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**ON SEMIPRIME P.I.-ALGEBRAS OVER COMMUTATIVE  
REGULAR RINGS**

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## ON SEMIPRIME P.I.-ALGEBRAS OVER COMMUTATIVE REGULAR RINGS

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**Let  $R$  be a commutative (von Neumann) regular ring with unit. This paper deals with algebras  $A$  over  $R$ , and following standard conventions  $A$  will be called a finitely generated  $R$ -algebra whenever  $A$  is a finitely generated  $R$ -module. One of the principal results obtained is that all semiprime finitely generated  $R$ -algebras are regular rings. Combining this with a result of J. Wehlen and a theorem of G. Michler and O. Villamayor shows that the finitely generated semiprime algebras over commutative regular rings are precisely the semiprime central separable algebras over regular rings.**

Since any finitely generated algebra over a commutative ring satisfies a polynomial identity, (is a P.I.-algebra), this leads to consideration of semiprime P.I.-algebras with regular center. In general, these will only be semisimple  $I$ -rings. However if the center is also a self-injective ring then the algebra is  $\pi$ -regular; this fact is a consequence of the observation that every semiprime P.I.-algebra is weakly algebraic over its center in the sense that every element is a root of a nonzero polynomial with central coefficients.

It will be assumed throughout that the polynomial identities occurring have at least one coefficient  $\pm 1$ .

**THEOREM 1.** *Let  $A$  be a semiprime finitely generated algebra over a commutative regular ring  $R$ . Then  $A$  is a regular ring.*

*Proof.* For the proof we appeal to a theorem of J. Fisher and R. Snider [5, Theorem 1.1] which says that  $A$  will be a regular ring provided

- (i)  $A/P$  is a regular ring for each prime ideal  $P$  of  $A$ ,
- (ii) the union of any chain of semiprime ideals of  $A$  is a semiprime ideal of  $A$ .

For (i), we first observe that any homomorphic image of  $R$  is a regular ring and hence we may assume that  $A$  is a faithful  $R$ -algebra with  $R$  lying in the center of  $A$ . Now if  $P$  is any prime ideal of  $A$  then  $P \cap R$  is a prime ideal of  $R$  hence is a maximal ideal of  $R$ . Consequently,  $A/P$  is a prime finite-dimensional algebra over the field  $R/P \cap R$  and so  $A/P$  is a simple Artinian ring. Thus (i) is satisfied. For (ii) suppose  $A = Ra_1 + \cdots + Ra_n$ , let  $\{B_\lambda \mid \lambda \in \Lambda\}$  be a chain of semiprime ideals of  $A$  and let  $B = \bigcup_{\lambda \in \Lambda} B_\lambda$ . Let  $x \in A$  with  $xAx \subseteq B$ . Then there is some

$\lambda \in \Lambda$  such that  $x_\lambda x \in B_\lambda$  for  $i = 1, \dots, t$  and so  $xAx \subseteq B_\lambda$ . Hence  $x \in B_\lambda \subseteq B$  since each  $B_\lambda$  is a semiprime ideal. Thus  $B$  is a semiprime ideal. It follows that  $A$  is a regular ring.

Combining Theorem 1 with [12, Theorem 2.3] and [10, Theorem 6.3] we have the following description of finitely generated semiprime algebras over commutative regular rings.

**THEOREM 2.** *Let  $A$  be a finitely generated algebra over a regular ring. The following conditions on  $A$  are equivalent:*

- (i)  $A$  is semiprime.
- (ii)  $A$  is regular.
- (iii)  $A$  is biregular.
- (iv)  $A$  is a semiprime central separable algebra.

*Proof.* The equivalence of (ii) and (iii) is [10, Theorem 6.3] while the equivalence of (iii) and (iv) is [12, Theorem 2.3].

Wehlen has also shown [12, Corollary 2.3.1] that a finitely generated biregular  $R$ -algebra  $A$  is separable over  $R$  if and only if its center is separable over  $R$ . If  $A$  is a finitely generated semiprime  $R$ -algebra with center  $K$  then since  $A$  is  $K$ -separable,  $K$  is a  $K$ -direct summand of  $A$  [4, Lemma 3.1, p. 51] hence  $K$  is an  $R$ -direct summand of  $A$ . Thus  $K$  is also a finitely generated  $R$ -module. On the other hand let  $A$  be a semiprime  $R$ -algebra with  $R$  regular whose center  $K$  is a finitely generated  $R$ -module. Then  $K$  is regular by Theorem 1. If in addition  $A$  is  $K$ -separable then  $A$  is biregular and finitely generated over  $K$  [12, Theorem 2.3]. We thus have

**COROLLARY 1.** *Let  $A$  be a semiprime  $R$ -algebra over a regular ring  $R$ . Then  $A$  is a finitely generated and hence regular  $R$ -algebra if and only if  $A$  is a central separable algebra whose center is a finitely generated  $R$ -module.*

In terms of Hochschild dimension,  $\dim_R A$ , of  $R$ -algebras  $A$  we have via [12, Corollary 2.4]

**COROLLARY 2.** *Let  $A$  be a finitely generated semiprime algebra over a regular ring  $R$ . Then  $\dim_R A = 0$  or  $\dim_R A = \infty$ .*

It may be of some interest to know whether or not commutative regular rings  $R$  are characterized by the property  $\dim_R A = 0$  or  $\infty$  for all semiprime finitely generated  $R$ -algebras  $A$ .

Wehlen has also noted that a finitely generated algebra  $A$  over a regular ring  $R$  has nilpotent and finitely generated Jacobson radical  $J(A)$  [12, Proposition 2.2].

COROLLARY 3. *If  $A$  is a finitely generated algebra over a regular ring  $R$  then  $J(A)$  is nilpotent and  $A/J(A)$  is a regular ring.*

In another direction we consider max-rings; a ring  $K$  is a (left) max-ring if and only if each nonzero left  $K$ -module has a maximal submodule. Commutative max-rings have been characterized by R. Hamsher [6] as rings  $K$  for which  $J(K)$  is  $T$ -nilpotent and  $K/J(K)$  is regular. We obtain a similar result for finitely generated algebras over max-rings, using Theorem 1.

COROLLARY 4. *If  $A$  is a finitely generated algebra over a max-ring,  $R$  then  $A$  is a left and right max-ring. Thus  $J(A)$  is left and right  $T$ -nilpotent and  $A/J(A)$  is a regular ring.*

*Proof.* We let  $M \neq 0$  be a left  $A$ -module and assume that  $A$  is a faithful  $R$ -algebra with  $R \subseteq \text{center } A$ . Since  $R$  is a max-ring,  $J(R)M \neq M$  [3], so we can assume that  $J(R)M = 0$ . Then  $M$  is a nonzero  $R/J(R)$ -module as well as an  $A/J(R)A$ -module. Since  $A/J(R)A$  is an  $R/J(R)$ -algebra we can assume  $J(R)A = 0$  so  $J(R) = 0$ . By Corollary 3,  $J(A)$  is nilpotent and so  $J(A)M \neq M$ . By Theorem 1,  $A/J(A)$  is regular and hence by [1, Theorem 2(a)]  $M/J(A)M$  has a maximal submodule. That  $J(A)$  is left and right  $T$ -nilpotent results from [3, pp. 470–471].

Now we consider semiprime P.I. rings having regular center. There exist semiprime P.I. rings which are  $\pi$ -regular but not regular [5, Example 1]; such P.I. rings have regular center. An example in [2] shows that semiprime P.I. rings can have regular center without being  $\pi$ -regular. However that example is not a ring with identity, so we now provide one with an identity.

Let  $F(t)$  be the rational functions over a field  $F$ , let

$$K = \begin{pmatrix} F(t) & F(t) \\ F(t) & F(t) \end{pmatrix}, \text{ and } L = \begin{pmatrix} F[t] & F(t) \\ 0 & F \end{pmatrix}.$$

Now if  $A$  consists of all sequences with entries from  $K$  which are eventually constant and in  $L$ , then  $A$  is semiprime with identity and has regular center. Also  $A$  is not  $\pi$ -regular since  $A$  maps onto  $F[t] \oplus F$  which is not  $\pi$ -regular.

We will show that if the center is also self-injective then  $\pi$ -regularity follows. This is a consequence of the following result which may be of some independent interest.

THEOREM 3. *Let  $A$  be a semiprime P.I. ring with center  $R$ . Then every element in  $A$  satisfies a nonzero polynomial with coefficients from*

*R.* Moreover if  $a \in A$  is nilpotent then  $p(x) = \sum_{i=0}^k r_i x^i \in R[X]$  can be chosen so that  $r_k a^k \neq 0$  and  $p(a) = 0$ .

*Proof.* Let  $Q(A)$  be the maximal quotient ring of  $A$  and  $Q(R)$  the maximal quotient ring of  $R$ . By [2, Theorem 2.5 and Theorem 3.7] we have  $Q(R) = \text{center of } Q(A)$  and  $Q(A)$  is a finitely generated  $Q(R)$ -module. Let  $a \in A$ ; if  $a$  is nilpotent then of course  $a$  satisfies a monic polynomial in  $R[X]$  so assume  $a$  is not nilpotent. We consider  $a$  as a  $Q(R)$ -homomorphism from  $Q(A)$  into  $Q(A)$  so by the Cayley–Hamilton theorem for finitely generated modules over a commutative ring,  $a$  satisfies a monic polynomial in  $Q(R)[x]$ , say  $a^k + u_{k-1}a^{k-1} + \cdots + u_1a + u_0 = 0$ . Since  $Q(R)$  is the maximal quotient ring of  $R$ , there is an essential ideal  $D$  of  $R$  such that  $Du_i \subseteq R$  for  $0 \leq i \leq k-1$ . Further  $AD$  is essential in  $A$  [19, Theorem 6] hence  $ADa^k \neq 0$  so  $Da^k \neq 0$ . Hence choosing  $r = r_k \in D$  so that  $ra^k \neq 0$  and letting  $r_i = ru_i$ ,  $1 \leq i \leq k-1$ , we see that  $a$  is a root of  $p(x) = \sum_{i=0}^k r_i x^i$  with  $r_k a^k \neq 0$ .

Observe that the degree  $k$  of the polynomial satisfied by  $a$  is independent of  $a$  since, as the proof shows,  $k$  is at most the number of generators for  $Q(A)$  over  $Q(R)$ . From the proof we deduce the following:

**THEOREM 4.** *If  $A$  is a semiprime P.I. ring whose center  $R$  is a self-injective ring then  $A$  is integral over  $R$ , hence  $A$  is a  $\pi$ -regular ring.*

*Proof.*  $Q(R) = R$  so  $A$  is integral over  $R$ . It follows that primes of  $A$  are maximal [7] so  $A$  is  $\pi$ -regular [5, Theorem 2.3].

We now show how to obtain rings as in Theorem 4. Let  $A$  be any semiprime P.I. ring with center  $R$  and let  $C = Q(R)$  so that  $C$  is a regular self-injective ring. In  $Q(A)$  we form the ring  $S = AC$ . Following Martindale [8] we call  $S$  the *central closure* of  $A$ . In general  $S \neq Q(A)$ .

**THEOREM 5.** *Let  $S = AC$  be the central closure of the semiprime P.I. ring  $A$ . Then  $C = \text{center } S$  hence  $S$  is a  $\pi$ -regular ring integral over  $C$ . Furthermore  $S = Q(A)$  if and only if  $S$  is a finitely generated  $C$ -module.*

*Proof.* Since  $A \subseteq S \subseteq Q(A)$  we have  $Q(S) = Q(A)$  and so  $C = \text{center } S$ , since  $C$  is the maximal quotient ring of the center of  $S$ . From Theorem 4,  $S$  is  $\pi$ -regular and integral over  $C$  and is a finitely generated  $C$ -module if  $S = Q(A)$  [2, Theorem 3.7]. On the other hand if  $S$  is finitely generated as a  $C$ -module then because  $S$  is a nonsingular  $C$ -module and  $C$  is a regular self-injective ring,  $S$  is then an injective

$C$ -module [11, Theorem 2.5]. However  $S$  is an essential  $C$ -submodule of  $Q(A)$  and so  $S = Q(A)$ .

It seems reasonable to conjecture that if  $A$  is a regular ring with a polynomial identity and  $M$  is a finitely generated  $A$ -module then  $\text{End}_A(M)$  is a  $\pi$ -regular ring. This is certainly true if  $A$  is a commutative or if  $A$  is finitely generated over its center. For  $\text{End}_A(M)$  is an integral extension of a regular subring of its center and so has its prime ideals maximal. Our concluding results are related to this conjecture.

**THEOREM 6.** *Let  $A$  be a semiprime ring with a polynomial identity and let  $M$  be finitely generated flat left  $A$ -module. Then every  $A$ -endomorphism of  $M$  which is onto is also one-to-one.*

*Proof.* Let  $Q$  denote the maximal quotient ring of  $A$  and let  $M^* = Q \otimes_A M$ . Since  $M$  is a flat left  $A$ -module the mapping from  $M$  to  $M^*$  given by  $m \rightarrow 1 \otimes m$  is an  $A$ -monomorphism. Since  $M$  is a finitely generated  $A$ -module,  $M^*$  is a finitely generated  $Q$ -module. However  $Q$  is a regular ring which is a finitely generated algebra over its center [2]. Thus  $E = \text{End}_O(M^*)$  is a  $\pi$ -regular ring. If  $f \in \text{End}_A(M)$  then  $f^* = 1 \otimes f \in E$  and the mapping  $f \rightarrow f^*$  is a ring isomorphism from  $\text{End}_A(M)$  into  $E$ , since  $m \rightarrow 1 \otimes m$  is an isomorphism. Finally if  $f$  is onto then  $f^*$  is also onto. Because  $E$  is  $\pi$ -regular,  $f^*$  is invertible in  $E$ . It follows then that  $f$  is one-to-one.

**COROLLARY 7.** *Let  $A$  be a regular ring with a polynomial identity and  $M$  a finitely generated  $A$ -module. Then every onto endomorphism of  $M$  is one-to-one.*

We remark that J. Fisher and R. Snider have constructed an example (unpublished) of a ring with a polynomial identity having a finitely generated module for which not all onto endomorphisms are one-to-one.

## REFERENCES

1. E. P. Armendariz and J. W. Fisher, *Regular P.I.-rings*, Proc. Amer. Math. Soc., **39** (1973), 247–251.
2. E. P. Armendariz and S. A. Steinberg, *Regular self-injective rings with a polynomial identity*, Trans. Amer. Math. Soc., **190** (1974), 417–425.
3. H. Bass, *Finitistic dimension and a homological generalization of semiprimary rings*, Trans. Amer. Math. Soc., **95** (1960), 466–488.
4. F. DeMeyer and E. C. Ingraham, *Separable algebras over commutative rings*, Lecture Notes in Mathematics, Vol. **181**, Springer-Verlag, New York and Berlin, 1971.
5. J. W. Fisher and R. L. Snider, *On the von Neumann regularity of rings with regular prime factor rings*, Pacific J. Math., **54** (1974), 135–144.

6. R. Hamsher, *Commutative rings over which every module has a maximal submodule*, Proc. Amer. Math. Soc., **18** (1967), 1133–1137.
7. A. G. Heinecke, *A remark about noncommutative integral extensions*, Canad. Math. Bull., **13** (1970), 359–361.
8. W. S. Martindale, III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra, **12** (1969), 576–584.
9. ———, *On semiprime P.I.-rings*, Proc. Amer. Math. Soc., **40** (1973), 364–369.
10. G. Michler and O. Villamayor, *On rings whose simple modules are injective*, J. Algebra, **25** (1973), 185–201.
11. F. L. Sandomierski, *Nonsingular rings*, Proc. Amer. Math. Soc., **19** (1968), 225–230.
12. J. A. Wehlen, *Algebras over absolutely flat commutative rings*, Trans. Amer. Math. Soc., **196** (1974), 149–160.

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Helen Elizabeth Adams, <i>Factorization-prime ideals in integral domains</i> .....	1
Patrick Robert Ahern and Robert Bruce Schneider, <i>The boundary behavior of Henkin's kernel</i> .....	9
Daniel D. Anderson, Jacob R. Matijevic and Warren Douglas Nichols, <i>The Krull intersection theorem. II</i> .....	15
Efraim Pacillas Armendariz, <i>On semiprime P.I.-algebras over commutative regular rings</i> .....	23
Robert H. Bird and Charles John Parry, <i>Integral bases for bicyclic biquadratic fields over quadratic subfields</i> .....	29
Tae Ho Choe and Young Hee Hong, <i>Extensions of completely regular ordered spaces</i> .....	37
John Dauns, <i>Generalized monofrom and quasi injective modules</i> .....	49
F. S. De Blasi, <i>On the differentiability of multifunctions</i> .....	67
Paul M. Eakin, Jr. and Avinash Madhav Sathaye, <i>R-endomorphisms of <math>R[[X]]</math> are essentially continuous</i> .....	83
Larry Quin Eifler, <i>Open mapping theorems for probability measures on metric spaces</i> .....	89
Garret J. Etgen and James Pawlowski, <i>Oscillation criteria for second order self adjoint differential systems</i> .....	99
Ronald Fintushel, <i>Local <math>S^1</math> actions on 3-manifolds</i> .....	111
Kenneth R. Goodearl, <i>Choquet simplexes and <math>\sigma</math>-convex faces</i> .....	119
John R. Graef, <i>Some nonoscillation criteria for higher order nonlinear differential equations</i> .....	125
Charles Henry Heiberg, <i>Norms of powers of absolutely convergent Fourier series: an example</i> .....	131
Les Andrew Karlovitz, <i>Existence of fixed points of nonexpansive mappings in a space without normal structure</i> .....	153
Gangaram S. Ladde, <i>Systems of functional differential inequalities and functional differential systems</i> .....	161
Joseph Michael Lambert, <i>Conditions for simultaneous approximation and interpolation with norm preservation in <math>C[a, b]</math></i> .....	173
Ernest Paul Lane, <i>Insertion of a continuous function</i> .....	181
Robert F. Lax, <i>Weierstrass points of products of Riemann surfaces</i> .....	191
Dan McCord, <i>An estimate of the Nielsen number and an example concerning the Lefschetz fixed point theorem</i> .....	195
Paul Milnes and John Sydney Pym, <i>Counterexample in the theory of continuous functions on topological groups</i> .....	205
Peter Johanna I. M. De Paepe, <i>Homomorphism spaces of algebras of holomorphic functions</i> .....	211
Judith Ann Palagallo, <i>A representation of additive functionals on <math>L^p</math>-spaces, <math>0 &lt; p &lt; 1</math></i> .....	221
S. M. Patel, <i>On generalized numerical ranges</i> .....	235
Thomas Thornton Read, <i>A limit-point criterion for expressions with oscillatory coefficients</i> .....	243
Elemer E. Rosinger, <i>Division of distributions</i> .....	257
Peter S. Shoenfeld, <i>Highly proximal and generalized almost finite extensions of minimal sets</i> .....	265
R. Sirois-Dumais and Stephen Willard, <i>Quotient-universal sequential spaces</i> .....	281
Robert Charles Thompson, <i>Convex and concave functions of singular values of matrix sums</i> .....	285
Edward D. Tymchatyn, <i>Some n-arc theorems</i> .....	291
Jang-Mei Gloria Wu, <i>Variation of Green's potential</i> .....	295