GENERALIZED MONOFORM AND QUASI INJECTIVE MODULES

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For a torsion radical $F$ (arising from an idempotent filter of right ideals) and a unital right $R$-module $M$ over a ring $R$, let $DM$ be the $F$-divisible hull $DM/M = F(EM/M)$, where $EM$ is the injective hull of $M$.

Let $0 \neq \beta: N \to M$ be any nonzero homomorphism whatever from any $F$-dense submodule $N \subseteq M$. Then $M$ is $F$-quasi-injective if each such $\beta$ extends to a homomorphism of $M \to M$; $M$ is $F$-monic if $\beta$ is monic; $M$ is $F$-co-monic if $\beta N \subseteq M$ is $F$-dense.

Each module $M$ has a natural $F$-quasi-injective envelope $JM$ inside $M \subseteq JM \subseteq DM$.

**Theorem III.** Form the $R$-endomorphism rings $\Delta = \text{End} JM$ and $\Lambda = \text{End} DM$, and $\Lambda^* = \{\lambda \in \Lambda | \lambda JM = 0\} \subseteq \Lambda$, the annihilator subring of $M$.

When $M$ is $F$-monic and $F$-co-monic and $FM = 0$, then

1. $\Lambda^*$ is exactly the annihilator $\Lambda^* = \{\lambda \in \Lambda | \lambda JM = 0\}$ of the submodule $JM \subseteq DM$ and $\Lambda^* \subseteq \Lambda$ is an ideal;
2. $\Delta \equiv \Lambda/\Lambda^*$;
3. $\Delta$ is a division ring.

For a torsion radical $F$ and a torsion preradical $G$, let $IM$ be the $(F,G)$-injective hull of $M$; and, more generally, $\Lambda$ the ring of all those $R$-endomorphisms of $IM$ with $G$-dense kernels. The above is derived as a special case where $G = 1$ is the identity functor and $IM = DM$.

**Theorem II.** (i) $M$ is $(F,G)$-quasi-injective $\Leftrightarrow \Lambda M \subseteq M$.

(ii) The $(F,G)$-quasi-injective hull $JM$ of $M$ exists and $JM = M + \Lambda M$.

(iii) $JM$ is the unique smallest $(F,G)$-quasi-injective module with $M \subseteq JM \subseteq IM$.

Simple modules over a ring were at first generalized to quasi-simple modules ([9]), and then these to strongly uniform or monoform ones ([15] and [6]).

Here the monoform modules are further generalized to the $F$-monic ones. The quasi-injective hull plays an important role in the theory of quasi-simple modules ([8], [9], [15], and [6]). Furthermore, there is a
sizeable amount of literature about quasi-injectivity ([7], [8], [5], [16] and [14]). For these two reasons, a quasi-injective envelope is constructed with respect to a torsion radical $F$ and a torsion preradical $G$. Several previously known facts about quasi-injectives and quasi-injective hulls are generalized so that they can be obtained as corollaries from the more general case. It has been shown ([16; p. 54, Theorem 4.4]) that for a torsion radical $F$, a module $M$ is $F$-quasi-injective if and only if it is invariant in its $F$-divisible hull $DM$ under the ring of endomorphisms of $DM$. If $IM$ is the $(F, G)$-injective hull of $M$, then $M$ is shown to be $(F, G)$-quasi-injective if and only if it is invariant under an appropriate usually proper subring $\Lambda$ in the full endomorphism ring of $IM$ (Theorem I). Furthermore, in the latter the hypothesis that $F$ be a torsion radical can be relaxed a little (see 2.2). Then by taking $G$ as the identity functor $G = 1$, the previously known case of the $F$-quasi-injective hull now becomes a corollary of the more general result.

A module is monoform ([6] and [15]) if and only if the endomorphism ring of its quasi-injective hull $JM$ is a division ring. An analogue of this is proved in Theorem III.

Attention is focused on the original results by labelling them as Theorems I, II, III and Propositions A, B, and C. Proposition C shows that $M$ and $JM$ have the same annihilator in $\Lambda$ if and only if $M \subseteq JM$ is a rational extension of modules.

For these purposes it was first necessary to use recent new developments in [2] about $(F, G)$-injectivity. A module $C$ has been called $(F, G)$-injective provided any homomorphism $\alpha: A \rightarrow C$ from an $F$-dense submodule $A \subseteq B$ of a module $B$ with kernel $\alpha = \alpha^{-0} \subseteq A$ being $G$-dense, extends to $B \rightarrow C$.

Since some of the concepts used here have been studied independently by several authors, in various guises, under totally different names, their interconnection should be clarified.

The present 1-monic (Definition 3.2), the monoform ([6]), and the strongly uniform modules ([15]) are one and the same; while the quasi-simples ([18] and [9]) are the compressible monoform modules.

A “prefilter” (Definition 1.2) here is the same as a “filter” in [16; p. 517, 1.7 and Definition 1.8]; and the present “idempotent filter” (Definition 1.2), the same as an “idempotent filter” in [15], or a “strongly complete filter” ([16; p. 521, 1.17]).

In [16; p. 517, Lemmas 1.7, 1.9] a one-to-one correspondence is established between
torsion preradicals $F$ ([2; p. 313, §1]),
prefilters (1.2), and
strongly complete, additive classes ([16; p. 515, 1.2]).
There is also a one to one correspondence ([16; p. 515, Definition 1.1, p. 521, Lemma 1.18]) between
torsion radicals \( F \) ([2; p. 314, §1]),
idempotent filters (1.2) and,
strongly complete Serre classes ([16; p. 515, 1.1]).
The following are the same
preradical ([2; p. 313, §1]),
subfunctor of the identity ([16; p. 548, §5]);
also
torsion preradical ([2; p. 313; p. 314]),
concordant functor ([16; p. 548, §5]),
left exact subfunctor of the identity.

1. Filters, torsion preradicals, and semi-endomorphisms. The objects that will be investigated are first defined. Those basic properties that are repeatedly used in subsequent proofs are established so that later very short proofs of the main results may be given.

**Noteation 1.1.** A module here means a right unital module \( M \) over a ring \( R \) with identity. The notation \( N \subseteq M \) (or \( N < M \)) always means that \( N \) is a right \( R \)-submodule (proper) of \( M \). For \( m \in M \), define \( m^{-1}N \subseteq R \) as the right ideal \( m^{-1}N = \{ a \in R \mid ma \in N \} \) of \( R \).

**Definition 1.2.** A set \( \mathcal{F} \) of right ideals of \( R \) is a *prefilter* provided that

(i) \( A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F} \);
(ii) \( R \supseteq A \supseteq B, B \in \mathcal{F} \Rightarrow A \in \mathcal{F} \);
(iii) \( A \in \mathcal{F}, b \in R \Rightarrow b^{-1}A \in \mathcal{F} \).

The prefilter \( \mathcal{F} \) has been called an *idempotent filter* if for any \( B \subseteq R, A \in \mathcal{F} \)

(iv) \( \{ a^{-1}B \mid a \in A \} \subseteq \mathcal{F} \Rightarrow B \in \mathcal{F} \).

1.3. Parentheses will be omitted when the meaning is clear from the context by taking all quotients \( M/N \) of modules \( M \) and \( N \) first and applying any functors \( F \) last, e.g. \( FM/N = F(M/N) \). For \( m \in M \), abbreviate \( m^{-1}(0) = m^{-1}0 = m^{\perp} = \{ r \in R \mid mr = 0 \} \); for any subset \( M \) whatever of a module, set \( M^{\perp} = \{ r \in R \mid Mr = 0 \} \).

1.4. There is a bijective correspondence between prefilters \( \mathcal{F} \) and
torsion preradicals \( F \) given by \( \mathcal{F} = \{ A \subseteq R \mid FR/A = R/A \} \). Conversely, the \( F \)-torsion submodule of a module \( M \) is \( FM = \{ m \in M \mid m^{-1}0 \in \mathcal{F} \} \).

Consequently, first, a torsion preradical \( F \) is idempotent, or \( FFM = FM \) for all modules \( M \). Secondly, for any two or finite number of \( F \)-dense submodules \( A \subseteq C, B \subseteq C \) their intersection \( A \cap B \subseteq C \) is also \( F \)-dense in \( C \).
Any prefilter $\mathcal{F}$ is idempotent if and only if $F$ is a torsion radical, if and only if density is transitive, i.e. for any three modules $N < M < W$

$$FM/N = M/N, FW/M = W/M \Rightarrow FW/N = W/N.$$ 

A special case is the zero and the identity functors $F = 0, G = 1$ corresponding to $\mathcal{F} = \{R\}$ and $(0) \in \mathcal{G}$.

Any homomorphism $\beta : N \rightarrow M$ of any submodule $N \leq M$ into $M$ is called a semi-endomorphism.

A preradical $G$ is a torsion preradical if and only if modules $V < W$ satisfy, either $GV = V \cap GW$, or the condition that all submodules of any $G$-torsion module are $G$-torsion.

For the injective hull $EM$ of $M$ the torsion submodule $F((EM)/M) = FEM/M = DM/M$ defines the $F$-divisible hull $DM \leq EM$. For preradicals $F$ and $G$, define $IM \leq DM$ by $IM = M + DM \cap GEM$.

Since $IM/M \subseteq DM/M \cap (M + GEM)/M$, and $(M + GEM)/M \equiv GEM/(M \cap GEM)$, it follows that if either $F$ or $G$ is a torsion preradical, then $M \leq IM$ will be $F$- or $G$-dense, respectively. When $G$ is a torsion preradical, $GEM \cap DM = GDM$ and $IM = M + GDM$.

For various facts, definitions, and terminology, [2] should be consulted. When $F$ is a torsion preradical and $G$ a preradical, then $IM$ is $(F, G)$-injective if and only if $IM \supseteq DM \cap GEM$. ([2; p. 320, 2.5 (d)]). In case $F$ is a torsion radical (while $G$ is still a preradical), $IM$ is the $(F, G)$-injective hull of $M$ ([2; p. 321, Cor. 2.7]).

For any two modules, the additive abelian group of $R$-homomorphisms $IM \rightarrow IM$ will be abbreviated $\text{Hom}(IM, IM) = (IM, IM)$.

For any right $R$-module $W$, and preradical $G$, $G$-End $W$ will denote the set $G$-End $W \subseteq (W, W)$ of all those $R$-endomorphisms whose kernels are $G$-dense ([2; p. 315, 1.4 (b)]) in $W$. When $G$ is a torsion preradical, $G$-End $W$ is a ring. Define $\Gamma$ and $\Lambda$ to be $\Gamma = G$-End $DM \subseteq (DM, DM)$, $\Lambda = G$-End $IM \subseteq (IM, IM)$.

**Definition 1.5.** For preradicals $F$ and $G$, a module $M$ is $(F, G)$-quasi-injective if any semi-endomorphism $\beta : N \rightarrow M$ of any $F$-dense submodule $N \leq M$, $FM/N = M/N$, and $G$-dense kernel $\beta^{-1}0 < M$, $GM/\beta^{-1}0 = M/\beta^{-1}0$, extends to an element of $\Lambda$, i.e. $\alpha : M \rightarrow M$ such that the restriction $\alpha | N = \beta : N \rightarrow M$.

**Definition 1.6.** An $(F, 1)$-injective (or $(F, 1)$-quasi-injective) module will be simply called $F$-injective (or $F$-quasi-injective); and similarly for $(F, 1)$-injective (or $(F, 1)$-quasi-injective) hulls of modules.
1.7. The ordinary concepts of injectivity (or quasi-injectivity) are included in this much more general framework as the special cases of 
(1, 1)-injectivity (or (1, 1)-quasi-injectivity).

Whenever \( M \) is injective, \( M = EM = DM = IM \) and \( \Lambda = \Gamma \subseteq (M, M) \). The case \( G = 0 \) is uninteresting, for then every module is \( (F, 0) \)-injective and \( \Lambda = \Gamma = 0 \).

1.8. For torsion preradicals \( F \) and \( G \) a module \( M \) satisfies the following:

(i) \( \exists 1 \in \Gamma \Leftrightarrow GDM = DM \Rightarrow IM = DM \Rightarrow \Gamma = \Lambda. \)

(ii) \( \exists 1 \in \Lambda \Leftrightarrow GIM = IM \Rightarrow GM = M \Leftrightarrow IM = GDM. \)

(iii) \( F = 0 \) or \( G = 1 \Rightarrow \Lambda = \Gamma. \)

(iv) When \( F \) and \( G \) assume the values below, then \( DM, IM, \Lambda \) and \( \Gamma \) are as follows (where a blank means the entry is arbitrary):

<table>
<thead>
<tr>
<th></th>
<th>DM</th>
<th>IM</th>
<th>( \Lambda = \Gamma )</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>( M )</td>
<td>( M )</td>
<td>( \Lambda = \Gamma = (M, M) )</td>
</tr>
<tr>
<td>0</td>
<td>( 1 )</td>
<td>( M )</td>
<td>( \Lambda = \Gamma = (DM, DM) )</td>
</tr>
<tr>
<td>1</td>
<td>( 1 )</td>
<td>( EM )</td>
<td>( \Lambda = \Gamma = (EM, EM) )</td>
</tr>
<tr>
<td>1</td>
<td>( EM )</td>
<td>( M + GEM )</td>
<td></td>
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</table>

**Lemma 1.9.** Suppose \( A \subseteq B, C \) are any modules, \( \psi: C \to B \) any homomorphism, and \( F \) any torsion preradical with filter \( \mathcal{F} \). Then

(i) \( A \subseteq B \) is \( F \)-dense \( \Rightarrow \forall D < B, A \cap D \subseteq D \) is \( F \)-dense;

(ii) \( A \cap \psi C \subseteq \psi C \) is \( F \)-dense \( \Leftrightarrow \psi^{-1} A \subseteq C \) is \( F \)-dense;

(iii) \( A \subseteq B \) is \( F \)-dense \( \Rightarrow \psi^{-1} A \subseteq C \) is \( F \)-dense.

**Proof.** (i). Since \( D/A \cap D = (A + D)/A \), (i) holds.

(ii). If \( y \in C \) is arbitrary, then

\[
F \frac{C}{\psi^{-1} A} = \frac{C}{\psi^{-1} A} \Leftrightarrow y^{-1} \psi^{-1} A = (\psi y)^{-1} A \in \mathcal{F} \Leftrightarrow \forall x \in \psi C, \]

\[
x^{-1} A = x^{-1} (A \cap \psi C) \in \mathcal{F} \Leftrightarrow F \frac{\psi C}{A \cap \psi C} = \frac{\psi C}{A \cap \psi C}. \]

(iii). (i) and (ii) \( \Rightarrow \) (iii).

A limited form of transitivity is formulated below for torsion preradicals. Note that below in 1.10, if \( M \subseteq W \subseteq M + HEM \), then \( W = M + W \cap HEM = M + HW \).
1.10. If $F$, $G$, and $H$ are any torsion preradicals and $M < W$ any submodule such that $M < W = M + HW$, then

(i) $\forall N < M$ such that $HM/N = M/N \Rightarrow$ also $HW/N = W/N.$

Whenever $M \subseteq W \subseteq IM = M + GDM$, the above property (i) holds when either

(ii) $H = G$, or

(iii) $H = F$ and $IM = M + FIM$.

Frequently in subsequent proofs only transitivity of density with respect to $F$ will be all that is required, which will be guaranteed by the following standard hypothesis.

HYPOTHESIS 1.11. Either the torsion preradical

(1) $F$ is a torsion radical; or

(2) $IM = M + FIM$.

(In the above, $GDM = F(GDM) \Rightarrow IM = M + FIM$. The converse is false.)

1.12. LEMMA $\alpha$. Given torsion preradicals $F$ and $G$, an extension $M < W$ of modules, any semi-endomorphism $V \leq W$, $\psi: V \rightarrow W$ of $W$ induces a semi-endomorphism $\beta: N \equiv M \cap \psi^{-1}M \rightarrow M$ of $M$. Consider the following hypotheses:

(a) $FW/M = W/M$,

(b) $FW/V = W/V$,

(c) $GW/\psi^{-1}0 = W/\psi^{-1}0$.

Then

(i) $\Rightarrow \beta^{-1}0 < M$ is $G$-dense.

(ii) (a) $\Rightarrow \psi^{-1}M \leq V$, $N \leq M \cap V$,

and $M \cap V \leq V$ are $F$-dense.

(b) $\Rightarrow M \cap V \leq M$ is $F$-dense.

(a) and (b) $\Rightarrow M \cap V \leq W$ is $F$-dense.

(iii) (a), (b), and $F$ is a torsion radical $\Rightarrow$ are all $F$-dense.

(iv) (a), (b), (c), hypothesis $\Rightarrow \psi^{-1}M \leq IM$, $N < IM$ are $F$-dense; $1.11$ and $M \leq V \leq IM \Rightarrow \beta^{-1}0 < IM$ is $G$-dense.

Proof. (i) By 1.9 (i), $\beta^{-1}0 = M \cap \psi^{-1}0 < M$ is dense.

(ii) (a). Use of (a) and 1.9 (iii) show that $\psi^{-1}M \leq V$ is $F$-dense. By 1.9 (i), $N = M \cap \psi^{-1}M \leq M \cap V$, and $M \cap V \leq V$ are too.

(ii) (b). Again, by 1.9 (i), (b) implies that $M \cap V \leq M$ is $F$-dense.

(ii) (a) and (b). For any torsion preradical, the intersection of any two dense modules is always dense.

(iii). By (ii) all of $\psi^{-1}M \leq V$, $N \leq M \cap V$, $M \cap V \leq V$, $V \leq W$ are $F$-dense, and (iii) follows by transitivity of density.
(iv). The density of $\psi^{-1}M < V$, $N < M$, $\beta^{-1}0 < M$, 1.10, and 1.11 show that these are also dense in $IM$.

1.13. Lemma (β). Starting with torsion preradicals $F, G$ and modules $M < W$ assume that
(a) $FW/M = W/M$ and
(b) $GW/M = W/M$.
Now, conversely suppose $\beta : N \rightarrow M$ is any semi-endomorphism from any $F$-dense submodule $N \subseteq M$ and with $\beta^{-1}0 < M$ G-dense. Consider the following two pairs of possible additional hypotheses
(f1) $F$ is a torsion radical;  (g1) $G$ is a torsion radical;
(f2) $W = M + FW$.  (g2) $W = M + GW$.
Then $\beta$, when regarded as a semi-endomorphism of $W$, has the following properties
(i) Either (f1) or (f2) $\Rightarrow N < W$ is $F$-dense.
(ii) Either (g1) or (g2) $\Rightarrow \beta^{-1}0 < W$ is $G$-dense.
In particular, if $M \subseteq W \subseteq IM$, then
(iii) Hypothesis 1.11 $\Rightarrow FW/N = W/N$, and $GW/\beta^{-1}0 = W/\beta^{-1}0$.

The next corollary establishes a correspondence between the endomorphisms of $IM$ with $G$-dense kernels and the semi-endomorphisms from $F$-dense submodules of $M$ with $G$-dense kernels.

1.14. Corollary to Lemma (α). For torsion preradicals $F, G$ each $\lambda \in \Lambda$ induces a semi-endomorphism $\lambda : M \cap \lambda^{-1}M \rightarrow M$ with the following properties
(i) $\lambda^{-1}M \subseteq IM$, $M \cap \lambda^{-1}M \subseteq M$ are $F$-dense, while $M \cap \lambda^{-1}0 \subseteq M$ is $G$-dense.
(ii) Conversely, for any semi-endomorphism $\beta : N \rightarrow M$ from an $F$-dense $N \subseteq M$ and with $\beta^{-1}0 < M$ G-dense, if $F$ is a torsion radical; then $\Rightarrow \exists \lambda \in \Lambda$ such that the restriction $\lambda | N = \beta$.

Proof. (i). Immediate from (α). (ii) A consequence of [2; p. 320, Theorem 2.5 (d)].

2. (F, G) quasi-injective hulls. For a torsion radical $F$ and a torsion preradical $G$, the $(F, G)$-quasi-injective hull of a unital module $M$ will be constructed. The next theorem generalizes a characterization, that is a necessary and sufficient condition for both ordinary quasi-injectivity and $F$-quasi-injectivity.

The notation of the previous section is continued. In this section it will be necessary to assume for most of the results that $F$ is a torsion radical and $G$ is a torsion preradical.
2.1. **Theorem I.** For a torsion radical $F$ and a torsion preradical $G$, form $\text{IM} \subseteq \text{DM}$ the $(F, G)$-injective module $\text{IM}$ and the $F$-divisible hull $\text{DM}$ of a unital module $M$ (see 1.4); let $\Lambda$ and $\Gamma$ be the rings of all endomorphisms of $\text{IM}$ and $\text{DM}$ whose kernels are $G$-dense in $\text{IM}$ and $\text{DM}$.

Then the following holds:

(i) $M$ is $(F, G)$-quasi-injective $\iff \Lambda M \subseteq M$.

The next corollary observes that the hypothesis that $F$ be a torsion radical is not needed.

2.2. **Corollary 1 to Theorem I.** For torsion preradicals $F$ and $G$

(ii) $M$ is $(F, G)$-quasi-injective $\implies \Lambda M \subseteq M$ and $\Gamma M \subseteq M$.

2.3. **Corollary 2 to Theorem I.** If in addition to the above hypotheses in the previous Theorem I, $G$ is a torsion radical and $G(\text{DM}/M) = \text{DM}/M$, then

(iii) $M$ is $(F, G)$-quasi-injective $\iff \Gamma M \subseteq M$.

**Proof.** 2.1 (i), 2.2 (ii), and 2.3 (iii) $\implies$ : If $M$ is $(F, G)$-quasi-injective and $\lambda \in \Lambda$ (or $\lambda \in \Gamma$), let $\beta: N = M \cap \lambda^{-1}M \to M$ be the semi-endomorphism induced by $\lambda$. By 1.14 (i) and hypothesis, $\beta$ extends to $\alpha: M \to M$ with the restriction $\alpha|N = \beta$. Assume $(\lambda - \alpha)M \neq 0$. Then since $M \subseteq EM$ is essential, let

$$0 \neq m = \lambda n - \alpha n \in (\lambda - \alpha)M \cap M \neq 0 \quad m, n \in M.$$ 

Since $\lambda n = m + \alpha n \in M$, we have $n \in N$. But $\lambda|N = \alpha|N = \beta$; hence $m = 0$, a contradiction. Thus $(\lambda - \alpha)M = 0$, $\lambda M = \alpha M \subseteq M$, and as required $\Lambda M \subseteq M$ (or $\Gamma M \subseteq M$).

2.1 (i), and 2.3 (iii) $\iff$ : Conversely, assume $\Lambda M \subseteq M$ (or $\Gamma M \subseteq M$) and take any semi-endomorphism $\beta: N \to M$ with $N < M$, $\beta^{-1}0 < M$ being $F, G$-dense. By 1.14 (ii) (or [12; p. 61]) $\beta$ extends to a map $\lambda: \text{IM} \to \text{IM}$ (or, since $F$ is a torsion radical, to $\lambda: \text{DM} \to \text{DM}$). Since $M < \text{IM}$ and $\beta^{-1}0 < M$ are $G$-dense it follows from Lemma ($\beta$) (ii), that $\beta^{-1}0 < \text{IM}$ is $G$-dense and hence that $\lambda \in \Lambda$. (Since $\beta^{-1}0 < M < \text{DM}$ are all $G$-dense in 2.3, it follows from Lemma ($\beta$) that again $\lambda^{-1}0 < \text{DM}$ is $G$-dense and thus $\lambda \in \Gamma$.) By hypothesis $\lambda M \subseteq M$. Finally, the restriction and corestriction $\alpha$ of $\lambda$ to $\alpha = \lambda| \text{M}: M \to M$ is the required extension of $\beta$ to $M \to M$ with $\alpha|N = \beta$.

For $G = 1$, $\text{IM}$ and $\Lambda$ become $\text{IM} = \text{DM}$ and thus $\Lambda = \Gamma$. A result in [16; p. 541, Theorem 4.4] is the special case when $G = 1$ in the previous Theorem I.
2.4. COROLLARY 3 TO THEOREM I. With the notation and hypotheses of Theorem I for $G = 1$:

$$M \text{ is } F\text{-quasi-injective} \iff \Gamma M \subseteq M.$$  

The following well-known fact is a special case of the previous Theorem I with $F = 1$ and $G = 1$ (or of 2.4).

2.5. COROLLARY 4 TO THEOREM I. If $\Lambda$ is the $R$-endomorphism ring of the injective hull of a module $M$, then

$$M \text{ is quasi-injective} \iff \Lambda M \subseteq M.$$  

2.6. For torsion preradicals $F$ and $G$ and any module $M$, $\Lambda M \subseteq GDM$ and in particular $\Lambda M \subseteq GDM$.

Proof. Since $\Lambda = G\text{-End } IM$, $\Lambda IM \subseteq IM \subseteq DM$.

Again by 1.9 (iii), under the map $R \rightarrow IM$, $r \rightarrow yr$, the inverse image of the $G$-dense submodule $\lambda^{-1}0 < IM$ is $y^{-1}(\lambda^{-1}0) = (\lambda y)^{-1}0 < R$ and it is $G$-dense in $R$. So $\lambda y \in GEM$, and $\Lambda IM \subseteq GEM$.

2.7. When $F$ and $G$ are preradicals and $V < W$ any extension of modules, then there is a natural inclusion $IV \subseteq IW$.

Proof. By the definition of the preradical functor, the maps $EV/V \rightarrow (EV + W)/W \rightarrow EW/W$ induce by restriction

$$F \frac{EV}{V} \rightarrow F \frac{EV + W}{W} \rightarrow F \frac{EW}{W},$$

and the image of $DV/V$ is

$$\frac{DV}{V} \rightarrow \frac{DV + W}{W} \subseteq F \frac{EV + W}{W} \subseteq \frac{DW}{W}.$$  

Hence there are natural inclusions $DV \subseteq DW$ and $IV \subseteq IW$.

2.8. THEOREM II. For a torsion radical $F$, a torsion preradical $G$, and a unital right $R$-module $M$, let $EM$ be the injective hull of $M$, and $DM$ be the module $DM/M = FEM/M$. Form the $(F, G)$-injective module $IM = M + GDM \subseteq EM$ ([2; p. 321, 2.7]), and the ring $\Lambda$ of all those $R$-endomorphisms of $IM$ whose kernels are $G$-dense in $IM$. 

(i) **Existence:** \( \exists \) an \((F,G)\)-quasi-injective module \( JM \) in \( M \subseteq JM \subseteq IM \).

(ii) **Intrinsic characterization:** \( JM = M + \Lambda M \) and thus \( JM \) is the unique smallest \( \Lambda \)-invariant submodule containing \( M \) inside \( IM \).

(iii) **Uniqueness inside \( IM \):** Any \((F,G)\)-quasi-injective submodule \( W \) with \( M \subseteq W \subseteq IM \) satisfies \( JM \subseteq W \); \( JM \) is the unique smallest \((F,G)\)-quasi-injective submodule of \( IM \) containing \( M \).

**Proof.** In general for a preradical \( F \), for any module \( W \) in \( M \subseteq W \subseteq DM \), also \( DM \subseteq DW \subseteq D(DM) \). When \( F \) is a torsion radical as is the case here, \( D(DM) = DM = DW \). When \( M \subseteq W \subseteq DM \), then \( EW = EM \), \( DW = DM \), and consequently \( IW = W + GDM = W + IM \supseteq IM \). If moreover, in addition \( W \) is further restricted to \( M \subseteq W \subseteq IM \), then \( IW = W + IM = IM \) for any torsion preradical \( G \); furthermore, the subring \( \Omega \subseteq (IW, IW) \) whose kernels are \( G \)-dense in \( IW \) is \( \Omega = \Lambda \) the same as that of \( IM \).

(i) and (ii). Since \( \Lambda \subseteq (IM, IM) \), the image \( \Lambda M \) is in \( \Lambda M \subseteq IM \). Use of the above for \( W = M + \Lambda M \) with \( M \subseteq W \subseteq IM \) shows that \( IM(M + \Lambda M) = IM \). Set \( JM = M + \Lambda M \). Those endomorphisms \( \Omega \subseteq (JM, JM) \) with \( G \)-dense kernels in \( JM = IM \) are just \( \Omega = \Lambda \). Because \( \Omega JM \subseteq JM \), Theorem I shows that \( JM \) is \((F,G)\)-quasi-injective.

(iii) For an \((F,G)\)-quasi-injective \( W \) with \( M \subseteq W \subseteq IM \) as in (iii), by the above \( IW = IM \) and \( \Omega = \Lambda \). By Theorem I, \( \Lambda W \subseteq W \). Thus \( \Lambda M \subseteq W \) and hence \( JM = M + \Lambda M \subseteq W \).

3. **Endomorphism rings of \( F \)-monic modules.** This section can be read independently from the previous ones, provided one only assumes the existence of an \( F \)-quasi-injective module \( JM \) where \( M \subseteq JM = (\text{End IM})M \subseteq DM \). Here \( G \) will be \( G = 1 \) so that all submodules are \( G \)-dense.

**Notation 3.1.** The previous notation is continued for arbitrary \( F \) but \( G = 1 \). When \( F \) is a torsion radical, then \( IM = DM \) is \((F,1)\)- or \( F \)-injective, or \( F \)-divisible, and \( \Lambda = (DM, DM) = \text{End DM} \) now is the full endomorphism ring. For any \( W \subseteq DM \), \( ^{-1} W \) denotes \( \{ \lambda \in \Lambda \mid \lambda W = 0 \} \). Let \( \Lambda^* \) be the subring \( \Lambda^* = ^{-1} M \subseteq \Lambda \). When \( F \) is a torsion radical, \( IM = DM \) is the \( F \)-injective hull of \( M \), and the \( F \)-quasi-injective hull \( JM \) of \( M \) exists and \( JM = \Lambda M \). Set \( \Delta = \text{End JM} = (JM, JM) \). Define \( \Sigma \subseteq \Delta \) as the multiplicative semigroup of all endomorphisms \( 0 \neq \alpha : JM \rightarrow JM \) such that the image \( \alpha V \) of every \( F \)-dense submodule \( V \subseteq JM \) is also an \( F \)-dense submodule \( \alpha V \subseteq JM \). Ideals (two-sided) are denoted by "\( \triangleleft \)".
DEFINITION 3.2. A module $M$ is $F$-monic with respect to a preradical $F$ provided any nonzero semi-endomorphism $0 \neq \beta: N \to M$ with an $F$-dense domain $N \subseteq M$ in $M$ is monic, i.e. $\beta^{-1}0 = 0$.

For $F = 1$, the 1-monic modules have already been called simply monoform ([6]) or strongly uniform.

DEFINITION 3.3. For a preradical $F$, a module $M$ is $F$-co-monic if for every nonzero semi-endomorphism $0 = \beta: N \to M$ from any $F$-dense submodule $N \subseteq M$, its image $\beta N$ is also $F$-dense $\beta N \subseteq M$ in $M$.

The module $M$ is weakly $F$-co-monic if for every $R$-endomorphism $0 \neq \alpha \in \text{End } M$ of $M$, the image $\alpha M \subseteq M$ is $F$-dense.

LEMMA 3.4. For a torsion radical $F$ and a module $W$ in $M \leq W \leq DM$, $W$ is $F$-co-monic if every semi-endomorphism $0 \neq \alpha: V \to W$ with $V \subseteq M$ satisfies

\[ V \leq M \text{ is } F\text{-dense } \Rightarrow \text{ also } \alpha V \leq JM \text{ is } F\text{-dense.} \]

The converse holds if $W$ is $F$-monic.

Proof. Let $V \leq M$ and $FM/V = M/V$. Firstly by transitivity of density, any extension that can be formed from any two modules below will be an $F$-dense extension

\[ V \leq M \leq W \leq DM, \quad V \leq M \leq JM \leq DM. \]

Secondly, any $\alpha: V \to W$ extends to a map $\lambda: DM \to DM$. Since $V \leq M$, $\alpha V \leq JM = \lambda M$. Thus $\alpha V \leq JM$.

$\Rightarrow$: Since $V \leq M$, $V \leq W$, and $W \leq DM$ are $F$-dense, and since $W$ is $F$-co-monic, it follows that $\alpha V \leq W$, $W \leq DM$, $\alpha V \leq DM$, and hence $\alpha V \leq JM$ are all $F$-dense.

$\Leftarrow$: For any semi-endomorphism $0 \neq \gamma: U \to W$ from an $F$-dense $U \leq W$, set $V = U \cap M$ and $0 \neq \alpha \equiv \gamma | V: V \to W$. By hypothesis $0 \neq \alpha V \leq JM$ is $F$-dense. Thus $JM \leq DM$, $\alpha V \leq Dm$, and in particular $\alpha V \leq W$ are all $F$-dense. But then $\alpha V \leq \gamma U \leq W$ shows that $\gamma U \leq W$ is $F$-dense. Thus $W$ is $F$-co-monic.

LEMMA 3.5. For a module $M$ that is $F$-monic with respect to a torsion radical $F$, any nonzero semi-endomorphism $0 \neq \alpha: V \to DM$ from an $F$-dense submodule $V \leq DM$ satisfies:

(i) either $\alpha$ is monic on all of $V$ or $\alpha (M \cap V) = 0$.

(ii) In particular, $M \leq V \Rightarrow$ either $\alpha$ is monic on $V$ or $\alpha M = 0$.

Proof. Note that $\alpha^{-1}M = \alpha^{-1}(M \cap \alpha V)$ by definition. Set $N = M \cap \alpha^{-1}M$ and let $\beta = \alpha | N: N \to M$ be the induced semi-endomorphism.
Since $M \leq DM$, $V \leq DM$ are $F$-dense, so are $M \cap V \leq DM$, $M \cap V \leq V$, and $\alpha^{-1} M \leq V$. Hence the intersection $N = M \cap V \cap \alpha^{-1} M \leq V$ of two $F$-dense submodules of $V$ is $F$-dense in $V$. The fact that $N \leq V$ and $V \leq DM$ are both $F$-dense with respect to a torsion radical, by transitivity implies that $N \leq DM$ is $F$-dense too. (Alternatively we may argue that $\alpha^{-1} M < V$, $N = M \cap \alpha^{-1} M \leq M \cap V$, $M \cap V \leq DM$ being $F$-dense imply — again by transitivity — that $N \leq DM$ is $F$-dense.)

Assume alternative (i) does not hold. So suppose that $\alpha(M \cap V) \neq 0$ and also that $\alpha^{-1} 0 \neq 0$. Because $M \leq DM$ is essential, also $\beta^{-1} 0 = M \cap \alpha^{-1} 0 \neq 0$, and there exists $0 \neq an \in M \cap \alpha(M \cap V)$ with $n \in M \cap V$. Hence $n \in N$, $\beta n \neq 0$, and $\beta \neq 0$. On the other hand, since $N \leq M$ is $F$-dense, since $M$ is $F$-monic, and $0 \neq \beta : N \rightarrow M$, it follows that $\beta^{-1} 0 = 0$, a contradiction. Hence either $\alpha(M \cap V) = 0$ or $\alpha^{-1} 0 = 0$.

3.6. PROPOSITION (A). When $M < W$ is $F$-dense with respect to a torsion radical $F$, then

(i) $W$ is $F$-co-monic $\Rightarrow$ $M$ is $F$-co-monic;

(ii) $W$ is $F$-monic $\Rightarrow$ $M$ is $F$-monic.

If, in addition to the above hypotheses, $M < W$ is essential, and $FM = 0$, then the converse holds:

(iii) $M$ is $F$-co-monic $\Rightarrow$ $W$ is $F$-co-monic;

(iv) $M$ is $F$-monic $\Rightarrow$ $W$ is $F$-monic.

Proof. (i) and (ii) $\Rightarrow$ : For (i) and (ii), given any $0 \neq \beta : N \rightarrow M$ with $N \leq M$, $FM/N = M/N$, also $N \leq W$ is $F$-dense in $W$ because $F$ is a torsion radical. Now regard $0 \neq \beta : N \rightarrow M$ as a semi-endomorphism of $W$. (i) Thus $\beta N \leq W$ is $F$-dense, and $M$ is $F$-co-monic in case of (i). (ii) By (ii), $\beta$ is monic, and $M$ is $F$-monic.

(iii) and (iv) $\Rightarrow$ : In both (iii) and (iv) for any $0 \neq \alpha : V \rightarrow W$ with $FW/V = W/V$, form $N = M \cap \alpha^{-1} M$ and $\beta = \alpha | N : N \rightarrow M$. Since $V \leq W$, $\alpha^{-1} M \leq V$ are $F$-dense, so are the intersections $N = M \cap \alpha^{-1} M \leq M \cap V$ and $M \cap V \leq M$. By transitivity $N \leq M$ is $F$-dense. Since $M < W$ is essential and $\alpha \neq 0$, there is a $v \in V$ with $0 \neq \alpha v \in \alpha V \cap M$. Then $\alpha^{-1} M \leq R$ is $F$-dense because $M < W$ is, and $FM = 0$ requires that there is a nonzero element $0 \neq an \in (av)^{-1} M \subseteq M \cap \alpha v R$ with $n \in V \cap M$. Thus $n \in N$, $\beta n = \alpha n \neq 0$, and $\beta \neq 0$.

(iii) In case of (iii), $\beta N = \alpha N \leq M$ is $F$-dense. Again by transitivity $\alpha N \leq W$ will be $F$-dense. Thus $W$ is $F$-co-monic.

(iv) By (iv), $\beta$ is monic $\beta^{-1} 0 = M \cap \alpha^{-1} 0 = 0$. Since $M \leq W$ is essential, also $\alpha^{-1} 0 = 0$ and $W$ is $F$-monic.

3.7. PROPOSITION (B). The following conclusions hold under the assumptions on a module $M$ and torsion radical $F$ that
(a) \( M \) is \( F \)-monic.

(i) \( \Lambda \# \subseteq \Lambda \).

(ii) \( \Lambda^\ast JM = 0 \), and \( \uparrow JM = \uparrow M = \Lambda^\ast \).

(iii) \( \Delta = \Lambda / \Lambda^\ast \).

(iv) The maps in \( \Lambda \setminus \Lambda^\ast \) and in \( \Delta \setminus \{0\} \) are monic; \( \Lambda \setminus \Lambda^\ast \) is a multiplicative semigroup and \( \Delta \) is an integral domain.

**Proof.** (i) Always \( \Lambda \Lambda^\ast \subseteq \Lambda^\ast \). For any \( 0 \neq \gamma \in \Lambda^\ast \) and any \( 0 \neq \lambda \in \Lambda \), we have \( 0 \neq M \cap \lambda M \subseteq \ker \gamma \). Since \( \gamma \lambda : DM \rightarrow DM \) is not monic, it follows from 3.5 (ii) (with \( V = DM \) and \( \alpha = \gamma \lambda \)), that \( \gamma \lambda M = 0 \), and hence that \( \gamma \lambda \in \Lambda^\ast \). Thus \( \Lambda^\ast \subseteq \Lambda \).

(ii) Since \( JM = \Lambda M \), by (i) \( \Lambda^\ast JM = 0 \). Always \( \Lambda^\ast = \uparrow M \supseteq \uparrow JM \), because \( M \subseteq JM \). Hence \( \uparrow JM = \uparrow M \).

(iii) The restriction map \( \Lambda \rightarrow \Delta \) of elements of \( \Lambda \) to the \( \Lambda \)-submodule \( JM \subseteq DM \) induces a ring isomorphism \( \Delta \cong \Lambda \setminus \Lambda^\ast \).

(iv) Take \( a, b \in \Lambda \setminus \Lambda^\ast \), \( \alpha = a + \Lambda^\ast \), \( \beta = b + \Lambda^\ast \in \Delta \). Note that \( ay = \alpha y \) for every \( y \in JM \). If \( a \) were not monic, then by 3.5 (ii) (with \( V = DM \)), \( aM = 0 \) and \( a \in \Lambda^\ast \). Thus \( aM \neq 0 \); \( a, b \) are monic, \( baM \neq 0 \), and \( ba \in \Lambda \setminus \Lambda^\ast \). Hence \( \Lambda \setminus \Lambda^\ast \) and \( \Delta \setminus \{0\} \) are semigroups of monic maps.

The next corollary is merely a continuation of the previous proposition. It attempts to determine how close \( \Delta \) is to being a division ring.

3.8. **Corollary 1 to Proposition (B).** Under the assumptions of the previous Proposition 3.7:

(v) \( \Sigma \) is a subgroup of the multiplicative group of units of \( \Delta \setminus \{0\} \); and \( \Sigma \) consists of automorphisms of \( JM \).

(vi) \( JM \) is weakly \( F \)-co-monic \( \Rightarrow \Delta \) is a division ring.

**Proof.** (v) Since any \( \alpha \in \Sigma \) is monic by 3.7 (iv), the semi-endo- morphism \( \alpha JM \rightarrow JM \), \( \alpha y \rightarrow y \) from the \( F \)-dense submodule \( \alpha JM \leq JM \), extends to \( \beta : JM \rightarrow JM \) with \( \beta \alpha = 1 \), and \( \beta \in \Delta \). In order to show that \( \beta \in \Sigma \), it will suffice to show that for any \( F \)-dense \( W \leq JM \) submodule, \( \beta (W \cap \alpha JM) \leq JM \) is \( F \)-dense. Because \( \beta \) is a left inverse of \( \alpha \), a straightforward verification shows that \( \beta (W \cap \alpha JM) = \alpha^{-1} W \). But \( \alpha^{-1} W \leq JM \) is \( F \)-dense. Thus \( \beta \in \Sigma \). Since \( \Sigma \) is a semigroup, \( \beta \alpha = \alpha \beta = 1 \). Thus \( \Sigma \) is a group of automorphisms of \( JM \).

(vi) By 3.7 (iv), \( \Delta \setminus \{0\} \) is a semigroup, and by the argument in 3.8 (v), each element has a left inverse. Thus \( \Delta \) is a division ring.

3.9. **Corollary 2 to Proposition (B).** For a torsion radical \( F \), define \( \Sigma^\ast = \{ \alpha \in \Lambda \mid \alpha JM \leq JM \) is \( F \)-dense \}. Then

(i) \( \Sigma^\ast \) is a semigroup with \( \Sigma \subseteq \Sigma^\ast \).

(ii) 3.7(a) \( \Rightarrow \) 3.8 holds for \( \Sigma^\ast \) in place of \( \Sigma \).
Proof. (i) For $\alpha, \beta \in \Sigma^*$, since $JM/\alpha JM \to \beta JM/\beta \alpha JM$ is epic, $\beta \alpha JM \subseteq \beta JM$ is $F$-dense. Hence $\Sigma^*$ is a semigroup. (ii) is clear from 3.8.

A few of the results of this section are combined and summarized below.

**Theorem III.** Starting with a torsion radical $F$, a unital right $R$-module $M$, its $F$-injective or $F$-divisible hull $DM$ and the $R$-endomorphism ring $\Lambda = (DM, DM)$ of $DM$, form the $F$-quasi-injective hull $JM = \Lambda M$ of $M$, and its $R$-endomorphism ring $\Delta = (JM, JM)$; let $\Lambda^* = \{\lambda \in \Lambda \mid \lambda M = 0\} = ^\perp M$, $\downarrow JM = \{\lambda \in \Lambda \mid \lambda JM = 0\}$ be the annihilator subrings of $M$, $JM$ in $\Lambda$.

(i) If $FM = 0$, then

$M$ is $F$-co-monic (respectively $F$-monic) $\Leftrightarrow$ $JM$ is likewise.

The subsequent conclusions hold under the assumption that

(a) $M$ is $F$-monic.

(ii) $\Lambda^*$ is exactly the annihilator of the $\Lambda$-$R$-bi-submodule $JM$ of $DM$; hence

$$\Lambda^* JM = 0, \quad ^\perp M = ^\perp JM, \quad \Lambda^* \subseteq \Lambda, \quad \text{and} \quad \Delta \cong \Lambda/\Lambda^*.$$  

(iii) $\Delta$ is an integral domain; furthermore, each nonzero map of $\Delta$ is monic.

Finally, when in addition to (a), it is assumed that

(b) $FM = 0$ and $M$ is $F$-co-monic,

(iv) then $\Delta$ is a division ring.

4. Arbitrary endomorphism rings. Without any restrictive assumptions on the unital module $M$, but some restrictions on the torsion preradicals $F$ and $G$, the subring $\Lambda \subseteq (IM, IM)$ will be considered. It should be stressed that

$$F = 1 \quad G = 1: IM = EM, \quad \Lambda = (EM, EM);$$

$$F \quad \text{arbitrary,} \quad G = 1: IM = DM, \quad \Lambda = (DM, DM);$$

are included as special cases. Here, some information is obtained about the $G$-endomorphism ring of the $(F, G)$-quasi-injective hull $JM$ of $M$. The simplest way of doing this is to consider any $\Lambda$-$R$-bimodule $W$ in $M \subseteq W \subseteq IM$. This approach also seems to be the shortest because $W = M, W = JM$, or $W = IM$ are frequently used special cases.

**Notation 4.1.** The notation of the previous sections will be continued.

For any subset $W \subseteq IM$, define $^\perp W = \{\lambda \in \Lambda \mid \lambda W = 0\}$. From now on $W$ will be a $\Lambda$-$R$-bimodule in $M \subseteq W \subseteq IM$; in this case define
\( \Omega = G - \text{End } W \) as the subring \( \Omega \subseteq (W, W) \) of all endomorphisms of \( W \) whose kernels are \( G \)-dense in \( W \). If \( \Lambda W \subseteq W \), then \( \Lambda^W \Lambda \), where "\( < \)" denotes (two-sided) ideals. Set \( \Omega^* = \{ \gamma \in \Omega \mid \gamma M = 0 \} \). Unless stated otherwise, in this section \( F \) will be a torsion radical and \( G \) a torsion preradical.

When \( F \) is a torsion radical, then \( W \) is \((F, G)\)-quasi-injective by Theorem II.

Note that (iii) below provides a means of constructing \((F, G)\)-quasi-injective \( \Lambda\text{-R-bimodules.} \)

**Lemma 4.2.** With the notation and the hypotheses of 4.1 (as well as 1.11), if \( \pi: \Lambda \to \Omega \) is the map induced by restricting \( \Lambda \) to \( W \), and if \( V \subseteq W \) is an \( \text{R-submodule} \), then

(i) the exact sequence of modules induces an exact commutative diagram of ring homomorphisms

\[
\begin{array}{ccc}
IM/W & \to & IM \to W \\
0 & \to & \Lambda & \to & \Omega & \to & 0 \\
0 & \to & \Lambda^* & \to & \Omega^* & \to & 0 \\
\end{array}
\]

(ii) In particular,

\[
\pi \Lambda = \Omega \equiv \Lambda/\Lambda \quad \quad \Omega^* = \pi \Lambda^* \equiv \Lambda^*/\Lambda^* \quad \text{and} \quad \Lambda W \equiv (IM/W, IM).
\]

(iii) \( \Lambda V = \Omega V \) and \( V + \Omega V \) is a \( \Lambda\text{-R-bimodule.} \)

**Proof.** In (i) and (ii) in general, that is when \( F \) as well as \( G \) are only torsion preradicals, the sequences need not be right exact and \( \pi \) induces only ring monomorphisms with \( \pi \Lambda \subseteq \Omega \), \( \pi \Lambda^* \subseteq \Omega^* \). In this case in (iii), \( \Lambda V \subseteq \Omega V \) and \( V + \Omega V \) is not necessarily a left \( \Lambda \)-module.

(i), (ii), and (iii). However, when \( F \) is a torsion radical, for any \( \beta \in \Omega \), use of \( M \subseteq W \) and \( \beta^{-1}0 \subseteq W \subseteq IM \subseteq W + GEM \) shows that \( \beta \) extends to a map \( \lambda \in \Lambda \) with \( \pi \lambda = \lambda \mid W = \beta \).

**Lemma 4.3.** Under the assumptions of 4.1 that \( G \) is a torsion preradical and \( F \) is a torsion radical, all of the five conditions (i)-(v) are equivalent.

(i) \( \Lambda^* = \Lambda^*(M + \Lambda M) \); 
(ii) \( \Lambda^* \Lambda \); 
(iii) \( \Omega^* < \Omega \);
(iv) $M \leq M + \Lambda M$ is a rational extension of modules;
(v) $((M + \Lambda M)/M, M + \Lambda M) = 0$.

Proof. First note that since $M \subseteq M + \Lambda M$, always $\Lambda^* \supseteq \Lambda^*(M + \Lambda M)$. Also $\Lambda^*$ is a left ideal $\Lambda^* \subseteq \Lambda^*$ always.

(i) $\Rightarrow$ (ii). Since $\Lambda^* \Lambda M = 0$, $\Lambda^* \Lambda \subseteq \Lambda^*$ and $\Lambda^* \Lambda^* \Lambda M = 0$. Thus $\Lambda^* = \Lambda^*(M + \Lambda M)$. So far for (i) $\Leftrightarrow$ (ii), the hypothesis that $F$ is a torsion radical was not needed.

(ii) $\Rightarrow$ (iii). However, the fact that $F$ is a torsion radical was used to show in 4.2 (i) that the image of the subring $\Lambda^*/\Lambda W \subseteq \Lambda^*/\Lambda W$ under the isomorphism $\Lambda^*/\Lambda W \cong \Omega$ is exactly $\Omega^*$. Thus if $\Lambda^* \Lambda \Lambda M = 0$, then necessarily also $\Omega^* \Lambda \Lambda = 0$.

(iii) $\Rightarrow$ (i). In 4.2 (i), $(\lambda - \pi \lambda) W = 0$ for $\lambda \in \Lambda$, and $\pi \Lambda^* = \Omega^*$. In 4.2 (iii) for $V = M$, since $M \subseteq W$, also $\Lambda M \subseteq \Lambda W \subseteq W$, and consequently $\Lambda^* \Lambda M = \pi \Lambda^* \pi \Lambda M = \Omega^* \Omega M = 0$, since $\Omega^* \Omega \subseteq \Omega^*$. Thus $\Lambda^* \subseteq \Lambda^*(M + \Lambda M)$ and hence $\Lambda^* = \Lambda^*(M + \Lambda M)$.

In the remainder, let $\beta$ represent any map $\beta : V \rightarrow M + \Lambda M$ for $M \leq V \leq M + \Lambda M$ with $\beta M = 0$.

(iii) $\Rightarrow$ (iv). Since $\beta$ extends to $\lambda \in \Lambda$ with $\lambda | V = \beta | V$, we get $\lambda \in \Lambda^*$. Thus (i) $\Lambda^* \Lambda \Lambda M = 0$. Hence $M \leq M + \Lambda M$ is rational. Always, (iv) $\Rightarrow$ (v).

(v) $\Rightarrow$ (iv). By Theorem II, $M + \Lambda M = JM$, the $(F, G)$-quasi-injective hull of $M$. Therefore a typical $\beta$ as above extends to $\gamma \in G$-End $JM$, $\gamma : M + \Lambda M \rightarrow M + \Lambda M$. Due to $\beta M = \gamma M = 0$, $\gamma$ induces a map $\alpha \in ((M + \Lambda M)/M, M + \Lambda M)$. Thus by (v) $\alpha = 0$, $\gamma = 0$, and $\beta = 0$. Hence $M \leq M + \Lambda M$ is rational. Clearly, (v) $\Rightarrow$ (i).

In the next proposition the results of this section are specialized to $W = JM$.

4.6. Proposition (C). For a torsion radical $F$, a torsion preradical $G$, and any unital right $R$-module $M$, form its quasi-injective and injective $(F, G)$-hulls $JM$, $IM$ with $M \subseteq JM \subseteq IM$. Abbreviate the $R$-homomorphisms between any two modules whatever by $\text{Hom}(IM, IM) = (IM, IM)$. Take the subring $\Lambda = G$-End $IM$, $\Lambda \subseteq (IM, IM)$ consisting of all endomorphisms with $G$-dense kernels and let $\Lambda^*$ be the annihilator of $M$ in any ring $\Lambda$ (i.e. $\Lambda^* M = 0$ and $\Lambda^* \subseteq \Lambda \subseteq (IM, IM)$). Similarly for $JM$, form $\Omega^* \subseteq \Omega = G$-End $JM \subseteq (JM, JM)$. The annihilator of any submodule such as $JM$ of $IM$ in $\Lambda$ is denoted by $\Lambda JM$ (e.g. $\Lambda JM = \Lambda^* = (IM, IM)$).

Then the following hold.

(i) $\Lambda^* \Lambda$ is an ideal $\Leftrightarrow \Omega^* \Omega = 0 \Leftrightarrow M \leq JM$ is rational $\Leftrightarrow (JM/M, JM) = 0$.

(ii) $\Omega \equiv \Lambda/\Lambda JM; \Lambda JM \equiv (IM, JM, JM)$.

(iii) If the conditions in (i) hold then $\Omega \equiv \Lambda/\Lambda^*$ and $\Lambda^* = (IM, JM, JM)$. 
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