Pacific Journal of Mathematics

GENERALIZED MONOFORM AND QUASI INJECTIVE MODULES

JOHN DAUNS

Vol. 66, No. 1 November 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol 66, No 1, 1976

GENERALIZED MONOFORM AND QUASI-INJECTIVE MODULES

JOHN DAUNS

For a torsion radical F (arising from an idempotent filter of right ideals) and a unital right R-module M over a ring R, let DM be the F-divisible hull $DM/M = F\,(EM/M)$, where EM is the injective hull of M.

Let $0 \neq \beta: N \to M$ be any nonzero homomorphism whatever from any F-dense submodule $N \subseteq M$. Then M is F-quasi-injective if each such β extends to a homomorphism of $M \to M$; M is F-monic if β is monic; M is F-co-monic if $\beta N \subseteq M$ is F-dense.

Each module M has a natural F-quasi-injective envelope $\mathrm{J}M$ inside $M \subseteq \mathrm{J}M \subseteq DM$.

THEOREM III. *Form the R-endomorphism rings $\Delta =$ End $\mathrm{J}M$ and $\Lambda =$ End DM, and $\Lambda^{\#} = \{\lambda \in \Lambda \mid \lambda M = 0\} \subseteq \Lambda$, the annihilator subring of M.*

When M is F-monic and F-co-monic and $FM = 0$, then

(1) $\Lambda^{\#}$ is exactly the annihilator $\Lambda^{\#} = \{\lambda \in \Lambda \mid \lambda \mathrm{J}M = 0\}$ of the submodule $\mathrm{J}M \subseteq DM$ and $\Lambda^{\#} \subseteq \Lambda$ is an ideal;

(2) $\Delta \cong \Lambda/\Lambda^{\#}$;

(3) Δ is a division ring.

For a torsion radical F and a torsion preradical G, let $\mathrm{I}M$ be the (F, G)-injective hull of M; and, more generally, Λ the ring of all those R-endomorphisms of $\mathrm{I}M$ with G-dense kernels. The above is derived as a special case where $G = 1$ is the identity functor and $\mathrm{I}M = DM$.

THEOREM II. (i) *M is (F, G)-quasi-injective $\Leftrightarrow \Lambda M \subseteq M$.*

(ii) *The (F, G)-quasi-injective hull $\mathrm{J}M$ of M exists and $\mathrm{J}M = M + \Lambda M$.*

(iii) *$\mathrm{J}M$ is the unique smallest (F, G)-quasi-injective module with $M \subseteq \mathrm{J}M \subseteq \mathrm{I}M$.*

Simple modules over a ring were at first generalized to quasi-simple modules (**[9]**), and then these to strongly uniform or monoform ones (**[15]** and **[6]**).

Here the monoform modules are further generalized to the F-monic ones. The quasi-injective hull plays an important role in the theory of quasi-simple modules (**[8]**, **[9]**, **[15]**, and **[6]**). Furthermore, there is a sizeable amount of literature about quasi-injectivity ([7], [8], [5], [16] and [14]). For these two reasons, a quasi-injective envelope is constructed with respect to a torsion radical F and a torsion preradical G. Several previously known facts about quasi-injectives and quasi-injective hulls are generalized so that they can be obtained as corollaries from the more general case. It has been shown ([16; p. 54, Theorem 4.4]) that for a torsion radical F, a module M is F-quasi-injective if and only if it is invariant in its F-divisible hull DM under the ring of endomorphisms of DM. If $\mathbf{I}M$ is the (F, G)-injective hull of M, then M is shown to be (F, G)-quasi-injective if and only if it is invariant under an appropriate usually proper subring Λ in the full endomorphism ring of $\mathbf{I}M$ (Theorem I). Furthermore, in the latter the hypothesis that F be a torsion radical can be relaxed a little (see 2.2). Then by taking G as the identity functor $G = 1$, the previously known case of the F-quasi-injective hull now becomes a corollary of the more general result.

A module is monoform ([6] and [15]) if and only if the endomorphism ring of its quasi-injective hull $\mathbf{J}M$ is a division ring. An analogue of this is proved in Theorem III.

Attention is focused on the original results by labelling them as Theorems I, II, III and Propositions A, B, and C. Proposition C shows that M and $\mathbf{J}M$ have the same annihilator in Λ if and only if $M \subset \mathbf{J}M$ is a rational extension of modules.

For these purposes it was first necessary to use recent new developments in [2] about (F, G)-injectivity. A module C has been called (F, G)-injective provided any homomorphism $\alpha: A \to C$ from an F-dense submodule $A \subset B$ of a module B with kernel $\alpha = \alpha^{-1}0 \subset A$ being G-dense, extends to $B \to C$.

Since some of the concepts used here have been studied independently by several authors, in various guises, under totally different names, their interconnection should be clarified.

The present 1-monic (Definition 3.2), the monoform ([6]), and the strongly uniform modules ([15]) are one and the same; while the quasi-simples ([8] and [9]) are the compressible monoform modules.

A "prefilter" (Definition 1.2) here is the same as a "filter" in [16; p. 517, 1.7 and Definition 1.8]; and the present "idempotent filter" (Definition 1.2), the same as an "idempotent filter" in [15], or a "strongly complete filter" ([16; p. 521, 1.17]).

In [16; p. 517, Lemmas 1.7, 1.9] a one-to-one correspondence is established between

torsion preradicals F ([2; p. 313, §1]),

prefilters (1.2), and

strongly complete, additive classes ([16; p. 515, 1.2]).

There is also a one to one correspondence ([16; p. 515, Definition 1.1, p. 521, Lemma 1.18]) between

torsion radicals F ([**2**; p. 314, §1]),
idempotent filters (1.2) and,
strongly complete Serre classes ([**16**; p. 515, 1.1]).

The following are the same
preradical ([**2**; p. 313, §1]),
subfunctor of the identity ([**16**; p. 548, §5]);

also
torsion preradical ([**2**; p. 313; p. 314]),
concordant functor ([**16**; p. 548, §5]),
left exact subfunctor of the identity.

1. Filters, torsion preradicals, and semi-endomorphisms. The objects that will be investigated are first defined. Those basic properties that are repeatedly used in subsequent proofs are established so that later very short proofs of the main results may be given.

NOTATION 1.1. A module here means a right unital module M over a ring $\mathbf{R}$ with identity. The notation $N \leqq M$ (or $N < M$) always means that N is a right $\mathbf{R}$-submodule (proper) of M. For $m \in M$, define $m^{-1}N \leqq \mathbf{R}$ as the right ideal $m^{-1}N = \{a \in \mathbf{R} \mid ma \in N\}$ of $\mathbf{R}$.

DEFINITION 1.2. A set $\mathcal{F}$ of right ideals of $\mathbf{R}$ is a *prefilter* provided that

(i) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;

(ii) $\mathbf{R} \geqq A \supset B,\ B \in \mathcal{F} \Rightarrow A \in \mathcal{F}$;

(iii) $A \in \mathcal{F},\ b \in \mathbf{R} \Rightarrow b^{-1}A \in \mathcal{F}$.

The prefilter $\mathcal{F}$ has been called an *idempotent filter* if for any $B \leqq \mathbf{R}$, $A \in \mathcal{F}$

(iv) $\{a^{-1}B \mid a \in A\} \subseteq \mathcal{F} \Rightarrow B \in \mathcal{F}$.

1.3. Parentheses will be omitted when the meaning is clear from the context by taking all quotients M/N of modules M and N first and applying any functors F last, e.g. $FM/N \equiv F(M/N)$. For $m \in M$, abbreviate $m^{-1}(0) = m^{-1}0 = m^{\perp} = \{r \in R \mid mr = 0\}$; for any subset M whatever of a module, set $M^{\perp} = \{r \in \mathbf{R} \mid Mr = 0\}$.

1.4. There is a bijective correspondence between prefilters $\mathcal{F}$ and torsion preradicals F given by $\mathcal{F} = \{A \leqq \mathbf{R} \mid F\mathbf{R}/A = \mathbf{R}/A\}$. Conversely, the F-torsion submodule of a module M is $FM = \{m \in M \mid m^{-1}0 \in \mathcal{F}\}$.

Consequently, first, a torsion preradical F is idempotent, or $FFM = FM$ for all modules M. Secondly, for any two or finite number of F-dense submodules $A \leqq C$, $B \leqq C$ their intersection $A \cap B \leqq C$ is also F-dense in C.

Any prefilter $\mathcal{F}$ is idempotent if and only if F is a torsion radical, if and only if density is transitive, i.e. for any three modules $N < M < W$

$$FM/N = M/N,\ FW/M = W/M \Rightarrow FW/N = W/N.$$

A special case is the zero and the identity functors $F = 0$, $G = 1$ corresponding to $\mathcal{F} = \{\mathbf{R}\}$ and $(0) \in \mathcal{G}$.

Any homomorphism $\beta\colon N \to M$ of any submodule $N \leqq M$ into M is called a semi-endomorphism.

A preradical G is a torsion preradical if and only if modules $V < W$ satisfy, either $GV = V \cap GW$, or the condition that all submodules of any G-torsion module are G-torsion.

For the injective hull EM of M the torsion submodule $F((EM)/M) \equiv FEM/M \equiv DM/M$ defines the F-divisible hull $DM \leqq EM$. For preradicals F and G, define $\mathbf{I}M \leqq DM$ by $\mathbf{I}M = M + DM \cap GEM$.

Since $\mathbf{I}M/M \subseteq DM/M \cap (M + GEM)/M$, and $(M + GEM)/M \cong GEM/(M \cap GEM)$, it follows that if either F or G is a torsion preradical, then $M \leqq \mathbf{I}M$ will be F- or G-dense, respectively. When G is a torsion preradical, $GEM \cap DM = GDM$ and $\mathbf{I}M = M + GDM$.

For various facts, definitions, and terminology, **[2]** should be consulted. When F is a torsion preradical and G a preradical, then $\mathbf{I}M$ is (F, G)-injective if and only if $\mathbf{I}M \supseteq DM \cap GEM$. ([**2**; p. 320, 2.5 (d)]). In case F is a torsion radical (while G is still a preradical), $\mathbf{I}M$ is the (F, G)-injective hull of M ([**2**; p. 321, Cor. 2.7]).

For any two modules, the additive abelian group of $\mathbf{R}$-homomorphisms $\mathbf{I}M \to \mathbf{I}M$ will be abbreviated $\operatorname{Hom}(\mathbf{I}M, \mathbf{I}M) = (\mathbf{I}M, \mathbf{I}M)$.

For any right R-module W, and preradical G, G-End W will denote the set $G\text{-End } W \subseteq (W, W)$ of all those $\mathbf{R}$-endomorphisms whose kernels are G-dense ([**2**; p. 315, 1.4 (b)]) in W. When G is a torsion preradical, G-End W is a ring. Define Γ and Λ to be $\Gamma = G\text{-End}\, DM \subseteq (DM, DM)$, $\Lambda = G\text{-End}\, \mathbf{I}M \subseteq (\mathbf{I}M, \mathbf{I}M)$.

DEFINITION 1.5. For preradicals F and G, a module M is (F, G)-*quasi-injective* if any semi-endomorphism $\beta\colon N \to M$ of any F-dense submodule $N \leqq M$, $FM/N = M/N$, and G-dense kernel $\beta^{-1}0 < M$, $GM/\beta^{-1}0 = M/\beta^{-1}0$, extends to an element of Λ, i.e. $\alpha\colon M \to M$ such that the restriction $\alpha \mid N = \beta\colon N \to M$.

DEFINITION 1.6. An $(F, 1)$-injective (or $(F, 1)$-quasi-injective) module will be simply called F-*injective* (or F-*quasi-injective*); and similarly for $(F, 1)$-injective (or $(F, 1)$-quasi-injective) hulls of modules.

1.7. The ordinary concepts of injectivity (or quasi-injectivity) are included in this much more general framework as the special cases of $(1, 1)$-injectivity (or $(1, 1)$-quasi-injectivity).

Whenever M is injective, $M = EM = DM = \mathbf{I}M$ and $\Lambda = \Gamma \subseteq (M, M)$. The case $G = 0$ is uninteresting, for then every module is $(F, 0)$-injective and $\Lambda = \Gamma = 0$.

1.8. For torsion preradicals F and G a module M satisfies the following:

(i) $\exists 1 \in \Gamma \Leftrightarrow GDM = DM \Rightarrow \mathbf{I}M = DM \Rightarrow \Gamma = \Lambda$.

(ii) $\exists 1 \in \Lambda \Leftrightarrow G\mathbf{I}M = \mathbf{I}M \Leftrightarrow GM = M \Leftrightarrow \mathbf{I}M = GDM$.

(iii) $F = 0$ or $G = 1 \Rightarrow \Lambda = \Gamma$.

(iv) When F and G assume the values below, then DM, $\mathbf{I}M$, Λ and Γ are as follows (where a blank means the entry is arbitrary):

F	G	DM	$\mathbf{I}M$	
0		M	M	$\Lambda = \Gamma$
0	1	M	M	$\Lambda = \Gamma = (M, M)$
	1		DM	$\Lambda = \Gamma = (DM, DM)$
1	1	EM	EM	$\Lambda = \Gamma = (EM, EM)$
1		EM	$M + GEM$	

LEMMA 1.9. *Suppose $A \leqq B$, C are any modules, $\psi: C \to B$ any homomorphism, and F any torsion preradical with filter $\mathcal{F}$. Then*

(i) *$A \leqq B$ is F-dense $\Rightarrow \forall D < B$, $A \cap D \leqq D$ is F-dense;*

(ii) *$A \cap \psi C \leqq \psi C$ is F-dense $\Leftrightarrow \psi^{-1}A \leqq C$ is F-dense;*

(iii) *$A \leqq B$ is F-dense $\Rightarrow \psi^{-1}A \leqq C$ is F-dense.*

Proof. (i). Since $D/A \cap D \simeq (A + D)/A$, (i) holds.

(ii). If $y \in C$ is arbitrary, then

$$F\frac{C}{\psi^{-1}A} = \frac{C}{\psi^{-1}A} \Leftrightarrow y^{-1}\psi^{-1}A = (\psi y)^{-1}A \in \mathcal{F} \Leftrightarrow \forall x \in \psi C,$$

$$x^{-1}A = x^{-1}(A \cap \psi C) \in \mathcal{F} \Leftrightarrow F\frac{\psi C}{A \cap \psi C} = \frac{\psi C}{A \cap \psi C}.$$

(iii). (i) and (ii) $\Rightarrow$ (iii).

A limited form of transitivity is formulated below for torsion preradicals. Note that below in 1.10, if $M \subseteq W \subseteq M + HEM$, then $W = M + W \cap HEM = M + HW$.

1.10. If F, G, and H are any torsion preradicals and $M < W$ any submodule such that $M < W = M + HW$, then

(i) $\forall N < M$ such that $HM/N = M/N \Rightarrow$ also $HW/N = W/N$.

Whenever $M \subseteq W \subseteq \mathbf{I}M = M + GDM$, the above property (i) holds when either

(ii) $H = G$, or

(iii) $H = F$ and $\mathbf{I}M = M + F\mathbf{I}M$.

Frequently in subsequent proofs only transitivity of density with respect to F will be all that is required, which will be guaranteed by the following standard hypothesis.

HYPOTHESIS 1.11. Either the torsion preradical

(1) F is a torsion radical; or

(2) $\mathbf{I}M = M + F\mathbf{I}M$.

(In the above, $GDM = F(GDM) \Rightarrow \mathbf{I}M = M + F\mathbf{I}M$. The converse is false.)

1.12. LEMMA (α). *Given torsion preradicals F and G, an extension $M < W$ of modules, any semi-endomorphism $V \leqq W$, ψ: $V \to W$ of W induces a semi-endomorphism β: $N \equiv M \cap \psi^{-1}M \to M$ of M. Consider the following hypotheses*:

(a) $FW/M = W/M$,

(b) $FW/V = W/V$,

(c) $GW/\psi^{-1}0 = W/\psi^{-1}0$.

Then

(i)	(c)	$\Rightarrow \beta^{-1}0 < M$ *is G-dense*.
(ii)	(a)	$\Rightarrow \psi^{-1}M \leqq V$, $N \leqq M \cap V$, *and* $M \cap V \leqq V$ *are F-dense*.
	(b)	$\Rightarrow M \cap V \leqq M$ *is F-dense*.
	(a) *and* (b)	$\Rightarrow M \cap V \leqq W$ *is F-dense*.
(iii)	(a), (b), *and F is a torsion radical*	$\Rightarrow$ $\psi^{-1}M \leqq W$, $M \cap V \leqq W$, $N < W$ *are all F-dense*.
(iv)	(a), (b), (c), *hypothesis* 1.11 *and* $M \leqq V \leqq \mathbf{I}M$	$\Rightarrow$ $\psi^{-1}M \leqq \mathbf{I}M$, $N < \mathbf{I}M$ *are F-dense*; $\beta^{-1}0 < \mathbf{I}M$ *is G-dense*.

Proof. (i) By 1.9 (i), $\beta^{-1}0 = M \cap \psi^{-1}0 < M$ is dense.

(ii) (a). Use of (a) and 1.9 (iii) show that $\psi^{-1}M \leqq V$ is F-dense. By 1.9 (i), $N = M \cap \psi^{-1}M \leqq M \cap V$, and $M \cap V \leqq V$ are too.

(ii) (b). Again, by 1.9 (i), (b) implies that $M \cap V \leqq M$ is F-dense.

(ii) (a) and (b). For any torsion preradical, the intersection of any two dense modules is always dense.

(iii). By (ii) all of $\psi^{-1}M \leqq V$, $N \leqq M \cap V$, $M \cap V \leqq V$, $V \leqq W$ are F-dense, and (iii) follows by transitivity of density.

(iv). The density of $\psi^{-1}M < V$, $N < M$, $\beta^{-1}0 < M$, 1.10, and 1.11 show that these are also dense in $\mathbf{I}M$.

1.13. LEMMA (β). *Starting with torsion preradicals F, G and modules $M < W$ assume that*

(a) $FW/M = W/M$ *and*

(b) $GW/M = W/M$.

Now, conversely suppose $\beta: N \to M$ is any semi-endomorphism from any F-dense submodule $N \leqq M$ and with $\beta^{-1}0 < M$ G-dense. Consider the following two pairs of possible additional hypotheses

(f1) *F is a torsion radical*; (g1) *G is a torsion radical*;

(f2) $W = M + FW$. (g2) $W = M + GW$.

Then β, when regarded as a semi-endomorphism of W, has the following properties

(i) *Either* (f1) *or* (f2) $\Rightarrow$ *$N < W$ is F-dense.*

(ii) *Either* (g1) *or* (g2) $\Rightarrow$ *$\beta^{-1}0 < W$ is G-dense.*

In particular, if $M \subseteq W \subseteq \mathbf{I}M$, then

(iii) *Hypothesis* 1.11 $\Rightarrow$ $FW/N = W/N$, *and* $GW/\beta^{-1}0 = W/\beta^{-1}0$.

The next corollary establishes a correspondence between the endomorphisms of $\mathbf{I}M$ with G-dense kernels and the semi-endomorphisms from F-dense submodules of M with G-dense kernels.

1.14. COROLLARY TO LEMMA (α). *For torsion preradicals F, G each $\lambda \in \Lambda$ induces a semi-endomorphism $\lambda: M \cap \lambda^{-1}M \to M$ with the following properties*

(i) *$\lambda^{-1}M \leqq \mathbf{I}M$, $M \cap \lambda^{-1}M \leqq M$ are F-dense, while $M \cap \lambda^{-1}0 \leqq M$ is G-dense.*

Conversely, for any semi-endomorphism $\beta: N \to M$ from an F-dense $N \leqq M$ and with $\beta^{-1}0 < M$ G-dense, if

(ii) *F is a torsion radical; then $\Rightarrow \exists \lambda \in \Lambda$ such that the restriction $\lambda \mid N = \beta$.*

Proof. (i). Immediate from (α). (ii) A consequence of [**2**; p. 320, Theorem 2.5 (d)].

2. (F, G) quasi-injective hulls. For a torsion radical F and a torsion preradical G, the (F, G)-quasi-injective hull of a unital module M will be constructed. The next theorem generalizes a characterization, that is a necessary and sufficient condition for both ordinary quasi-injectivity and F-quasi-injectivity.

The notation of the previous section is continued. In this section it will be necessary to assume for most of the results that F is a torsion radical and G is a torsion preradical.

2.1. THEOREM I. *For a torsion radical F and a torsion preradical G, form* $\mathbf{I}M \subseteq DM$ *the* (F, G)*-injective module* $\mathbf{I}M$ *and the* F*-divisible hull* DM *of a unital module* M (see 1.4); *let* Λ *and* Γ *be the rings of all endomorphisms of* $\mathbf{I}M$ *and* DM *whose kernels are* G*-dense in* $\mathbf{I}M$ *and* DM.

Then the following holds:
(i) M is (F, G)-quasi-injective $\Leftrightarrow \Lambda M \subseteq M$.

The next corollary observes that the hypothesis that F be a torsion radical is not needed.

2.2. COROLLARY 1 TO THEOREM I. *For torsion preradicals F and G*
(ii) *M is (F, G)-quasi-injective* $\Rightarrow \Lambda M \subseteq M$ *and* $\Gamma M \subseteq M$.

2.3. COROLLARY 2 TO THEOREM I. *If in addition to the above hypotheses in the previous Theorem I, G is a torsion radical and* $G(DM/M) = DM/M$, *then*
(iii) *M is (F, G)-quasi-injective* $\Leftrightarrow \Gamma M \subseteq M$.

Proof. 2.1 (i), 2.2 (ii), and 2.3 (iii) $\Rightarrow$: If M is (F, G)-quasi-injective and $\lambda \in \Lambda$ (or $\lambda \in \Gamma$), let $\beta: N \equiv M \cap \lambda^{-1}M \to M$ be the semi-endomorphism induced by λ. By 1.14 (i) and hypothesis, β extends to $\alpha: M \to M$ with the restriction $\alpha \mid N = \beta$. Assume $(\lambda - \alpha)M \neq 0$. Then since $M \leqq EM$ is essential, let

$$0 \neq m = \lambda n - \alpha n \in (\lambda - \alpha)M \cap M \neq 0 \qquad m, n \in M.$$

Since $\lambda n = m + \alpha n \in M$, we have $n \in N$. But $\lambda \mid N = \alpha \mid N = \beta$; hence $m = 0$, a contradiction. Thus $(\lambda - \alpha)M = 0$, $\lambda M = \alpha M \subseteq M$, and as required $\Lambda M \subseteq M$ (or $\Gamma M \subseteq M$).

2.1 (i), and 2.3 (iii) $\Leftarrow$: Conversely, assume $\Lambda M \subseteq M$ (or $\Gamma M \subseteq M$) and take any semi-endomorphism $\beta: N \to M$ with $N < M$, $\beta^{-1}0 < M$ being F, G-dense. By 1.14 (ii) (or [**12**; p. 61]) β extends to a map $\lambda: \mathbf{I}M \to \mathbf{I}M$ (or, since F is a torsion radical, to $\lambda: DM \to DM$). Since $M < \mathbf{I}M$ and $\beta^{-1}0 < M$ are G-dense it follows from Lemma (β) (ii), that $\beta^{-1}0 < \mathbf{I}M$ is G-dense and hence that $\lambda \in \Lambda$. (Since $\beta^{-1}0 < M < DM$ are all G-dense in 2.3, it follows from Lemma (β) that again $\lambda^{-1}0 < DM$ is G-dense and thus $\lambda \in \Gamma$.) By hypothesis $\lambda M \subseteq M$. Finally, the restriction and corestriction α of λ to $\alpha \equiv \lambda \mid M: M \to M$ is the required extension of β to $M \to M$ with $\alpha \mid N = \beta$.

For $G = 1$, $\mathbf{I}M$ and Λ become $\mathbf{I}M = DM$ and thus $\Lambda = \Gamma$. A result in [**16**; p. 541, Theorem 4.4] is the special case when $G = 1$ in the previous Theorem I.

2.4. COROLLARY 3 TO THEOREM I. *With the notation and hypotheses of Theorem I for $G = 1$:*

$$M \text{ is } F\text{-quasi-injective} \Leftrightarrow \Gamma M \subseteq M.$$

The following well-known fact is a special case of the previous Theorem I with $F = 1$ and $G = 1$ (or of 2.4).

2.5. COROLLARY 4 TO THEOREM I. *If Λ is the $\mathbf{R}$-endomorphism ring of the injective hull of a module M, then*

$$M \text{ is quasi-injective} \Leftrightarrow \Lambda M \subseteq M.$$

2.6. For torsion preradicals F and G and any module M, $\Lambda \mathbf{I}M \subseteq GDM$ and in particular $\Lambda M \subseteq GDM$.

Proof. Since $\Lambda = G\text{-End}\, IM$, $\Lambda \mathbf{I}M \subseteq \mathbf{I}M \subseteq DM$.

Again by 1.9 (iii), under the map $\mathbf{R} \to \mathbf{I}M$, $r \to yr$, the inverse image of the G-dense submodule $\lambda^{-1}0 < \mathbf{I}M$ is $y^{-1}(\lambda^{-1}0) = (\lambda y)^{-1}0 < \mathbf{R}$ and it is G-dense in $\mathbf{R}$. So $\lambda y \in GEM$, and $\Lambda \mathbf{I}M \subseteq GEM$.

2.7. When F and G are preradicals and $V < W$ any extension of modules, then there is a natural inclusion $\mathbf{I}V \subseteq \mathbf{I}W$.

Proof. By the definition of the preradical functor, the maps $EV/V \to (EV + W)/W \to EW/W$ induce by restriction

$$F\frac{EV}{V} \to F\frac{EV + W}{W} \to F\frac{EW}{W},$$

and the image of DV/V is

$$\frac{DV}{V} \twoheadrightarrow \frac{DV + W}{W} \subseteq F\frac{EV + W}{W} \subseteq \frac{DW}{W}.$$

Hence there are natural inclusions $DV \subseteq DW$ and $\mathbf{I}V \subseteq \mathbf{I}W$.

2.8. THEOREM II. *For a torsion radical F, a torsion preradical G, and a unital right R-module M, let EM be the injective hull of M, and DM be the module $DM/M = FEM/M$. Form the (F, G)-injective module $\mathbf{I}M = M + GDM \subseteq EM$ ([**2**; p. 321, 2.7]), and the ring Λ of all those $\mathbf{R}$-endomorphisms of $\mathbf{I}M$ whose kernels are G-dense in $\mathbf{I}M$.*

(i) *Existence*: $\exists$ *an* (F, G)*-quasi-injective module* (see 1.5) $\mathbf{J}M$ *in* $M \subseteq \mathbf{J}M \subseteq \mathbf{I}M$.

(ii) *Intrinsic characterization*: $\mathbf{J}M = M + \Lambda M$ *and thus* $\mathbf{J}M$ *is the unique smallest* Λ*-invariant submodule containing* M *inside* $\mathbf{I}M$.

(iii) *Uniqueness inside* $\mathbf{I}M$: *Any* (F, G)*-quasi-injective submodule* W *with* $M \subseteq W \subseteq \mathbf{I}M$ *satisfies* $\mathbf{J}M \subseteq W$; $\mathbf{J}M$ *is the unique smallest* (F, G)*-quasi-injective submodule of* $\mathbf{I}M$ *containing* M.

Proof. In general for a preradical F, for any module W in $M \leqq W \leqq DM$, also $DM \subseteq DW \subseteq D(DM)$. When F is a torsion radical as is the case here, $D(DM) = DM = DW$. When $M \leqq W \leqq DM$, then $EW = EM$, $DW = DM$, and consequently $\mathbf{I}W = W + GDM = W + \mathbf{I}M \supseteq \mathbf{I}M$. If moreover, in addition W is futher restricted to $M \leqq W \leqq \mathbf{I}M$, then $\mathbf{I}W = W + \mathbf{I}M = \mathbf{I}M$ for any torsion preradical G; furthermore, the subring $\Omega \subseteq (\mathbf{I}W, \mathbf{I}W)$ whose kernels are G-dense in $\mathbf{I}W$ is $\Omega = \Lambda$ the same as that of $\mathbf{I}M$.

(i) and (ii). Since $\Lambda \subseteq (\mathbf{I}M, \mathbf{I}M)$, the image ΛM is in $\Lambda M \subseteq \mathbf{I}M$. Use of the above for $W = M + \Lambda M$ with $M \leqq W \leqq \mathbf{I}M$ shows that $\mathbf{I}(M + \Lambda M) = \mathbf{I}M$. Set $\mathbf{J}M \equiv M + \Lambda M$. Those endomorphisms $\Omega \subseteq (\mathbf{IJ}M, \mathbf{IJ}M)$ with G-dense kernels in $\mathbf{IJ}M = \mathbf{I}M$ are just $\Omega = \Lambda$. Because $\Omega\mathbf{J}M \subseteq \mathbf{J}M$, Theorem I shows that $\mathbf{J}M$ is (F, G)-quasi-injective.

(iii) For an (F, G)-quasi-injective W with $M \leqq W \leqq \mathbf{I}M$ as in (iii), by the above $\mathbf{I}W = \mathbf{I}M$ and $\Omega = \Lambda$. By Theorem I, $\Lambda W \subseteq W$. Thus $\Lambda M \subseteq W$ and hence $\mathbf{J}M = M + \Lambda M \subseteq W$.

3. Endomorphism rings of F-monic modules. This section can be read independently from the previous ones, provided one only assumes the existence of an F-quasi-injective module $\mathbf{J}M$ where $M \subseteq \mathbf{J}M = (\operatorname{End} \mathbf{I}M)M \subseteq DM$. Here G will be $G = 1$ so that all submodules are G-dense.

NOTATION 3.1. The previous notation is continued for arbitrary F but $G = 1$. When F is a torsion radical, then $\mathbf{I}M = DM$ is $(F, 1)$- or F-injective, or F-divisible, and $\Lambda = (DM, DM) = \operatorname{End} DM$ now is the full endomorphism ring. For any $W \leqq DM$, ${}^{\perp}W$ denotes ${}^{\perp}W = \{\lambda \in \Lambda \mid \lambda W = 0\}$. Let $\Lambda^{\#}$ be the subring $\Lambda^{\#} = {}^{\perp}M \subseteq \Lambda$. When F is a torsion radical, $\mathbf{I}M = DM$ is the F-injective hull of M, and the F-quasi-injective hull $\mathbf{J}M$ of M exists and $\mathbf{J}M = \Lambda M$. Set $\Delta = \operatorname{End} \mathbf{J}M = (\mathbf{J}M, \mathbf{J}M)$. Define $\Sigma \subseteq \Delta$ as the multiplicative semigroup of all endomorphisms $0 \neq \alpha : \mathbf{J}M \to \mathbf{J}M$ such that the image αV of every F-dense submodule $V \leqq \mathbf{J}M$ is also an F-dense submodule $\alpha V \leqq \mathbf{J}M$. Ideals (two-sided) are denoted by "$\triangleleft$".

DEFINITION 3.2. A module M is *F-monic* with respect to a preradical F provided any nonzero semi-endomorphism $0 \neq \beta \colon N \to M$ with an F-dense domain $N \leqq M$ in M is monic, i.e. $\beta^{-1}0 = 0$.

For $F = 1$, the 1-monic modules have already been called simply monoform ([6]) or strongly uniform.

DEFINITION 3.3. For a preradical F, a module M is *F-co-monic* if for every nonzero semi-endomorphism $0 = \beta \colon N \to M$ from any F-dense submodule $N \leqq M$, its image βN is also F-dense $\beta N \leqq M$ in M.

The module M is *weakly F-co-monic* if for every R-endomorphism $0 \neq \alpha \in \operatorname{End} M$ of M, the image $\alpha M \leqq M$ is F-dense.

LEMMA 3.4. *For a torsion radical F and a module W in $M \leqq W \leqq DM$, W is F-co-monic if every semi-endomorphism $0 \neq \alpha \colon V \to W$ with $V \subseteq M$ satisfies*

$$V \leqq M \text{ is } F\text{-dense} \Rightarrow \text{also } \alpha V \leqq \mathbf{J}M \text{ is } F\text{-dense}.$$

The converse holds if W is F-monic.

Proof. Let $V \leqq M$ and $FM/V = M/V$. Firstly by transitivity of density, any extension that can be formed from any two modules below will be an F-dense extension

$$V \leqq M \leqq W \leqq DM, \qquad V \leqq M \leqq \mathbf{J}M \leqq DM.$$

Secondly, any $\alpha \colon V \to W$ extends to a map $\lambda \colon DM \to DM$. Since $V \leqq M$, $\alpha V \leqq \mathbf{J}M = \Lambda M$. Thus $\alpha V \leqq \mathbf{J}M$.

$\Rightarrow$: Since $V \leqq M$, $V \leqq W$, and $W \leqq DM$ are F-dense, and since W is F-co-monic, it follows that $\alpha V \leqq W$, $W \leqq DM$, $\alpha V \leqq DM$, and hence $\alpha V \leqq \mathbf{J}M$ are all F-dense.

$\Leftarrow$: For any semi-endomorphism $0 \neq \gamma \colon U \to W$ from an F-dense $U \leqq W$, set $V \equiv U \cap M$ and $0 \neq \alpha \equiv \gamma \mid V \colon V \to W$. By hypothesis $0 \neq \alpha V \leqq \mathbf{J}M$ is F-dense. Thus $\mathbf{J}M \leqq DM$, $\alpha V \leqq Dm$, and in particular $\alpha V \leqq W$ are all F-dense. But then $\alpha V \leqq \gamma U \leqq W$ shows that $\gamma U \leqq W$ is F-dense. Thus W is F-co-monic.

LEMMA 3.5. *For a module M that is F-monic with respect to a torsion radical F, any nonzero semi-endomorphism $0 \neq \alpha \colon V \to DM$ from an F-dense submodule $V \leqq DM$ satisfies*:

(i) *either α is monic on all of V or $\alpha(M \cap V) = 0$.*

(ii) *In particular, $M \leqq V \Rightarrow$ either α is monic on V or $\alpha M = 0$.*

Proof. Note that $\alpha^{-1}M = \alpha^{-1}(M \cap \alpha V)$ by definition. Set $N \equiv M \cap \alpha^{-1}M$ and let $\beta \equiv \alpha \mid N \colon N \to M$ be the induced semi-endomorphism.

Since $M \leqq DM$, $V \leqq DM$ are F-dense, so are $M \cap V \leqq DM$, $M \cap V \leqq V$, and $\alpha^{-1}M \leqq V$. Hence the intersection $N = M \cap V \cap \alpha^{-1}M \leqq V$ of two F-dense submodules of V is F-dense in V. The fact that $N \leqq V$ and $V \leqq DM$ are both F-dense with respect to a torsion radical, by transitivity implies that $N \leqq DM$ is F-dense too. (Alternatively we may argue that $\alpha^{-1}M < V$, $N = M \cap \alpha^{-1}M \leqq M \cap V$, $M \cap V \leqq DM$ being F-dense imply — again by transitivity — that $N \leqq DM$ is F-dense.)

Assume alternative (i) does not hold. So suppose that $\alpha(M \cap V) \neq 0$ and also that $\alpha^{-1}0 \neq 0$. Because $M \leqq DM$ is essential, also $\beta^{-1}0 = M \cap \alpha^{-1}0 \neq 0$, and there exists $0 \neq \alpha n \in M \cap \alpha(M \cap V)$ with $n \in M \cap V$. Hence $n \in N$, $\beta n \neq 0$, and $\beta \neq 0$. On the other hand, since $N \leqq M$ is F-dense, since M is F-monic, and $0 \neq \beta\colon N \to M$, it follows that $\beta^{-1}0 = 0$, a contradiction. Hence either $\alpha(M \cap V) = 0$ or $\alpha^{-1}0 = 0$.

3.6. Proposition (A). *When $M < W$ is F-dense with respect to a torsion radical F, then*

(i) *W is F-co-monic $\Rightarrow$ M is F-co-monic;*

(ii) *W is F-monic $\Rightarrow$ M is F-monic.*

If, in addition to the above hypotheses, $M < W$ is essential, and $FM = 0$, then the converse holds:

(iii) *M is F-co-monic $\Rightarrow$ W is F-co-monic;*

(iv) *M is F-monic $\Rightarrow$ W is F-monic.*

Proof. (i) and (ii) $\Rightarrow$: For (i) and (ii), given any $0 \neq \beta\colon N \to M$ with $N \leqq M$, $FM/N = M/N$, also $N \leqq W$ is F-dense in W because F is a torsion radical. Now regard $0 \neq \beta\colon N \to M$ as a semi-endomorphism of W. (i) Thus $\beta N \leqq W$ is F-dense, and M is F-co-monic in case of (i). (ii) By (ii), β is monic, and M is F-monic.

(iii) and (iv) $\Rightarrow$: In both (iii) and (iv) for any $0 \neq \alpha\colon V \to W$ with $FW/V = W/V$, form $N = M \cap \alpha^{-1}M$ and $\beta \equiv \alpha \mid N\colon N \to M$. Since $V \leqq W$, $\alpha^{-1}M \leqq V$ are F-dense, so are the intersections $N = M \cap \alpha^{-1}M \leqq M \cap V$ and $M \cap V \leqq M$. By transitivity $N \leqq M$ is F-dense. Since $M < W$ is essential and $\alpha \neq 0$, there is a $v \in V$ with $0 \neq \alpha v \in \alpha V \cap M$. Then $v^{-1}M \leqq R$ is F-dense because $M < W$ is, and $FM = 0$ requires that there is a nonzero element $0 \neq \alpha n \in (\alpha v)v^{-1}M \subseteq M \cap \alpha vR$ with $n \in V \cap M$. Thus $n \in N$, $\beta n = \alpha n \neq 0$, and $\beta \neq 0$.

(iii) In case of (iii), $\beta N = \alpha N \leqq M$ is F-dense. Again by transitivity $\alpha N \leqq W$ will be F-dense. Thus W is F-co-monic.

(iv) By (iv), β is monic $\beta^{-1}0 = M \cap \alpha^{-1}0 = 0$. Since $M \leqq W$ is essential, also $\alpha^{-1}0 = 0$ and W is F-monic.

3.7. Proposition (B). *The following conclusions hold under the assumptions on a module M and torsion radical F that*

(a) *M is F-monic.*

(i) $\Lambda^{\#} \lhd \Lambda$;

(ii) $\Lambda^{\#}\mathbf{J}M = 0$, *and* ${}^{\perp}\mathbf{J}M = {}^{\perp}M = \Lambda^{\#}$;

(iii) $\Delta \cong \Lambda/\Lambda^{\#}$;

(iv) *The maps in* $\Lambda\backslash\Lambda^{\#}$ *and in* $\Delta\backslash\{0\}$ *are monic*; $\Lambda\backslash\Lambda^{\#}$ *is a multiplicative semigroup and* Δ *is an integral domain.*

Proof. (i) Always $\Lambda\Lambda^{\#} \subseteq \Lambda^{\#}$. For any $0 \neq \gamma \in \Lambda^{\#}$ and any $0 \neq \lambda \in \Lambda$, we have $0 \neq M \cap \lambda M \subseteq \ker \gamma$. Since $\gamma\lambda : DM \to DM$ is not monic, it follows from 3.5 (ii) (with $V = DM$ and $\alpha = \gamma\lambda$), that $\gamma\lambda M = 0$, and hence that $\gamma\lambda \in \Lambda^{\#}$. Thus $\Lambda^{\#} \lhd \Lambda$.

(ii) Since $\mathbf{J}M = \Lambda M$, by (i) $\Lambda^{\#}\mathbf{J}M = 0$. Always $\Lambda^{\#} \equiv {}^{\perp}M \supseteq {}^{\perp}\mathbf{J}M$, because $M \subseteq \mathbf{J}M$. Hence ${}^{\perp}\mathbf{J}M = {}^{\perp}M$.

(iii) The restriction map $\Lambda \to \Delta$ of elements of Λ to the Λ-submodule $\mathbf{J}M \subseteq DM$ induces a ring isomorphism $\Delta \cong \Lambda/{}^{\perp}\mathbf{J}M = \Lambda/\Lambda^{\#}$.

(iv) Take a, $b \in \Lambda\backslash\Lambda^{\#}$, $\alpha = a + \Lambda^{\#}$, $\beta = b + \Lambda^{\#} \in \Delta$. Note that $ay = \alpha y$ for every $y \in \mathbf{J}M$. If a were not monic, then by 3.5 (ii) (with $V = DM$), $aM = 0$ and $a \in \Lambda^{\#}$. Thus $aM \neq 0$; a, b are monic, $baM \neq 0$, and $ba \in \Lambda\backslash\Lambda^{\#}$. Hence $\Lambda\backslash\Lambda^{\#}$ and $\Delta\backslash\{0\}$ are semigroups of monic maps.

The next corollary is merely a continuation of the previous proposition. It attempts to determine how close Δ is to being a division ring.

3.8. Corollary 1 to Proposition (B). *Under the assumptions of the previous Proposition* 3.7.:

(v) Σ *is a subgroup of the multiplicative group of units of* $\Delta\backslash\{0\}$; *and* Σ *consists of automorphisms of* $\mathbf{J}M$.

(vi) $\mathbf{J}M$ *is weakly F-co-monic* $\Rightarrow$ Δ *is a division ring.*

Proof. (v) Since any $\alpha \in \Sigma$ is monic by 3.7 (iv), the semi-endomorphism $\alpha\mathbf{J}M \to \mathbf{J}M$, $\alpha y \to y$ from the F-dense submodule $\alpha\mathbf{J}M \leqq \mathbf{J}M$, extends to $\beta : \mathbf{J}M \to \mathbf{J}M$ with $\beta\alpha = 1$, and $\beta \in \Delta$. In order to show that $\beta \in \Sigma$, it will suffice to show that for any F-dense $W \leqq \mathbf{J}M$ submodule, $\beta(W \cap \alpha\mathbf{J}M) \leqq \mathbf{J}M$ is F-dense. Because β is a left inverse of α, a straightforward verification shows that $\beta(W \cap \alpha\mathbf{J}M) = \alpha^{-1}W$. But $\alpha^{-1}W \leqq \mathbf{J}M$ is F-dense. Thus $\beta \in \Sigma$. Since Σ is a semigroup, $\beta\alpha = \alpha\beta = 1$. Thus Σ is a group of automorphisms of $\mathbf{J}M$.

(vi) By 3.7 (iv), $\Delta\backslash\{0\}$ is a semigroup, and by the argument in 3.8 (v), each element has a left inverse. Thus Δ is a division ring.

3.9. Corollary 2 to Proposition (B). *For a torsion radical F, define* $\Sigma^* \equiv \{\alpha \in \Lambda \mid \alpha\mathbf{J}M \leqq \mathbf{J}M$ *is F-dense*$\}$. *Then*

(i) Σ^* *is a semigroup with* $\Sigma \subseteq \Sigma^*$.

(ii) 3.7(a) $\Rightarrow$ 3.8 *holds for* Σ^* *in place of* Σ.

Proof. (i) For $\alpha, \beta \in \Sigma^*$, since $\mathbf{J}M/\alpha\mathbf{J}M \to \beta\mathbf{J}M/\beta\alpha\mathbf{J}M$ is epic, $\beta\alpha\mathbf{J}M \subseteq \beta\mathbf{J}M$ is F-dense. Hence Σ^* is a semigroup. (ii) is clear from 3.8.

A few of the results of this section are combined and summarized below.

THEOREM III. *Starting with a torsion radical F, a unital right $\mathbf{R}$-module M, its F-injective or F-divisible hull DM and the R-endomorphism ring $\Lambda = (DM, DM)$ of DM, form the F-quasi-injective hull $\mathbf{J}M = \Lambda M$ of M, and its R-endomorphism ring $\Delta = (\mathbf{J}M, \mathbf{J}M)$; let $\Lambda^\# = \{\lambda \in \Lambda \mid \lambda M = 0\} = {}^\perp M$, ${}^\perp\mathbf{J}M = \{\lambda \in \Lambda \mid \lambda\mathbf{J}M = 0\}$ be the annihilator subrings of M, $\mathbf{J}M$ in Λ.*

(i) *If $FM = 0$, then*

M is F-co-monic (respectively F-monic) $\Leftrightarrow$ $\mathbf{J}M$ is likewise.

The subsequent conclusions hold under the assumption that

(a) M is F-monic.

(ii) $\Lambda^\#$ is exactly the annihilator of the Λ-$\mathbf{R}$-bi-submodule $\mathbf{J}M$ of DM; hence

$$\Lambda^\#\mathbf{J}M = 0, \quad {}^\perp M = {}^\perp\mathbf{J}M, \quad \Lambda^\# \lhd \Lambda, \quad \text{and} \quad \Delta \cong \Lambda/\Lambda^\#.$$

(iii) Δ is an integral domain; furthermore, each nonzero map of Δ is monic.

Finally, when in addition to (a), it is assumed that

(b) $FM = 0$ and M is F-co-monic,

(iv) then Δ is a division ring.

4. Arbitrary endomorphism rings. Without any restrictive assumptions on the unital module M, but some restrictions on the torsion preradicals F and G, the subring $\Lambda \subseteq (\mathbf{I}M, \mathbf{I}M)$ will be considered. It should be stressed that

$$F = 1 \qquad G = 1\colon \mathbf{I}M = EM, \quad \Lambda = (EM, EM);$$

$$F \text{ arbitrary}, \quad G = 1\colon \mathbf{I}M = DM, \quad \Lambda = (DM, DM);$$

are included as special cases. Here, some information is obtained about the G-endomorphism ring of the (F, G)-quasi-injective hull $\mathbf{J}M$ of M. The simplest way of doing this is to consider any Λ-$\mathbf{R}$-bimodule W in $M \leqq W \leqq \mathbf{I}M$. This approach also seems to be the shortest because $W = M$, $W = \mathbf{J}M$, or $W = \mathbf{I}M$ are frequently used special cases.

NOTATION 4.1. The notation of the previous sections will be continued.

For any subset $W \subseteq \mathbf{I}M$, define ${}^\perp W = \{\lambda \in \Lambda \mid \lambda W = 0\}$. From now on W will be a Λ-$\mathbf{R}$-bimodule in $M \leqq W \leqq \mathbf{I}M$; in this case define

$\Omega = G - \text{End } W$ as the subring $\Omega \subseteq (W, W)$ of all endomorphisms of W whose kernels are G-dense in W. If $\Lambda W \subseteq W$, then ${}^{\perp}W \lhd \Lambda$, where "$\lhd$" denotes (two-sided) ideals. Set $\Omega^{\#} = \{\gamma \in \Omega \mid \gamma M = 0\}$. Unless stated otherwise, in this section F will be a torsion radical and G a torsion preradical.

When F is a torsion radical, then W is (F, G)-quasi-injective by Theorem II.

Note that (iii) below provides a means of contructing (F, G)-quasi-injective Λ-$\mathbf{R}$-bimodules.

LEMMA 4.2. *With the notation and the hypotheses of* 4.1 (as well as 1.11), *if* $\pi: \Lambda \to \Omega$ *is the map induced by restricting* Λ *to* W, *and if* $V \leqq W$ *is an* $\mathbf{R}$-*submodule, then*

(i) *the exact sequence of modules induces an exact commutative diagram of ring homomorphisms*

$$\mathbf{I}M/W \leftarrow \mathbf{I}M \leftarrow W$$

$$\begin{array}{ccccccccc} 0 & \to & {}^{\perp}W & \to & \Lambda & \xrightarrow{\pi} & \Omega & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & {}^{\perp}W & \to & \Lambda^{\#} & \xrightarrow[\pi]{} & \Omega^{\#} & \to & 0 \end{array}$$

(ii) *In particular,*

$$\pi\Lambda = \Omega \cong \Lambda/{}^{\perp}W \qquad \Omega^{\#} = \pi\Lambda^{\#} \cong \Lambda^{\#}/{}^{\perp}W; \textit{ and}$$
$${}^{\perp}W \cong (\mathbf{I}M/W, \mathbf{I}M).$$

(iii) $\Lambda V = \Omega V$ *and* $V + \Omega V$ *is a* Λ-$\mathbf{R}$-*bimodule.*

Proof. In (i) and (ii) in general, that is when F as well as G are only torsion preradicals, the sequences need not be right exact and π induces only ring monomorphisms with $\pi\Lambda \subseteq \Omega$, $\pi\Lambda^{\#} \subseteq \Omega^{\#}$. In this case in (iii), $\Lambda V \subseteq \Omega V$ and $V + \Omega V$ is not necessarily a left Λ-module.

(i), (ii), and (iii). However, when F is a torsion radical, for any $\beta \in \Omega$, use of $M \subseteq W$ and $\beta^{-1}0 \leqq W \leqq \mathbf{I}M \subseteq W + GEM$ shows that β extends to a map $\lambda \in \Lambda$ with $\pi\lambda = \lambda \mid W = \beta$.

LEMMA 4.3. *Under the assumptions of* 4.1 *that* G *is a torsion preradical and* F *is a torsion radical, all of the five conditions* (i)–(v) *are equivalent.*

(i) $\Lambda^{\#} = {}^{\perp}(M + \Lambda M)$;

(ii) $\Lambda^{\#} \lhd \Lambda$;

(iii) $\Omega^{\#} \lhd \Omega$;

(iv) $M \leqq M + \Lambda M$ *is a rational extension of modules*;
(v) $((M + \Lambda M)/M, M + \Lambda M) = 0$.

Proof. First note that since $M \subseteq M + \Lambda M$, always $\Lambda^{\#} \supseteq {}^{\perp}(M + \Lambda M)$. Also $\Lambda^{\#}$ is a left ideal $\Lambda\Lambda^{\#} \subseteq \Lambda^{\#}$ always.

(i) $\Rightarrow$ (ii). Since $\Lambda^{\#}\Lambda M = 0$, $\Lambda^{\#}\Lambda \subseteq \Lambda^{\#}$ and $\Lambda^{\#} \lhd \Lambda$. (ii) $\Rightarrow$ (i): If $\Lambda^{\#} \lhd \Lambda$, then $\Lambda^{\#}\Lambda \subseteq \Lambda^{\#}$ and $\Lambda^{\#}(M + \Lambda M) = 0$. Thus $\Lambda^{\#} = {}^{\perp}(M + \Lambda M)$. So far for (i) $\Leftrightarrow$ (ii), the hypothesis that F is a torsion radical was not needed.

(ii) $\Rightarrow$ (iii). However, the fact that F is a torsion radical was used to show in 4.2 (i) that the image of the subring $\Lambda^{\#}/{}^{\perp}W \subseteq \Lambda/{}^{\perp}W$ under the isomorphism $\Lambda/{}^{\perp}W \cong \Omega$ is exactly $\Omega^{\#}$. Thus if $\Lambda^{\#} \lhd \Lambda$, then necessarily also $\Omega^{\#} \lhd \Omega$.

(iii) $\Rightarrow$ (i). In 4.2 (i), $(\lambda - \pi\lambda)W = 0$ for $\lambda \in \Lambda$, and $\pi\Lambda^{\#} = \Omega^{\#}$. In 4.2 (iii) for $V = M$, since $M \subset W$, also $\Lambda M \subseteq \Lambda W \subseteq W$, and consequently $\Lambda^{\#}\Lambda M = \pi\Lambda^{\#}\pi\Lambda M = \Omega^{\#}\Omega M = 0$, since $\Omega^{\#}\Omega \subseteq \Omega^{\#}$. Thus $\Lambda^{\#} \subseteq {}^{\perp}(M + \Lambda M)$ and hence $\Lambda^{\#} = {}^{\perp}(M + \Lambda M)$.

In the remainder, let β represent any map $\beta\colon V \to M + \Lambda M$ for $M < V \leqq M + \Lambda M$ with $\beta M = 0$.

(iii) $\Rightarrow$ (iv). Since β extends to $\lambda \in \Lambda$ with $\lambda \mid V = \beta \mid V$, we get $\lambda \in \Lambda^{\#}$. Thus (i) $\Lambda^{\#} \lhd \Lambda$ implies that $\lambda(M + \Lambda M) = 0$. Hence $M \leqq M + \Lambda M$ is rational. Always, (iv) $\Rightarrow$ (v).

(v) $\Rightarrow$ (iv). By Theorem II, $M + \Lambda M = \mathbf{J}M$, the (F, G)-quasi-injective hull of M. Therefore a typical β as above extends to $\gamma \in G$-End $\mathbf{J}M$, $\gamma\colon M + \Lambda M \to M + \Lambda M$. Due to $\beta M = \gamma M = 0$, γ induces a map $\alpha \in ((M + \Lambda M)/M, M + \Lambda M)$. Thus by (v) $\alpha = 0$, $\gamma = 0$, and $\beta = 0$. Hence $M \leqq M + \Lambda M$ is rational. Clearly, (v) $\Rightarrow$ (i).

In the next proposition the results of this section are specialized to $W = \mathbf{J}M$.

4.6. PROPOSITION (C). *For a torsion radical F, a torsion preradical G, and any unital right R-module M, form its quasi-injective and injective (F, G)-hulls* $\mathbf{J}M$, $\mathbf{I}M$ *with* $M \subseteq \mathbf{J}M \subseteq \mathbf{I}M$. *Abbreviate the R-homomorphisms between any two modules whatever by* $\operatorname{Hom}(\mathbf{I}M, \mathbf{I}M) = (\mathbf{I}M, \mathbf{I}M)$. *Take the subring $\Lambda = G$-End$\,\mathbf{I}M$, $\Lambda \subseteq (\mathbf{I}M, \mathbf{I}M)$ consisting of all endomorphisms with G-dense kernels and let $\Lambda^{\#}$ be the annihilator of M in any ring Λ* (i.e. $\Lambda^{\#}M = 0$ and $\Lambda^{\#} \subseteq \Lambda \subseteq (\mathbf{I}M, \mathbf{I}M)$). *Similarly for* $\mathbf{J}M$, *form* $\Omega^{\#} \subseteq \Omega = G$-End$\,\mathbf{J}M \subseteq (\mathbf{J}M, \mathbf{J}M)$. *The annihilator of any submodule such as $\mathbf{J}M$ of $\mathbf{I}M$ in Λ is denoted by ${}^{\perp}\mathbf{J}M$* (e.g. ${}^{\perp}M = \Lambda^{\#}$). *Then the following hold.*

(i) $\Lambda^{\#} \lhd \Lambda$ *is an ideal* $\Leftrightarrow \Omega^{\#} \lhd \Omega \Leftrightarrow \Lambda^{\#}\mathbf{J}M = 0 \Leftrightarrow M \leqq \mathbf{J}M$ *is rational* $\Leftrightarrow (\mathbf{J}M/M, \mathbf{J}M) = 0$.

(ii) $\Omega \cong \Lambda/{}^{\perp}\mathbf{J}M$; ${}^{\perp}\mathbf{J}M \cong (\mathbf{I}M/\mathbf{J}M, \mathbf{J}M)$.

(iii) *If the conditions in* (i) *hold then* $\Omega \cong \Lambda/\Lambda^{\#}$ *and* ${}^{\perp}M = {}^{\perp}\mathbf{J}M = \Lambda^{\#} \cong (\mathbf{I}M/\mathbf{J}M, \mathbf{J}M)$.

REFERENCES

1. J. A. Beachy, *Generating and cogenerating structures*, Trans. Amer. Math. Soc., **158** (1971), 75–92.

2. ———, *A generalization of injectivity*, Pacific J. Math., **41** (1972), 313–327.

3. J. Dauns, *Simple modules and centralizers*, Trans. Amer. Math. Soc., **166** (1972), 457–477.

4. ———, *One sided prime ideals*, Pacific J. Math., **47** (1973), 401–412.

5. L. Fuchs, *On quasi-injective modules*, Annali della Scuola Normale Superiore di Pisa Classe di Scienze, **23** (1969), 541–546.

6. R. Gordon and J. C. Robson, *Krull dimension*, Amer. Math. Soc., Memoir No. 133.

7. R. E. Johnson and E. T. Wong, *Quasi-injective modules and irreducible rings*, J. London Math. Soc., **36** (1961), 260–268.

8. K. Koh, *Quasi-simple modules and other topics in ring theory*, Tulane Ring Year Lecture Notes, Lecture Notes in Mathematics No. 246, Springer (1972), New York.

9. K. Koh and A. Mewborn, *A class of prime rings*, Canad. Math. Bull., **9** (1966), 63–72.

10. J. Lambek, *Lectures on Modules and Rings*, Blaisdell, Waltham, Mass., 1966.

11. ———, *Torsion Theories, Additive Semantics, and Rings of Quotients*, Springer Verlag, Berlin, 1971.

12. ———, *Bicommutators of nice injectives*, J. Algebra, **21** (1972), 60–73.

13. ———, *Localization and completion*, J. Pure and Appl. Algebra, **2** (1972), 343–370.

14. ———, *Localization at epimorphisms and quasi-injectives*, to appear.

15. H. Storrer, *On Goldman's primary decomposition*, Tulane Ring Year Lecture Notes, Lecture Notes in Mathematics No. 246, Springer (1972), New York.

16. C. L. Walker and E. A. Walker, *Quotient categories and rings of quotients*, Rocky Mountain J. Math., **2** (1972), 513–555.

Received July 25, 1974.

TULANE UNIVERSITY

Pacific Journal of Mathematics

Vol. 66, No. 1 November, 1976