THE BOUNDARY BEHAVIOR OF HENKIN’S KERNEL

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In this paper, the boundary behaviour of a reproducing kernel, introduced by Henkin, for strictly pseudoconvex domains is studied. As an application, an improved version of a known result about generators of certain maximal ideals is given.

The boundary behaviour of the Bergmann kernel $B(z, \zeta)$ for a strictly pseudoconvex domain has been studied by Bergmann [1] and Hörmander [5]. Among other things, they determine the rate at which $B(z, z)$ goes to infinity as $z$ approaches a boundary point of the domain. Another type of reproducing kernel has been introduced by Henkin [3] for bounded strictly pseudoconvex domains $D$, in $\mathbb{C}^n$. Henkin’s kernel is of the form $K(\zeta, z)/\Phi^n(\zeta, z)$, where $K$ and $\Phi$ are holomorphic in a neighborhood of $\bar{D}$ for each $\zeta$ in $\partial D$, the boundary of $D$. The denominator $\Phi$ has the properties that $\Phi(\zeta, \zeta) = 0$ for all $\zeta \in \partial D$ and that $\Phi(\zeta, z) \neq 0$ if $z \in \bar{D}\setminus\{\zeta\}$. For $z$ near $\zeta$, $\Phi$ is given explicitly (up to a nonvanishing factor) in terms of the plurisubharmonic function $\rho$ that defines the domain $D$. Precise statements about the way $\Phi(\zeta, z)$ approaches zero as $z$ approaches $\zeta$ from inside $D$ are given in Henkin’s paper [3]. We show that this determines the behaviour of the kernel $K/\Phi^n$ by showing that $K(\zeta, \zeta) \neq 0$.

It has been proven in [4], [6], [7] and [9], that if $f$ is in the space $A(D)$ of functions continuous on $\bar{D}$ and holomorphic in $D$ and if $a \in D$ then there exist functions $g_1, \cdots, g_n \in A(D)$ such that

$$f(z) - f(a) = \sum_{j=1}^{n} (z_j - a_j)g_j(z).$$

This is a solution to a problem originally posed by Gleason [2] for the unit ball in $\mathbb{C}^n$. Using Henkin’s integral formula and our result on the behaviour of Henkin’s kernel we can improve the result just stated in two ways. Firstly, we show that the $g_i$ can be chosen in such a way that the association between $f$ and the $n$-tuple of functions $(g_1, \cdots, g_n)$ is linear, and secondly we show that the $g_i$ may be also chosen to depend analytically on $a$ as well as on $z$.

1. Notation. $D$ will always denote a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ defined as $D = \{z : \rho(z) < 0\}$, where $\rho$ is defined and strictly plurisubharmonic in a neighborhood $U$ of $\bar{D}$, such that the gradient of $\rho$ is not zero on the boundary of $D$. For $\epsilon > 0$ we let
\(D_e = \{z \in U : \rho(z) < \epsilon\}\) and if \(V\) is a neighborhood of \(\partial D\) we let \(V_e = V \cap D_e\). We denote by \(C^k(V_e, H(D_e))\) the space of \(C^k\) functions on \(V_e\) with values in the space \(H(D_e)\) of functions holomorphic in \(D_e\). In other words, functions that are \(C^k\) on \(V_e \times D_e\) and holomorphic in \(D_e\) for each fixed \(\zeta \in V_e\). Finally, we let \(S_{\zeta, \delta} = \{\zeta : |\zeta - z| < \delta\}\).

2. The work of Henkin [3], modified slightly by Ovrelid [8], shows that if \(D\) has a \(C^3\) boundary then there are functions \(K\) and \(\Phi\) and a neighborhood \(V\) of \(\partial D\) and an \(\epsilon > 0\) such that:

2.1. (a) \(K \in C^1(V_e, H(D_e))\) and 

\[\Phi \in C^2(V_e, H(D_e)).\]

(b) \(\Phi(\zeta, z) \neq 0\) if \(z \in \overline{D} \setminus \{\zeta\}\).

2.2. If \(f \in A(D)\) then 

\[f(z) = \int_{\partial D} f(\zeta) \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} d\sigma(\zeta), \quad \text{for all } z \in D,
\]

where \(d\sigma\) is \(2n - 1\) dimensional volume measure on \(\partial D\).

2.3. There are constants \(\gamma, \delta_0 > 0\) such that for all \(z \in \overline{D}\) and \(0 < \delta < \delta_0\),

\[\int_{\partial D \cap S_{\zeta, \delta}} \frac{|\zeta - z|}{|\Phi^n(\zeta, z)|} d\sigma(\zeta) \leq \gamma \delta \log \frac{1}{\delta}.
\]

**Theorem A.** Suppose \(K\) and \(\Phi\) satisfy properties 2.1, 2.2, and 2.3, then \(K(\zeta_0, \zeta_0) \neq 0\) for any \(\zeta_0 \in \partial D\).

**Proof.** We assume that \(K(\zeta_0, \zeta_0) = 0\) and arrive at a contradiction. If \(K(\zeta_0, \zeta_0)\) were zero then, from property 2.1, there would be a constant \(M\) such that

(a) \(|K(\zeta, \zeta_0)| \leq M |\zeta - \zeta_0|\),

(b) \(|K(\zeta, z) - K(\zeta_0, z)| \leq M |\zeta - \zeta_0|\),

(c) \(|K(\zeta_0, z)| \leq M |z - \zeta_0|\).

Now it follows from (a) and 2.3 that 

\[\int_{\partial D} \frac{|K(\zeta, \zeta_0)|}{|\Phi^n(\zeta, \zeta_0)|} d\sigma(\zeta) < \infty.
\]

We will show that if \(f \in A(D)\), then
Due to the remark just made, the right hand side of 2.4 is well-defined. To prove 2.4 we show that as \( z \) approaches \( \zeta_0 \) in a certain way, the expression,

\[
f(z) = \int_{\partial D} f(\zeta) \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} \, d\sigma(\zeta)
\]

converges to 0. Now by 2.2 we have,

\[
f(z) - \int_{\partial D} f(\zeta) \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} \, d\sigma(\zeta) = \int_{\partial D} f(\zeta) \left[ \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} - \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \right] \, d\sigma(\zeta)
\]

\[
= \int_{\partial D \cap S_{\zeta_0}} f(\zeta) \left[ \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} - \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \right] \, d\sigma(\zeta)
\]

\[
+ \int_{\partial D \cap S_{\zeta_0}} f(\zeta) \left[ \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} - \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \right] \, d\sigma(\zeta).
\]

Now for any fixed \( \delta > 0 \), the first integral above approaches zero as \( z \) approaches \( \zeta_0 \), since we can take the limit under the integral sign. As for the second integral, its absolute value is not greater than

\[
\int_{\partial D \cap S_{\zeta_0}} |f(\zeta)| \left| \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} \right| \, d\sigma(\zeta) + \int_{\partial D \cap S_{\zeta_0}} |f(\zeta)| \left| \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \right| \, d\sigma(\zeta).
\]

Now by (a) and 2.3, the first of these integrals is majorized by \( M \| f \|_\infty \gamma \delta \log 1/\delta \). To estimate the second of these integrals we let \( z \) approach \( \zeta_0 \) along the inward normal to \( \partial D \). Now if \( z \) lies on this normal and if \( \delta \) is sufficiently small then there is a constant \( C \) such that \( |z - \zeta_0| \leq C |z - \zeta| \) and \( |\zeta - \zeta_0| \leq C |z - \zeta| \) as long as \( |z - \zeta_0| < \delta \) and \( |\zeta - \zeta_0| < \delta \), and hence \( |K(\zeta, z)| \leq |K(\zeta, z) - K(\zeta_0, z)| + |K(\zeta_0, z)| \leq M |\zeta - \zeta_0| + M |z - \zeta_0| \leq 2MC |\zeta - z| \). So with these assumptions,

\[
\int_{\partial D \cap S_{\zeta_0}} |f(\zeta)| \left| \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \right| \, d\sigma(\zeta) \leq \| f \|_\infty 2MC \int_{\partial D \cap S_{\zeta_0}} \left| \frac{z - \zeta}{\Phi^n(\zeta, z)} \right| \, d\sigma(\zeta)
\]

\[
\leq 2MC \| f \|_\infty \gamma 2\delta \log \frac{1}{2\delta}, \quad \text{if} \quad 2\delta < \delta_0.
\]

So now if we first choose \( \delta \) sufficiently small and then let \( z \) approach \( \zeta_0 \)
along the inward normal we see that 2.5 approaches zero. This proves 2.4. Now it is easy to finish the proof of the theorem. We take \( f \in A(D) \) such that \( f(\zeta_0) = 1 \) and \( |f(\zeta)| < 1 \) for \( \zeta \in \overline{D} \setminus \{\zeta_0\} \). Applying 2.4 to \( f^N \) we get

\[
1 = f^N(\zeta_0) = \int_{\partial D} f^N(\zeta) \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} \, d\sigma(\zeta).
\]

However the right hand side approaches zero, by the bounded convergence theorem. This contradiction completes the proof of Theorem A.

We now apply Theorem A to obtain

**Theorem B.** Suppose \( D \) is a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) with a \( C^3 \) boundary. There is a linear mapping \( T: A(D) \rightarrow H(D \times D)^n \) such that \( (Tf)(z,\bar{z}) \in C((D \times \overline{D}) \setminus \{(z, z): z \in \partial D\}) \) for every \( f \in A(D) \) and such that

\[
f(z) - f(\omega) = \sum (z_i - \omega_i)(Tf)(z, \omega).
\]

**Proof.** From Henkin’s integral formula we see that

\[
f(z) - f(\omega) = \int f(\zeta) \frac{\Phi^n(\zeta, \omega)K(\zeta, z) - \Phi^n(\zeta, z)K(\zeta, \omega)}{\Phi^n(\zeta, z)\Phi^n(\zeta, \omega)} \, d\sigma(\zeta).
\]

If \( L(\zeta, z, \omega) = \Phi^n(\zeta, \omega)K(\zeta, z) - \Phi^n(\zeta, z)K(\zeta, \omega) \), then \( L \in C^1(V_\epsilon, H(D_\epsilon \times D_\epsilon)) \) and \( L(\zeta, z, z) \equiv 0 \), so by the argument given as a remark on page 148 of [8] there are functions \( L_i \in C^1(V_\epsilon, H(D_\epsilon \times D_\epsilon)) \) (for some \( \epsilon' < \epsilon \)) such that

\[
L(\zeta, z, \omega) = \sum_{i=1}^n (z_i - \omega_i)L_i(\zeta, z, \omega).
\]

Hence, we have

\[
f(z) - f(\omega) = \sum_{i=1}^n (z_i - \omega_i) \int f(\zeta) \frac{L_i(\zeta, z, \omega)}{\Phi^n(\zeta, z)\Phi^n(\zeta, \omega)} \, d\sigma(\zeta).
\]

So it remains to show that

\[
f_i(z, \omega) = \int f(\zeta) \frac{L_i(\zeta, z, \omega)}{\Phi^n(\zeta, z)\Phi^n(\zeta, \omega)} \, d\Phi(\zeta)
\]
satisfies the statement of the theorem. Certainly \( f \in H(D \times D) \) so we need only show that \( f \in C[(\overline{D} \times \overline{D}) \setminus \{(z, z): z \in \partial D\}] \). Suppose \((z, \omega) \in D \times D \) and \((z, \omega) \rightarrow (\zeta_0, \omega_0) \in \overline{D} \times \overline{D} \setminus \{(z, z): z \in \partial D\} \). We wish to show that \( f(z, \omega) \) has a limit. We will assume that \( \zeta_0 \in \partial D \) and \( \omega_0 \in \partial D \) and \( \zeta_0 \neq \omega_0 \). The other possibilities are treated in a similar fashion (and are easier). By Theorem A, \( K(\zeta_0, \zeta_0) \neq 0 \), and \( K(\omega_0, \omega_0) \neq 0 \). Hence there is a \( \delta > 0 \) such that if \(|z - \zeta_0| \leq 2\delta \) and \(|\zeta - \zeta_0| \leq 2\delta \) then \( K(z, \zeta) \neq 0 \), and if \(|z - \omega_0| \leq 2\delta \) and \(|\zeta - \omega_0| \leq 2\delta \) then \( K(z, \zeta) \neq 0 \). We also assume \( 4\delta < |\zeta_0 - \omega_0| \). Let \( \varphi(z) \) be a \( C^\infty \) function that is identically equal to 1 if \(|z| \leq \delta^2 \) and identically 0 if \(|z| \geq (2\delta)^2 \). Now we write

\[
f(z, \omega) = \int f(\xi) \frac{L_i(\zeta, z, \omega)\varphi(|z - \xi|^2)}{K(\xi, z)\Phi^*(\zeta, \omega)} \frac{K(\xi, z) K(\zeta, \omega)}{\Phi^*(\zeta, \omega)} d\sigma(\xi)
\]

for \(|z - \zeta_0| < \delta \) and \(|\omega - \omega_0| < \delta \). The third term has a limit as \((z, \omega) \rightarrow (\zeta_0, \omega_0) \) since we may take the limit under the integral sign. We write the first term as

\[
2.6. \quad \int f(\xi) \chi(\xi, z, \omega) \frac{K(\xi, z)}{\Phi^*(\zeta, \omega)} d\sigma(\xi),
\]

where all we need to know about \( \chi \) is that it is continuous on \( \partial D \times \overline{D} \times S^s \) and that there is a constant \( C \) such that \(|\chi(\zeta, z, \omega) - \chi(\zeta', z, \omega)| \leq C|\zeta - \zeta'| \), for all \( z, \omega, \zeta, \) and \( \zeta' \). Now we just imitate the proof of Lemma 4.3 of [3] to see that 2.6 has a limit as \((z, \omega) \rightarrow (\zeta_0, \omega_0) \). The second term is handled in the same way as the first. This completes the proof.

Note that if

\[
f(z) - f(\omega) = \sum_{i=1}^{n} (z_i - \omega_i)g_i(z, \omega) \quad \text{then} \quad \frac{\partial f}{\partial z_i}(z) = g_i(z, z),
\]

so that \( g_i \) need not be in \( A(D \times D) \) when \( f \in A(D) \).

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