h, h' \in H \) with \( h(p) = h'(p) \). If
\( q \in \mathcal{S}(M) \) then
\[
q((Uh)(p)) = (p \otimes q)(Uh) = \tilde{U}(p \otimes q)((p \otimes q)(h)) \\
= \tilde{U}(p \otimes q)(q(h(p))) \\
= \tilde{U}(p \otimes q)(q(h'(p))) = q((Uh')(p)) .
\]
Thus \((Uh)(p) = (Uh')(p)\). We may thus define a linear operator \( \pi(p) \) on \( F \) by \( \pi(p)y = (Uh)(p) \) whenever \( h(p) = y \). \( \pi(p) \) is clearly linear, is well defined, and has domain the whole of \( F \) since we may take \( h = e \otimes y \) where \( e(p) = 1 \).

\( \pi(p) \) has norm at most \( ||U|| \), for we may find \( e_n \in E \) with \( e_n(p) = 1 \), \( ||e_n|| \leq (n + 1)/n \), and then
\[
||\pi(p)y|| = ||U(e_n \otimes y)(p)|| \leq ||U(e_n \otimes y)|| \\
\leq ||U|| ||e_n \otimes y|| = ||U|| ||y|| (n + 1)/n .
\]
Thus \( ||\pi(p)y|| \leq ||U|| ||y|| \). In fact \( \pi(p) \in \mathcal{S}(F) \) because if \( y \in F \), \( q \in \mathcal{S}(M) \) and \( e \in E \) with \( p(e) = 1 \) then
\[
q(\pi(p)y) = q(U(e \otimes y)(p)) = (p \otimes q)(U(e \otimes y)) \\
= \tilde{U}(p \otimes q)(p \otimes q)(e \otimes y) = \tilde{U}(p \otimes q)q(y) .
\]
We thus have a function \( \pi: \mathcal{S}(K) \rightarrow \mathcal{S}(F) \) with \( (Uh)(p) = \pi(p)h(p) (p \in \mathcal{S}(K)) \). Also \( \pi \) is norm bounded, and we let \( ||\pi|| \) denote \( \sup\{||\pi(p)||: p \in \mathcal{S}(K)\} \).

\( \pi \) is continuous for the structure topology on \( \mathcal{S}(K) \) and the weak operator topology on \( \mathcal{S}(F) \). Suppose \( y \in F \), \( g \in F^* \) and \( x \in E \) then \( k \mapsto g(U(x \otimes y)(k)) \) is a continuous affine function on \( K \) vanishing at 0, so may be identified with an element of \( E \). If \( p \in \mathcal{S}(K) \) then
\[
g(U(x \otimes y)(p)) = g(\pi(p)(x \otimes y)(p)) \\
= g(\pi(p)x(p)y) = x(p)(g(\pi(p)y)) .
\]
Thus \( x \mapsto g(U(x \otimes y)) \) is an element of \( \mathcal{S}(E) \), so the function \( p \mapsto g(\pi(p)y) \) is structurally continuous.

By [2], Proposition 3.10 \( \pi \) has an extension, \( \tilde{\pi} \), to \( \overline{\mathcal{S}(K)}\{0\} \) which is continuous for the weak* topology on \( \overline{\mathcal{S}(K)}\{0\} \) and the weak operator topology on \( \overline{\mathcal{S}(F)} \) (the result there is stated for real valued functions but the proof remains valid in this context). We note for later reference that \( \pi(\mathcal{S}(K)) = \tilde{\pi}(\overline{\mathcal{S}(K)}\{0\}) \). We propose now to show \( \tilde{\pi} \) is still continuous when \( \overline{\mathcal{S}(F)} \) is given its strong operator topology.

Provisionally we define \( \tilde{\pi}(k) \), for \( k \in \overline{\mathcal{S}(K)}\{0\} \), to be that linear operator on \( F \) such that
\[
\tilde{\pi}(k)y = U(x \otimes y)(k)/k(x)
\]
with $x \in E$, $k(x) > 0$. This definition coincides with that of $\pi$ if $k \in \mathcal{C}(K)$, and is well defined because if $k_r \in \mathcal{C}(K)$ and $k_r \to k$ for the weak* topology then

$$\tilde{\pi}(k)y = U(x \otimes y)(k)/k(x) = \lim U(x \otimes y)(k_r)/k_r(x) = \lim \pi(k_r)y.$$  

Clearly $\tilde{\pi}(k)$ acts linearly on $F$, and it is bounded because

$$\| (\tilde{\pi}(k)y) \| = \| U(x \otimes y)(k) \|/k(x) = \lim \| U(x \otimes y)(k_r) \|/k_r(x) = \lim \| \pi(k_r)y \| \leq \| \pi \| \| y \|.$$  

Also $\| \tilde{\pi} \| = \sup \{ \| \pi(k) \| : k \in \mathcal{C}(K) \setminus \{0\} \} = \| \pi \|$. $\tilde{\pi}$ is locally a quotient of a function that is clearly strong operator continuous and a non-vanishing scalar function, so is strong operator continuous. In fact $\tilde{\pi}$ is the same as $\pi$ as both are extensions of $\pi$ to $\mathcal{C}(K) \setminus \{0\}$ which are continuous for the weak* topology on $\mathcal{C}(K) \setminus \{0\}$ and the weak operator topology on $\mathcal{K}(F)$.

We do not know if $\pi$ itself is continuous when $\mathcal{K}(F)$ is given the strong operator topology. All that we shall require is that if $D \subset \mathcal{C}(K)$ and $0$ does not lie in the weak* closure of $D$, then $\pi|_D$ is continuous for the structure topology on $D$ and the strong operator topology on $\mathcal{K}(F)$. For suppose $d_r, d \in D$ and $d_r \to d$ for the structure topology, then $\pi(d_r) \to \pi(d)$ for the weak operator topology whenever $(d_r)$ is a subnet of $(d_r)$. Let $(d_r)$ be a weak* convergent subnet of $(d_r)$ with limit $d' \neq 0$, which exists as $K$ is weak* compact. Then $\pi(d_r) \to \pi(d)$ for the weak operator topology whilst $\pi(d_r) = \tilde{\pi}(d_r) \to \tilde{\pi}(d')$ for the strong operator topology, and hence also for the weak operator topology. Thus $\pi(d) = \tilde{\pi}(d')$ and $\pi(d_r) \to \pi(d)$ for the strong operator topology. I.e. every subnet of $(\pi(d_r))$ has a subnet converging to $\pi(d)$, so in fact $\pi(d_r) \to \pi(d)$ for the strong operator topology.

We now seek, given $h_i \in H(i = 1, 2, \ldots, n)$ and $\varepsilon > 0$, to find $\pi': \mathcal{C}(E) \to \mathcal{K}(F)$ which is of finite dimensional range and continuous for the structure topology, such that

$$\| \pi'(p)h_i(p) - \pi(p)h_i(p) \| \leq \varepsilon \quad (p \in \mathcal{C}(K), 1 \leq i \leq n).$$  

$\pi'$ is the image of an element of $\mathcal{K}(E) \otimes \mathcal{K}(F)$ so defines an element $U'$ of the copy of $\mathcal{K}(E) \otimes \mathcal{K}(F)$ in $\mathcal{C}(E \otimes \lambda F)$. We then have

$$\| (U'h_i)(p) - (Uh_i)(p) \| \leq \varepsilon \quad (p \in \mathcal{C}(K), 1 \leq i \leq n).$$  

The function $k \mapsto \| (U'h_i)(k) - (Uh_i)(k) \|$ on $K$ is continuous and convex, so by [1], Lemma II.7.1, $\| (U'h_i) - (Uh_i) \| \leq \varepsilon(1 \leq i \leq n)$. This will show that $U$ is in the strong operator closure of the copy of $\mathcal{K}(E) \otimes$
We first prove that \([3], \text{Proposition 4.8}\) remains valid in this context. I.e. if \(x \in E\) then \(P = \{p \in \mathcal{E}(K): |p(x)| \geq \alpha\}\) is structurally compact provided \(\alpha > 0\). If \((C_s)_{s \in S}\) is a family of nonempty structurally closed subsets of \(P\) with the finite intersection property, let \(C_s = P \cap F_s\) with each \(F_s\) a weak* closed \(L\)-ideal in \(E^*\). Set \(Q = \{k \in K: |k(x)| \geq \alpha\}\) then each \(F_s \cap Q\) is nonempty and this family has the finite intersection property. As \(Q\) is weak* compact and these sets are weak* closed, \(\bigcap (F_s \cap Q) = (\bigcap F_s) \cap Q \neq \emptyset\). \(\bigcap F_s\) is a weak* closed \(L\)-ideal and for some \(k \in K \cap (\bigcap F_s)|k(x)| \geq \alpha\). But \(x\) attains its supremum at an extreme point, \(p\), of \(K \cap (\bigcap F_s)\) which is an extreme point of \(K\) by \([2]\), Proposition 1.15. As \(K \cap (\bigcap F_s)\) is symmetric, \(p(x) \geq \alpha\) so that \(p \in E(K) \cap (\bigcap F_s) = (\bigcap (p \cap F_s)) = \bigcap C_s\). We note also that such a set \(P\) does not contain 0 in its weak* closure, so \(\pi|_P\) is continuous for the strong operator topology.

Given \(h \in H\), \(\delta > 0\), we may find a weak* closed subset \(Q_i\) of \(\mathcal{E}(K)\), not containing 0 and with \(Q_i \cap \mathcal{E}(K)\) structurally compact, such that \(||h_i(k)|| < \delta\) if \(k \in \mathcal{E}(K)\backslash Q_i\). For we can find \(\sum_{j=1}^m e_j \otimes f_j \in E \otimes F\) with \(\sum_{j=1}^m k(e_j)f_j - h_i(k)\| < \delta/2(k \in K)\). Now let \(P_i = \{k \in \mathcal{E}(K): \|k(e_j)||f_j|| \geq \delta/2m\}\), which is weak* closed, does not contain 0, and is such that \(P_i \cap \mathcal{E}(K)\) is structurally compact. Define \(Q_i = \bigcup_{j=1}^n P_j\), then \(Q_i\) will have all the desired properties except possibly that on the norm. If \(k \in \mathcal{E}(K)\backslash Q_i\) then

\[
||h_i(k)|| \leq \left\| \sum_{j=1}^m k(e_j)f_j \right\| + \left\| \sum_{j=1}^m k(e_j)f_j - h_i(k) \right\|
\]

\[
< \sum_{j=1}^m |k(e_j)||f_j|| + \delta/2
\]

\[
\leq m(\delta/2m) + \delta/2 = \delta.
\]

We may thus find a weak* open neighbourhood of 0 in \(\mathcal{E}(K)\), \(O_0\), with structurally compact complement in \(\mathcal{E}(K)\), such that \(O_0 \subset \{k \in \mathcal{E}(K): ||h_i(k)|| < \varepsilon/(2||\pi|| + 1)(1 \leq i \leq n)\}\). Indeed if we take \(\delta = \varepsilon/(2||\pi|| + 1)\) and choose \(Q_i\) as above we take \(O_0\) to be \(\mathcal{E}(K)\backslash \bigcup_{i=1}^n Q_i\), which has the desired properties. If \(k \in \mathcal{E}(K)\) we let \(U_k = \{T \in \mathcal{E}(F): ||T(h_i(k))|| < \varepsilon/3(1 \leq i \leq n)\}\), an open symmetric neighbourhood of the origin in \(\mathcal{E}(F)\) for the strong operator topology. Thus \(\pi^{-1}(\mathcal{E}(K) \backslash O_0)\) is an open subset of \(\mathcal{E}(K)\backslash \{0\}\) (by the continuity of \(\pi\) for the strong operator topology) and hence of \(\mathcal{E}(K)\). The set \(\mathcal{E}(K) \cap \bigcap_{i=1}^n h_i^{-1}(h_i(k) + B)\) (where \(B\) is the open ball in \(F\) of centre the origin and radius \(\varepsilon/(3(||\pi|| + 1))\)) is also weak* open, hence so is

\[
O_k = (\pi^{-1}(\pi(k) + U_k)) \cap \bigcap_{i=1}^n h_i^{-1}(h_i(k) + B)
\]
for each \( k \in \mathcal{S}(K) \setminus \{0\} \), and we have \( k \in O_k \). Now let \( \{0, k_1, k_2, \ldots, k_r\} \) be a finite set of distinct points of \( \mathcal{S}(K) \) with \( \mathcal{S}(K) = O_0 \cup \bigcup_{j=1}^r O_{k_j} \).

Let \( W = \bigcap_{j=1}^r U_{k_j} \), an open convex symmetric neighbourhood of the origin in \( \mathcal{H}(F) \) for the strong operator topology. Because \( \mathcal{S}(K) \setminus O_0 \) is structurally compact and \( \pi \) is continuous on this for the strong operator topology on \( \mathcal{H}(F) \), \( \pi(\mathcal{S}(K) \setminus O_0) \) is strong operator compact. Thus there exist \( \{T_i, T_2, \ldots, T_s\} \subset \mathcal{H}(F) \) such that \( \bigcup_{i=1}^s (T_i + W/2) \supset \pi(\mathcal{S}(K) \setminus O_0) \). Define \( G \) to be the linear span of \( \{T_i : 1 \leq i \leq s\} \) in \( \mathcal{H}(F) \), and let \( \Phi \) be defined on \( \pi(\mathcal{S}(K) \setminus O_0) \) with values in \( 2^G \) by

\[
\Phi(S) = \{g \in G : \|g\| < \|\pi\| + 1, g - S \in W/2\}.
\]

For some \( i, T_i - S \in W/2 \) and \( T_i \in \pi(\mathcal{S}(K) \setminus O_0) \) so \( \|T_i\| \leq \|\pi\| \), so that \( \Phi(S) \) is certainly nonempty. It is clear that \( \Phi(S) \) is closed and convex.

We show that \( \Phi \) is lower semi-continuous, for the unique vector topology on \( G \), and the weak and strong operator topologies on \( \pi(\mathcal{S}(K) \setminus O_0) \) which coincide by the compactness of \( \pi(\mathcal{S}(K) \setminus O_0) \) for the latter topology. If \( D \subset G \) is open we must show that \( \{S \in \pi(\mathcal{S}(K) \setminus O_0) : \Phi(S) \cap D \neq \emptyset\} \) is open. Suppose \( S_0 \in \pi(\mathcal{S}(K) \setminus O_0) \) with \( \Phi(S_0) \cap D \neq \emptyset \). By the definition of \( \Phi \), we can find \( x_0 \in D \) with \( \|x_0\| < \|\pi\| + 1 \), \( x_0 - S_0 \in W/2 \). As \( W \) is open, there is a symmetric strong operator neighbourhood of the origin in \( \mathcal{H}(F) \), \( V \), such that \( x_0 - S_0 + V \subset W/2 \). Now if \( S \in (S_0 + V) \cap \pi(\mathcal{S}(K) \setminus O_0) \) we claim \( \Phi(S) \cap D \neq \emptyset \), for \( x_0 - S = (x_0 - S_0) + (S_0 - S) \in (x_0 - S_0) + V \subset W/2 \). It is now clear that \( x_0 \in \Phi(S) \cap D \), completing the proof that \( \Phi \) is lower semi-continuous.

As \( G \) is finite dimensional we can apply a selection theorem (e.g. \cite{4}, Theorem 3.2') to assert the existence of a continuous selection for \( \Phi, \phi \). We note that \( \phi(\pi(\mathcal{S}(K) \setminus O_0)) \) is contained in the closed ball in \( G \) of centre the origin and radius \( \|\pi\| + 1 \). We extend \( \phi \) to \( \psi \) defined on the whole of \( \pi(\mathcal{S}(K)) \) with values in the same ball and with \( \psi \) continuous for the weak operator topology on \( \pi(\mathcal{S}(K)) \). Let \( \beta(\pi(\mathcal{S}(K))) \) be the Stone-Čech compactification of \( \pi(\mathcal{S}(K)) \) (for the weak operator topology), and \( \rho \) the natural injection of \( \pi(\mathcal{S}(K)) \) into \( \beta(\pi(\mathcal{S}(K))) \). Since the weak operator topology is uniformisable \( \rho \) is a homeomorphism, so that \( \phi \circ \rho^{-1} \) is a continuous function from the closed set \( \rho(\pi(\mathcal{S}(K) \setminus O_0)) \) into \( G \). Let \( \sigma \) be a continuous extension of \( \phi \circ \rho^{-1} \) to the whole of \( \beta(\pi(\mathcal{S}(K))) \) with values in the required ball in \( G \), which exists by Tietze's extension theorem. Now \( \psi = \sigma \circ \rho \) is the desired function. Define \( \pi' = \psi \circ \pi \), a function from \( \mathcal{S}(K) \) into \( G \) that is bounded and continuous for the structure topology on \( \mathcal{S}(K) \), since \( \pi \) is continuous for the structure topology on \( \mathcal{S}(K) \) and the weak operator topology on \( \mathcal{H}(F) \) whilst \( \psi \) is continuous for the
weak operator topology on $\pi(\mathcal{E}(K))$. We claim $\pi'$ has the required property.

If $p \in \mathcal{E}(K)|O_0$ then $p \in O_{k_j}$ for some $j$. Then $||h_i(p) - h_i(k_j)|| < \varepsilon/3(||\pi|| + 1)$ and we also have $\pi'(p) - \pi(p) \in \overline{W/2} \subset W$. Thus for $1 \leq i \leq n$,

$$||\pi(p)h_i(p) - \pi'(p)h_i(p)|| \leq ||\pi(p)h_i(p) - \pi(p)h_i(k_j)|| + ||\pi(p)h_i(k_j) - \pi'(p)h_i(k_j)||$$
$$+ ||\pi'(p)h_i(k_j) - \pi'(p)h_i(p)||$$
$$\leq ||\pi(p)|| ||h_i(p) - h_i(k_j)|| + (\varepsilon/3) + ||\pi'(p)|| ||h_i(k_j) - h_i(p)||$$
(since $\pi(p) - \pi'(p) \in W \subset U_{k_j}$)
$$\leq ||\pi||((\varepsilon/3(||\pi|| + 1)) + (\varepsilon/3) + (||\pi|| + 1)(\varepsilon/3(||\pi|| + 1))$$
$$< \varepsilon .$$

On the other hand if $p \in O_0 \cap \mathcal{E}(K)$ then

$$||\pi(p)h_i(p) - \pi'(p)h_i(p)|| \leq (||\pi'(p)|| + ||\pi(p)||)||h_i(p)||$$
$$\leq (2||\pi|| + 1)(\varepsilon/(2||\pi|| + 1)) = \varepsilon .$$

Thus $\pi'$ has the desired properties.

So far we have shown that $\mathcal{Z}(E \otimes_1 F)$ is contained in the strong operator closure in $\mathcal{B}(E \otimes_1 F)$ of the copy of $\mathcal{Z}(E) \odot \mathcal{Z}(F)$ there. It remains only to show that for any Banach space, $X$, $\mathcal{Z}(X)$ is strong operator closed in $\mathcal{B}(X)$. Indeed if $T_\tau \rightarrow T$ for the strong operator topology with $T_\tau \in \mathcal{Z}(X)$, $p$ is an extreme point of the unit ball of $X^*$ and $x \in X$, then

$$(T^*p)(x) = \lim (T_\tau^*p)(x) = \lim \overline{T_\tau}(p)x .$$

Thus $\lim \overline{T_\tau}(p)$ exists and $T^*p = (\lim \overline{T_\tau}(p))p$, so $T \in \mathcal{Z}(X)$.

REFERENCES


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