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Let K be a compact F-space such that $|C^*(K)| = 2^\omega$. Using the continuum hypothesis we characterize those subspaces of K that are C^*-embedded in K. We also characterize the class of extremally disconnected Tychonoff spaces of countable cellularity. As corollaries of these theorems, using various set-theoretic hypotheses we characterize the C^*-embedded, and the extremally disconnected C^*-embedded, subspaces of $\beta \underline{N}$.

1. Introdution. Our notation and terminology follows that of the Gillman-Jerison text [4]. All hypothesized topological spaces are assumed to be completely regular and Hausdorff (i.e., Tychonoff). As usual βX denotes the Stone-Čech compactification of the Tychonoff space X, and $\underline{N}$ denotes the countable discrete space. $C^*(X)$ denotes the family of bounded real-valued continuous functions on X. A subspace S of X is *C^*-embedded* in X if given $f \in C^*(S)$ there exists $g \in C^*(X)$ such that $g|S = f$. A *cozero-set* of X is a set of the form $X - f^{\leftarrow}(0)$ where $f \in C^*(X)$. The collection of cozero-sets of X is denoted by $\operatorname{coz}(X)$. A space X is *zero-dimensional* if its open-and-closed (clopen) sets form a base for its open sets. X is *strongly zero-dimensional* if βX is zero-dimensional.

A space X is *weakly Lindelöf* if given an open cover $\mathscr{V}$ of X, there is a countable subfamily $\mathscr{C}$ of $\mathscr{V}$ such that $\bigcup \mathscr{C}$ is dense in X (if $\mathscr{C}$ is a collection of subsets of a set we denote $\bigcup \{C: C \in \mathscr{C}\}$ by $\bigcup \mathscr{C}$). A space X has the *countable chain condition*, or *countable cellularity*, if each family of pairwise disjoint nonempty open subsets of X is countable. We abbreviate this by writing "X has c.c.c." The following lemma, which came to the attention of the author through a letter from W.W. Comfort, is easily proved.

LEMMA 1.1. *A space has* c.c.c. *iff each of its open subsets is weakly Lindelöf.*

A space X is *extremally disconnected* if disjoint open subsets have disjoint closures. It is an *F-space* if its cozero-sets are C^*-embedded. It is an *F'-space* if disjoint cozero-sets have disjoint closures. Each extremally disconnected space is an F-space, and each F-space is an F'-space. Proofs of these facts, plus other information on these classes of spaces, may be found in [1] and [4]. We shall need the following facts.

THEOREM 1.2 (1H and 6M of [4]). *The following are equivalent for a space X.*

(1) *X is extremally disconnected.*
(2) *Each dense subspace of X is extremally disconnected.*
(3) *Each open subspace of X is extremally disconnected.*
(4) *Each dense subspace of X is C^*-embedded in X.*
(5) *Each open subspace of X is C^*-embedded in X.*

THEOREM 1.3 (14.25 and 14.26 of [4]).

(1) *Each C^*-embedded subspace of an F-space is an F-space.*
(2) *X is an F-space iff βX is an F-space.*

The following lemma appears as the "note added on September 16, 1968" on page 494 of [1].

LEMMA 1.4. *If X is an F'-space and if each open subset of X is weakly Lindelöf, then X is extremally disconnected.*

LEMMA 1.5 (Corollary 1.7 of [1]). *Each weakly Lindelöf subspace of an F'-space is C^*-embedded in its own closure.*

The symbol [CH] preceding the statement of a theorem indicates that the continuum hypothesis ($2^\omega = \omega_1$) is used in the proof of the theorem. The cardinality of a set S is denoted by $|S|$. The *weight* of a topological space X, denoted by $w(X)$, is the least cardinal of a base for the open subsets of X. If α is a cardinal number then $D(\alpha)$ is the discrete space of cardinality α and $\log \alpha = \min\{\gamma: 2^\gamma \geqq \alpha\}$.

Finally, we shall use the following theorem, which appears as Remark 8, page 274 of [2].

THEOREM 1.6. *Each compact extremally disconnected space K such that $w(K) \leqq 2^\alpha$ can be topologically embedded in $\beta D(\alpha)$.*

2. C^*-embedded subsets of $\beta \underline{N}$. The proof of the implication in Theorem 2.2 that requires the continuum hypothesis—namely (3)$\rightarrow$(1)—relies heavily on a theorem, and a technique of proof, due to Fine and Gillman [3]. We first state the theorem.

THEOREM 2.1 (4.1(c) of [3]). *Let X be an F-space, let $\{S_\alpha: \alpha < \omega_1\}$ be a family of ω_1 cozero-sets of X, and put $S = \bigcup_{\alpha<\omega_1} S_\alpha$. If $G \subset S$ and $G \cap S_\alpha \in \operatorname{coz}(S_\alpha)$ for each $\alpha < \omega_1$, then G is C^*-embedded in S.*

We now state and prove the main theorem of this section.

THEOREM 2.2 [CH]. *Let K be a compact F-space such that $|C^*(K)| = 2^\omega$. Let X be a subspace of K. The following are equivalent:*

(1) *X is weakly Lindelöf.*
(2) *X is C^*-embedded in K.*
(3) $|C^*(X)| = 2^\omega$.

Proof.
(1) $\rightarrow$ (2): By 1.5 X is C^*-embedded in $\mathrm{cl}_K X$, which in turn is C^*-embedded in K by the Urysohn extension theorem (see 3.11(c) of [4]).

(2) $\rightarrow$ (3): Since $|C^*(K)| = 2^\omega$ this is obvious.

(3) $\rightarrow$ (1): Assume (1) fails; we shall prove that (3) fails also. Let X be a subspace of K that is not weakly Lindelöf. Let $\mathscr{V}$ be an open cover of X which has no countable subcollection whose union is dense in X. By writing each member of $\mathscr{V}$ as the intersection of X with a union of cozero sets of $\mathrm{cl}_K X$, and noting that $\mathrm{cl}_K X$ has only $2^\omega(=\omega_1)$ cozero subsets, we see that without loss of generality we may assume that $\mathscr{V} = \{A_\alpha \cap X : \alpha < \omega_1\}$, where each A_α is a cozero subset of $\mathrm{cl}_K X$. Put $U = \bigcup\{A_\alpha : \alpha < \omega_1\}$. Fix $\alpha_0 < \omega_1$, and inductively assume that for each $\alpha < \alpha_0$, we have found a nonempty cozero-set B_α of $\mathrm{cl}_K X$ such that $B_\alpha \subset U$ and $\gamma < \alpha < \alpha_0$ implies that $B_\alpha \cap (A_\gamma \cup B_\gamma) = \varnothing$. Now $\bigcup_{\alpha<\alpha_0} A_\alpha \cup B_\alpha$ is a cozero-set of $\mathrm{cl}_K X$ contained in U. If it were dense in U, then as cozero-sets of compact spaces are Lindelöf there would be a countable subcollection $\mathscr{C}$ of $\{A_\alpha : \alpha < \omega_1\}$ whose union covers $\bigcup_{\alpha<\alpha_0} A_\alpha \cup B_\alpha$. Thus $\bigcup \mathscr{C}$ would be dense in U, and so $\{C \cap X : C \in \mathscr{C}\}$ would be a countable subfamily of $\mathscr{V}$ whose union is dense in X, contradicting hypothesis. Thus assume that $U - \mathrm{cl}_K (\bigcup_{\alpha<\alpha_0} A_\alpha \cup B_\alpha) \neq \varnothing$. Hence a nonempty cozero-set B_{α_0} of $\mathrm{cl}_K X$ can be found such that $B_{\alpha_0} \cap (\bigcup_{\alpha<\alpha_0} A_\alpha \cup B_\alpha) = \varnothing$ and $B_{\alpha_0} \subset U$. Now let $B = \bigcup_{\alpha<\omega_1} B_\alpha$. As $\gamma > \alpha$ implies $B_\gamma \cap A_\alpha = \varnothing$, evidently $B \cap A_\alpha = \bigcup_{\gamma \leqq \alpha} B_\gamma \cap A_\alpha \in \mathrm{coz}\,(U)$. Thus by 2.1 B is C^*-embedded in U. But B is the union of ω_1 pairwise disjoint nonempty open subsets of $\mathrm{cl}_K X$, so $|C^*(B)| \geqq 2^{\omega_1}$. Thus $|C^*(U)| \geqq 2^{\omega_1}$ and as X is dense in U, $|C^*(X)| \geqq 2^{\omega_1} > 2^\omega$. Thus (3) fails, and the proof is complete.

REMARKS 2.3. (1) The hypotheses on K in Theorem 2.2 are satisfied by a large class of spaces. One such class is the class of extremally disconnected spaces of weight no greater than 2^ω, such as $\beta\underline{N}$, or the absolute of a compact separable space (see [2] for details concerning absolutes of compact spaces). Another such class is the class of spaces of the form $\beta X - X$, where X is a locally compact σ-compact non-compact space with $|C^*(X)| = 2^\omega$ (see 14.27 of [4]); $\beta \boldsymbol{R} - \boldsymbol{R}$ is such a space, where $\boldsymbol{R}$ denotes the space of real numbers. Under assumption of the continuum hypothesis Theorem 2.2 gives a characterization of the C^*-embedded subspaces of each of these spaces.

(2) Let K satisfy the hypotheses imposed in 2.2. One consequence of 2.2 is that the question of whether a subspace X of K is C^*-embedded in K depends only on the topology of X and not on "how X is placed" in K. In the general case, by contrast, a space T can contain two homeomorphic subspaces, one C^*-embedded in T and the other not. For example the space Q of rational numbers is C^*-embedded in βQ, but its homeomorphs $Q - \{0\}$ and $Q \cap (0, \infty)$, for example, are not C^*-embedded in βQ.

(3) If $2^{\omega_1} = 2^\omega$ then Theorem 2.2 fails; for by 1.6 $\beta D(\omega_1)$ could be topologically embedded in $\beta \underline{N}$. Hence βN would contain a C^*-embedded copy of $D(\omega_1)$, which certainly is not weakly Lindelöf. I do not know whether Theorem 2.2 holds only if the continuum hypothesis holds; neither do I know whether the (possibly weaker) implication (3) $\rightarrow$ (2) can hold in the absence of the continuum hypothesis.

Theorem 2.2 tells us when a subspace of $\beta \underline{N}$ will be C^*-embedded in $\beta \underline{N}$. A slight generalization of a theorem of Louveau (stated below) allows us to characterize (assuming the continuum hypothesis) those Tychonoff spaces that are homeomorphic to some C^*-embedded subspace of $\beta \underline{N}$. The following theorem appears in [5].

THEOREM 2.4 [CH]. *A compact space K is homeomorphic to a subspace of $\beta \underline{N}$ iff K is a zero-dimensional F-space and $w(K) \leqq 2^\omega$.*

THEOREM 2.5 [CH]. *The following are equivalent for a space X:*
(1) *X is a strongly zero-dimensional F-space and $|C^*(X)| = 2^\omega$.*
(2) *X is homeomorphic to a C^*-embedded subspace of $\beta \underline{N}$.*

Proof.
(1) $\rightarrow$ (2): By 1.3 βX is a compact zero-dimensional F-space and $|C^*(\beta X)| = 2^\omega$. Thus $w(\beta X) \leqq 2^\omega$ so by 2.4 there is a compact subspace K of $\beta \underline{N}$ and a homeomorphism $h: \beta X \rightarrow K$. Evidently $h[X]$ is homeomorphic to X and C^*-embedded in $\beta \underline{N}$.

(2) $\rightarrow$ (1): By hypothesis $\mathrm{cl}_{\beta \underline{N}} X = \beta X$. Thus βX is zero-dimensional so X is strongly zero-dimensional. As βX is C^*-embedded in $\beta \underline{N}$, by 1.3 βX is an F-space and $|C^*(\beta X)| = 2^\omega$. Hence $|C^*(X)| = 2^\omega$ and by 1.3 X is an F-space.

One interesting consequence of 2.2 and 2.5 is that if the continuum hypothesis is assumed, if $X \subset \beta \underline{N}$ and $|C^*(X)| = 2^\omega$ then X is a strongly zero-dimensional F-space.

3. Extremally disconnected spaces of countable cellularity. By combining 1.1, 1.5, and 1.6 we obtain the following.

THEOREM 3.1. *Let X be a Tychonoff space of countable cellularity. The following are equivalent:*

(1) *X is extremally disconnected.*

(2) *X is homeomorphic to a subspace of $\beta D(\log w(\beta X))$.*

Further, if X is homeomorphic to a subspace Y of $\beta D(\alpha)$ for some α, then Y is C^*-embedded in $\beta D(\alpha)$.

Proof. Let $\beta D(\log w(\beta X)) = K$.

(1) $\rightarrow$ (2): βX is extremally disconnected (see 6M of [4]), so by 1.6 βX can be embedded in K.

(2) $\rightarrow$ (1): We may assume $X \subset K$. As K is extremally disconnected and hence an F-space, its C^*-embedded subspace $\mathrm{cl}_K X$ is an F-space. But $\mathrm{cl}_K X$ has c.c.c. as X has; hence by 1.1 and 1.4 $\mathrm{cl}_K X$ is extremally disconnected. Thus by 1.2 X is extremally disconnected. The final statement of the theorem follows from 1.1 and 1.5.

COROLLARY 3.2. *A separable Tychonoff space is extremally disconnected iff it is homeomorphic to a subspace of $\beta \underline{N}$.*

Proof. If X is separable then $w(\beta X) \leqq 2^\omega$ (as βX will have no more than 2^ω regular open subsets), so $\log (w(\beta X)) = \omega$.

We now consider extremally disconnected C^*-embedded subspaces of $\beta \underline{N}$. Note that 3.1 says that a subspace $\beta \underline{N}$ having c.c.c. will be extremally disconnected and C^*-embedded in $\beta \underline{N}$. The following theorem describes when the converse holds.

THEOREM 3.3. *The following are equivalent:*

(1) $2^{\omega_1} > 2^\omega$.

(2) *Each extremally disconnected C^*-embedded subspace of $\beta \underline{N}$ has* c.c.c.

Proof.

(1) $\rightarrow$ (2): Suppose X were an extremally disconnected C^*-embedded subspace of $\beta \underline{N}$ but that X does not have c.c.c. Let $\mathscr{M}$ be a family of ω_1 pairwise disjoint open subsets of X. By 1.2 $\bigcup \mathscr{M}$ is C^*-embedded in X and hence in $\beta \underline{N}$. But evidently $|C^*(\bigcup \mathscr{M})| \geqq 2^{\omega_1}$; thus $|C^*(\beta \underline{N})| \geqq 2^{\omega_1}$. But $|C^*(\beta \underline{N})| = 2^\omega$ so $2^\omega = 2^{\omega_1}$. Hence if (2)

fails, so does (1).

(2) $\rightarrow$ (1): If $2^{\omega_1} = 2^{\omega}$ then $w(\beta D(\omega_1)) = 2^{\omega}$. Thus by 1.6 $\beta D(\omega_1)$ can be embedded in $\beta \underline{N}$. Hence there will be a C^*-embedded copy of $D(\omega_1)$ in $\beta \underline{N}$, and $D(\omega_1)$ is extremally disconnected but does not have c.c.c. Hence if (1) fails, so does (2).

REMARKS 3.4. (1) Part of 3.3 appears as Corollary 10 of [2], where it is shown that $2^{\omega} < 2^{\omega_1}$ iff each compact extremally disconnected space of weight 2^{ω} has c.c.c.

(2) Not every compact subspace of $\beta \underline{N}$ with c.c.c. is separable. Let B denote the Boolean algebra of Lebesgue measurable subsets of the unit interval, modulo sets of measure zero, and let X denote the Stone space of B. Then X is compact, extremally disconnected, has c.c.c., is not separable, and $w(X) = 2^{\omega}$. Hence X can be embedded in $\beta \underline{N}$. A discussion of X, together with references to further sources of information about it, may be found in Example 7.5 of [7].

(3) In Remark 2.3 (2) we have seen that if $2^{\omega_1} = 2^{\omega}$ then $\beta \underline{N}$ has *some* discrete C^*-embedded subspaces of cardinality ω_1. It would be interesting to know whether it is consistent that *every* discrete subspace of $\beta \underline{N}$ of cardinality ω_1 is C^*-embedded in $\beta \underline{N}$. More generally, if one assumes, say, Martin's axiom [MA] but not CH, is it true that each discrete subspace of $\beta \underline{N}$ of cardinality less than 2^{ω} is C^*-embedded in $\beta \underline{N}$? (It is known that MA plus not CH implies that if $\kappa < 2^{\omega}$ then $2^{\kappa} = 2^{\omega}$; see, for example, page 21 of [6].)

(4) There is an interesting parallel between Theorems 2.2 and 3.3 as follows. Lemma 1.2 of [1] asserts that each cozero-set of a weakly Lindelöf space is weakly Lindelöf. Hence assuming the continuum hypothesis, a subspace of $\beta \underline{N}$ is C^*-embedded in $\beta \underline{N}$ iff all its cozero-sets are weakly Lindelöf; it is extremally disconnected and C^*-embedded in $\beta \underline{N}$ iff all its open subsets are weakly Lindelöf.

REFERENCES

1. W. W. Comfort, N. Hindman, and S. Negrepontis, *F'-spaces and their products with P-spaces*, Pacific J. Math., **28** (1969), 489-502.

2. B. A. Efimov, *Extremally disconnected compact spaces and absolutes*, Trans. Moscow Math. Soc., **23** (1970), 243-285.

3. N. J. Fine and L. Gillman, *Extensions of continuous functions in βN*, Bull. Amer. Math. Soc., **66** (1960), 376-381.

4. L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton: Van Nostrand (1960).

5. A. Louveau, *Caracterisation des sous-espaces compacts de βN*, Bull. Sci. Math. 2e série, **97** (1973), 259-263.

6. M. E. Rudin, *Lectures on set-theoretic topology*, C.B.M.S. Regional Conference series in mathematics No. 23 (1974).

7. F. D. Tall, *The countable chain condition versus separability-applications of Martin's axiom*, General Topology and its Applications, **4** (1974), 315-339.

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