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**THE BOUNDARY BEHAVIOR OF HENKIN'S KERNEL**

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**In this paper, the boundary behaviour of a reproducing kernel, introduced by Henkin, for strictly pseudoconvex domains is studied. As an application, an improved version of a known result about generators of certain maximal ideals is given.**

The boundary behaviour of the Bergmann kernel  $B(z, \zeta)$  for a strictly pseudoconvex domain has been studied by Bergmann [1] and Hörmander [5]. Among other things, they determine the rate at which  $B(z, z)$  goes to infinity as  $z$  approaches a boundary point of the domain. Another type of reproducing kernel has been introduced by Henkin [3] for bounded strictly pseudoconvex domains  $D$ , in  $\mathbb{C}^n$ . Henkin's kernel is of the form  $K(\zeta, z)/\Phi^n(\zeta, z)$ , where  $K$  and  $\Phi$  are holomorphic in a neighborhood of  $\bar{D}$  for each  $\zeta$  in  $\partial D$ , the boundary of  $D$ . The denominator  $\Phi$  has the properties that  $\Phi(\zeta, \zeta) = 0$  for all  $\zeta \in \partial D$  and that  $\Phi(\zeta, z) \neq 0$  if  $z \in \bar{D} \setminus \{\zeta\}$ . For  $z$  near  $\zeta$ ,  $\Phi$  is given explicitly (up to a nonvanishing factor) in terms of the plurisubharmonic function  $\rho$  that defines the domain  $D$ . Precise statements about the way  $\Phi(\zeta, z)$  approaches zero as  $z$  approaches  $\zeta$  from inside  $D$  are given in Henkin's paper [3]. We show that this determines the behaviour of the kernel  $K/\Phi^n$  by showing that  $K(\zeta, \zeta) \neq 0$ .

It has been proven in [4], [6], [7] and [9], that if  $f$  is in the space  $A(D)$  of functions continuous on  $\bar{D}$  and holomorphic in  $D$  and if  $a \in D$  then there exist functions  $g_1, \dots, g_n \in A(D)$  such that

$$f(z) - f(a) = \sum_{j=1}^n (z_j - a_j) g_j(z).$$

This is a solution to a problem originally posed by Gleason [2] for the unit ball in  $\mathbb{C}^n$ . Using Henkin's integral formula and our result on the behaviour of Henkin's kernel we can improve the result just stated in two ways. Firstly, we show that the  $g_i$  can be chosen in such a way that the association between  $f$  and the  $n$ -tuple of functions  $(g_1, \dots, g_n)$  is linear, and secondly we show that the  $g_i$  may be also chosen to depend analytically on  $a$  as well as on  $z$ .

**1. Notation.**  $D$  will always denote a bounded strictly pseudoconvex domain in  $\mathbb{C}^n$  defined as  $D = \{z : \rho(z) < 0\}$ , where  $\rho$  is defined and strictly plurisubharmonic in a neighborhood  $U$  of  $\bar{D}$ , such that the gradient of  $\rho$  is not zero on the boundary of  $D$ . For  $\epsilon > 0$  we let

$D_\epsilon = \{z \in U : \rho(z) < \epsilon\}$  and if  $V$  is a neighborhood of  $\partial D$  we let  $V_\epsilon = V \cap D_\epsilon$ . We denote by  $C^k(V_\epsilon, H(D_\epsilon))$  the space of  $C^k$  functions on  $V_\epsilon$  with values in the space  $H(D_\epsilon)$  of functions holomorphic in  $D_\epsilon$ . In other words, functions that are  $C^k$  on  $V_\epsilon \times D_\epsilon$  and holomorphic in  $D_\epsilon$  for each fixed  $\zeta \in V_\epsilon$ . Finally, we let  $S_{z,\delta} = \{\zeta : |\zeta - z| < \delta\}$ .

2. The work of Henkin [3], modified slightly by Øvrelid [8], shows that if  $D$  has a  $C^3$  boundary then there are functions  $K$  and  $\Phi$  and a neighborhood  $V$  of  $\partial D$  and an  $\epsilon > 0$  such that:

2.1. (a)  $K \in C^1(V_\epsilon, H(D_\epsilon))$  and

$$\Phi \in C^2(V_\epsilon, H(D_\epsilon)).$$

(b)  $\Phi(\zeta, z) \neq 0$  if  $z \in \bar{D} \setminus \{\zeta\}$ .

2.2. If  $f \in A(D)$  then

$$f(z) = \int_{\partial D} f(\zeta) \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} d\sigma(\zeta), \quad \text{for all } z \in D,$$

where  $d\sigma$  is  $2n - 1$  dimensional volume measure on  $\partial D$ .

2.3. There are constants  $\gamma, \delta_0 > 0$  such that for all  $z \in \bar{D}$  and  $0 < \delta < \delta_0$ ,

$$\int_{\partial D \cap S_{z,\delta}} \frac{|\zeta - z|}{|\Phi^n(\zeta, z)|} d\sigma(\zeta) \leq \gamma \delta \log \frac{1}{\delta}.$$

**THEOREM A.** *Suppose  $K$  and  $\Phi$  satisfy properties 2.1, 2.2, and 2.3, then  $K(\zeta_0, \zeta_0) \neq 0$  for any  $\zeta_0 \in \partial D$ .*

*Proof.* We assume that  $K(\zeta_0, \zeta_0) = 0$  and arrive at a contradiction. If  $K(\zeta_0, \zeta_0)$  were zero then, from property 2.1, there would be a constant  $M$  such that

- (a)  $|K(\zeta, \zeta_0)| \leq M|\zeta - \zeta_0|$ ,
- (b)  $|K(\zeta, z) - K(\zeta_0, z)| \leq M|\zeta - \zeta_0|$ ,
- (c)  $|K(\zeta_0, z)| \leq M|z - \zeta_0|$ .

Now it follows from (a) and 2.3 that

$$\int_{\partial D} \frac{|K(\zeta, \zeta_0)|}{|\Phi^n(\zeta, \zeta_0)|} d\sigma(\zeta) < \infty.$$

We will show that if  $f \in A(D)$ , then

$$2.4. \quad f(\zeta_0) = \int_{\partial D} f(\zeta) \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} d\sigma(\zeta).$$

Due to the remark just made, the right hand side of 2.4 is well-defined. To prove 2.4 we show that as  $z$  approaches  $\zeta_0$  in a certain way, the expression,

$$2.5. \quad f(z) - \int_{\partial D} f(\zeta) \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} d\sigma(\zeta)$$

converges to 0. Now by 2.2 we have,

$$\begin{aligned} f(z) - \int_{\partial D} f(\zeta) \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} d\sigma(\zeta) &= \int_{\partial D} f(\zeta) \left[ \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} - \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \right] d\sigma(\zeta) \\ &= \int_{\partial D \setminus S_{\zeta_0, \delta}} f(\zeta) \left[ \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} - \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \right] d\sigma(\zeta) \\ &\quad + \int_{\partial D \cap S_{\zeta_0, \delta}} f(\zeta) \left[ \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} - \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \right] d\sigma(\zeta). \end{aligned}$$

Now for any fixed  $\delta > 0$ , the first integral above approaches zero as  $z$  approaches  $\zeta_0$ , since we can take the limit under the integral sign. As for the second integral, its absolute value is not greater than

$$\int_{\partial D \cap S_{\zeta_0, \delta}} |f(\zeta)| \frac{|K(\zeta, \zeta_0)|}{|\Phi^n(\zeta, \zeta_0)|} d\sigma(\zeta) + \int_{\partial D \cap S_{\zeta_0, \delta}} |f(\zeta)| \frac{|K(\zeta, z)|}{|\Phi^n(\zeta, z)|} d\sigma(\zeta).$$

Now by (a) and 2.3, the first of these integrals is majorized by  $M\|f\|_{\infty}\gamma\delta \log 1/\delta$ . To estimate the second of these integrals we let  $z$  approach  $\zeta_0$  along the inward normal to  $\partial D$ . Now if  $z$  lies on this normal and if  $\delta$  is sufficiently small then there is a constant  $C$  such that  $|z - \zeta_0| \leq C|z - \zeta|$  and  $|\zeta - \zeta_0| \leq C|z - \zeta|$  as long as  $|z - \zeta_0| < \delta$  and  $|\zeta - \zeta_0| < \delta$ , and hence  $|K(\zeta, z)| \leq |K(\zeta, z) - K(\zeta_0, z)| + |K(\zeta_0, z)| \leq M|\zeta - \zeta_0| + M|z - \zeta_0| \leq 2MC|\zeta - z|$ . So with these assumptions,

$$\begin{aligned} \int_{\partial D \cap S_{\zeta_0, \delta}} |f(\zeta)| \frac{|K(\zeta, z)|}{|\Phi^n(\zeta, z)|} d\sigma(\zeta) &\leq \|f\|_{\infty} 2MC \int_{\partial D \cap S_{\zeta_0, \delta}} \frac{|z - \zeta|}{|\Phi^n(\zeta, z)|} d\sigma(\zeta) \\ &\leq 2MC \|f\|_{\infty} \int_{\partial D \cap S_{z, 2\delta}} \frac{|z - \zeta|}{|\Phi^n(\zeta, z)|} d\sigma(\zeta) \\ &\leq 2MC \|f\|_{\infty} \gamma 2\delta \log \frac{1}{2\delta}, \text{ if } 2\delta < \delta_0. \end{aligned}$$

So now if we first choose  $\delta$  sufficiently small and then let  $z$  approach  $\zeta_0$

along the inward normal we see that 2.5 approaches zero. This proves 2.4. Now it is easy to finish the proof of the theorem. We take  $f \in A(D)$  such that  $f(\zeta_0) = 1$  and  $|f(\zeta)| < 1$  for  $\zeta \in \bar{D} \setminus \{\zeta_0\}$ . Applying 2.4 to  $f^N$  we get

$$1 = f^N(\zeta_0) = \int_{\partial D} f^N(\zeta) \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} d\sigma(\zeta).$$

However the right hand side approaches zero, by the bounded convergence theorem. This contradiction completes the proof of Theorem A.

We now apply Theorem A to obtain

**THEOREM B.** *Suppose  $D$  is a bounded strictly pseudoconvex domain in  $\mathbf{C}^n$  with a  $C^3$  boundary. There is a linear mapping  $T: A(D) \rightarrow H(D \times D)^n$  such that  $(Tf)_i \in C[(\bar{D} \times \bar{D}) \setminus \{(z, z) : z \in \partial D\}]$  for every  $f \in A(D)$  and such that*

$$f(z) - f(\omega) = \sum (z_i - \omega_i)(Tf)_i(z, \omega).$$

*Proof.* From Henkin's integral formula we see that

$$f(z) - f(\omega) = \int f(\zeta) \frac{\Phi^n(\zeta, \omega)K(\zeta, z) - \Phi^n(\zeta, z)K(\zeta, \omega)}{\Phi^n(\zeta, z)\Phi^n(\zeta, \omega)} d\sigma(\zeta).$$

If  $L(\zeta, z, \omega) = \Phi^n(\zeta, \omega)K(\zeta, z) - \Phi^n(\zeta, z)K(\zeta, \omega)$ , then  $L \in C^1(V_{\epsilon'}, H(D_{\epsilon'} \times D_{\epsilon'}))$  and  $L(\zeta, z, z) \equiv 0$ , so by the argument given as a remark on page 148 of [8] there are functions  $L_i \in C^1(V_{\epsilon'}, H(D_{\epsilon'} \times D_{\epsilon'}))$  (for some  $\epsilon' < \epsilon$ ) such that

$$L(\zeta, z, \omega) = \sum_{i=1}^n (z_i - \omega_i)L_i(\zeta, z, \omega).$$

Hence, we have

$$f(z) - f(\omega) = \sum_{i=1}^n (z_i - \omega_i) \int f(\zeta) \frac{L_i(\zeta, z, \omega)}{\Phi^n(\zeta, z)\Phi^n(\zeta, \omega)} d\sigma(\zeta).$$

So it remains to show that

$$f_i(z, \omega) = \int f(\zeta) \frac{L_i(\zeta, z, \omega)}{\Phi^n(\zeta, z)\Phi^n(\zeta, \omega)} d\Phi(\zeta)$$

satisfies the statement of the theorem. Certainly  $f_i \in H(D \times D)$  so we need only show that  $f_i \in C[(\bar{D} \times \bar{D}) \setminus \{(z, z) : z \in \partial D\}]$ . Suppose  $(z, \omega) \in D \times D$  and  $(z, \omega) \rightarrow (\zeta_0, \omega_0) \in \bar{D} \times \bar{D} \setminus \{(z, z) : z \in \partial D\}$ . We wish to show that  $f_i(z, \omega)$  has a limit. We will assume that  $\zeta_0 \in \partial D$  and  $\omega_0 \in \partial D$  and  $\zeta_0 \neq \omega_0$ . The other possibilities are treated in a similar fashion (and are easier). By Theorem A,  $K(\zeta_0, \zeta_0) \neq 0$ , and  $K(\omega_0, \omega_0) \neq 0$ . Hence there is a  $\delta > 0$  such that if  $|z - \zeta_0| \leq 2\delta$  and  $|\zeta - \zeta_0| \leq 2\delta$  then  $K(z, \zeta) \neq 0$ , and if  $|z - \omega_0| \leq 2\delta$  and  $|\zeta - \omega_0| \leq 2\delta$  then  $K(z, \zeta) \neq 0$ . We also assume  $4\delta < |\zeta_0 - \omega_0|$ . Let  $\varphi(z)$  be a  $C^\infty$  function that is identically equal to 1 if  $|z| \leq \delta^2$  and identically 0 if  $|z| \geq (2\delta)^2$ . Now we write

$$\begin{aligned} f_i(z, \omega) = & \int f(\zeta) \frac{L_i(\zeta, z, \omega)\varphi(|z - \zeta|^2)}{K(\zeta, z)\Phi^n(\zeta, \omega)} \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} d\sigma(\zeta) \\ & + \int f(\zeta) \frac{L_i(\zeta, z, \omega)\varphi(|\omega - \zeta|^2)}{K(\zeta, \omega)\Phi^n(\zeta, z)} \frac{K(\zeta, \omega)}{\Phi^n(\zeta, \omega)} d\sigma(\zeta) \\ & + \int f(\zeta) \frac{L_i(\zeta, z, \omega)}{\Phi^n(\zeta, z)\Phi^n(\zeta, \omega)} [1 - \varphi(|z - \zeta|^2) - \varphi(|\omega - \zeta|^2)] d\sigma(\zeta), \end{aligned}$$

for  $|z - \zeta_0| < \delta$  and  $|\omega - \omega_0| < \delta$ . The third term has a limit as  $(z, \omega) \rightarrow (\zeta_0, \omega_0)$  since we may take the limit under the integral sign. We write the first term as

$$2.6. \quad \int f(\zeta)\chi(\zeta, z, \omega) \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} d\sigma(\zeta),$$

where all we need to know about  $\chi$  is that it is continuous on  $\partial D \times \bar{D} \times S_{\omega, \delta}$  and that there is a constant  $C$  such that  $|\chi(\zeta, z, \omega) - \chi(\zeta', z, \omega)| \leq C|\zeta - \zeta'|$ , for all  $z, \omega, \zeta$ , and  $\zeta'$ . Now we just imitate the proof of Lemma 4.3 of [3] to see that 2.6 has a limit as  $(z, \omega) \rightarrow (\zeta_0, \omega_0)$ . The second term is handled in the same way as the first. This completes the proof.

Note that if

$$f(z) - f(\omega) = \sum_{i=1}^n (z_i - \omega_i)g_i(z, \omega) \quad \text{then} \quad \frac{\partial f}{\partial z_i}(z) = g_i(z, z),$$

so so that  $g_i$  need not be in  $A(D \times D)$  when  $f \in A(D)$ .

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