

Pacific Journal of Mathematics

ON THE DIFFERENTIABILITY OF MULTIFUNCTIONS

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A new concept of differential for a multifunction is introduced. Here by a multifunction we mean a map from a Banach space X to some specified family of non void subsets of a Banach space Y . The comparison with another definition due to Lasota and Strauss shows that if a multifunction admits both differentials, these must be equal. The results are applicable to the perturbation theory for multivalued differential equations in a Banach space

$$\dot{x} \in F(x)$$

in a neighborhood of a rest position.

1. Introduction. The concept of differentiability for multifunctions has been considered by many authors from different points of view ([1], [3], [8], [11], [13], [17], [18]). Of all these approaches, that developed by Lasota and Strauss [17] seems to be more useful in perturbation theory for ordinary differential equations in the real Euclidean space R^n . Further applications along this same direction were obtained in [10] (see also [9]). In the present paper, moving from an idea of Bridgland [3], a new concept of differentiability for a multifunction is studied. This notion seems to be useful in perturbation theory. In [7] an application to problems of stability for multivalued differential equations in Banach spaces is given.

The definitions and the main properties of a multivalued differential (i.e. the differential of a multifunction or, in particular, of a function) are contained in §§2 and 3. Now, it is perhaps better to start by giving an answer to the preliminary question: where one may encounter a multivalued differential. To this end we recall the well known Theorems 1.1 and 1.2, due to Lyapunov.

THEOREM 1.1 ([20] p. 222). *Let $f: R^n \rightarrow R^n$, $f(0) = 0$, be continuously differentiable in a neighborhood of the origin with Fréchet differential f' at the origin. Let all the eigenvalues of f' have negative real parts, i.e. the origin is asymptotically stable for*

$$(1.1) \quad \dot{x} = f'(x).$$

Then the origin is asymptotically stable for

$$(1.2) \quad \dot{x} = f(x).$$

Of the possible extensions of the above result we mention the following two:

(I) f is single valued but not Fréchet differentiable at the origin. However f has a *multivalued differential* D at the origin, and so the variational equation which corresponds to (1.2) becomes

$$(1.3) \quad \dot{x} \in D(x).$$

(II) f is multivalued and admits at the origin a *multivalued differential* D . Thus, instead of (1.2) we have

$$(1.4) \quad \dot{x} \in f(x),$$

with corresponding variational equation (1.3).

In either case the problem arises whether the knowledge of a certain property of (1.3), for instance that the origin is a global attractor for this equation, implies that (1.2) (or (1.4)) possesses a similar, possibly weaker, property (see [17]).

THEOREM 1.2 ([20] p. 285). *Let $f: R^+ \times R^n \rightarrow R^n$, $R^+ = [0, \infty)$, be continuously differentiable, periodic in t with period $p > 0$. Let the equation*

$$(1.5) \quad \dot{x} = f(t, x)$$

have a periodic solution y of period p . Let all the characteristic numbers of the variational equation (along the periodic solution y)

$$(1.6) \quad \dot{x} = f'(t, y(t))x$$

have moduli strictly less than 1. Then the periodic solution y is asymptotically stable for (1.5).

A possible extension of Theorem 1.2 is the following:

(III) f is single valued but not Fréchet differentiable along y . However f has a *multivalued differential* D at any point $(t, y(t))$. Thus (1.6) is replaced by

$$(1.7) \quad \dot{x} \in D(t, y(t); x).$$

Then the problem arises whether, the fact that all solutions of (1.7) approach the origin for $t \rightarrow \infty$, implies that the periodic solution y is asymptotically stable for (1.5) (see [10]).

In §2 the definition of the multivalued differential D_x for a multifunction is introduced. Several elementary consequences of this definition are reviewed in §3. In the following one we consider, in infinite

dimension, another definition of differential Δ_x for a multifunction. (This was introduced by Lasota and Strauss [17] for mappings from R^n to R^n .) In Section 5 we consider γ -Lipschitz maps (γ is the Hausdorff measure of noncompactness [22]). Then we show that the multivalued differential of a γ -Lipschitz map, with constant k , is γ -Lipschitz with the same constant. In the subsequent paper [7] an application of the above theory to a problem of stability, by the first approximation method, for a multivalued differential equation in Banach space is presented.

2. Notation and preliminaries. Let Y be a Banach space. For any $a \in Y$, define $S(a, r) = \{y : \|y - a\| < r\}$ $r > 0$, $\bar{S}(a, r) = \{y : \|y - a\| \leq r\}$, $r \geq 0$. We write S, \bar{S} in place of $S(0, 1), \bar{S}(0, 1)$. Denote by: $\mathcal{B}(Y)$ (resp. $\mathcal{C}(Y)$, $\mathcal{C}_b(Y)$, $\mathcal{K}(Y)$, $\mathcal{K}_b(Y)$) the family of all non void bounded (resp. bounded closed, bounded closed convex, compact, compact convex) subsets of Y ; $N = \{1, 2, \dots\}$; \bar{A} the closure, $\overline{\text{co}} A$ the closed convex hull of $A \subset Y$. Let $A, B \in \mathcal{B}(Y)$. Define

$$d(A, B) = \inf\{t > 0 : A \subset B + tS, B \subset A + tS\}.$$

We review a number of well known properties of d , some of which will be used in the sequel. We have:

$$\begin{aligned} d(A, B) &\geq 0, & d(A, A) &= 0 \\ d(A, B) &= d(B, A) \\ d(A, B) &\leq d(A, C) + d(C, B). \end{aligned}$$

To conclude that d is a metric one has to show that $d(A, B) = 0$ implies $A = B$. This is not true in $\mathcal{B}(Y)$, but it is in $\mathcal{C}(Y)$. The restriction of d to couples of elements of $\mathcal{C}(Y)$ is called the *Hausdorff metric* in $\mathcal{C}(Y)$. We write $\|A\|$ in place of $d(A, 0)$.

The following lemma is fundamental.

LEMMA 2.1 (Rådström [21]). *Let A, B, C be non void subsets of Y . Suppose B closed and convex C bounded and $A + C \subset B + C$. Then $A \subset B$.*

LEMMA 2.2. *Let $A, A_1, B, B_1 \in \mathcal{B}(Y)$. Then*

- (i) $d(tA, tB) = td(A, B)$ $t \geq 0$
- (ii) $d(A + B, A_1 + B_1) \leq d(A, A_1) + d(B, B_1)$.

If $A, B \in \mathcal{C}_b(Y)$ and $C \in \mathcal{B}(Y)$ we have

- (iii) $d(A + C, B + C) = d(A, B)$.

Proof. Property (i) is obvious. To prove (ii) let $t > d(A, A_1)$, $t_1 > d(B, B_1)$. Then

$$\begin{aligned} A &\subset A_1 + tS & B &\subset B_1 + t_1S \\ A_1 &\subset A + tS & B_1 &\subset B + t_1S \end{aligned}$$

and $A + B \subset A_1 + B_1 + (t + t_1)S$, $A_1 + B_1 \subset A + B + (t + t_1)S$ which imply $d(A + B, A_1 + B_1) \leq t + t_1$. Letting $t \rightarrow d(A, A_1)$, $t_1 \rightarrow d(B, B_1)$ we get (ii).

Let us prove (iii). By (ii) $d(A + C, B + C) \leq d(A, B)$. Suppose the strict inequality holds and let t be such that $d(A + C, B + C) < t < d(A, B)$. Then

$$\begin{aligned} A + C &\subset B + C + tS \subset \overline{B + tS} + C \\ B + C &\subset A + C + tS \subset \overline{A + tS} + C \end{aligned}$$

and, since $\overline{B + tS}$, $\overline{A + tS}$ are closed convex while C is bounded, Lemma 2.1 yields $A \subset \overline{B + tS}$, $B \subset \overline{A + tS}$. On the other hand

$$\overline{B + tS} = \bigcap_{n=1}^{\infty} [(B + tS) + 2^{-n}S], \quad \overline{A + tS} = \bigcap_{n=1}^{\infty} [(A + tS) + 2^{-n}S],$$

thus if we choose n such that $t + 2^{-n} < d(A, B)$ we obtain $A \subset B + (t + 2^{-n})S$, $B \subset A + (t + 2^{-n})S$. These imply $d(A, B) \leq t + 2^{-n} < d(A, B)$, a contradiction.

Property (iii) is proved in [21] under different hypotheses (see also [8]).

Let X, Y be Banach spaces. Let U be a non void open subset of X .

DEFINITION 2.3. $F: U \rightarrow \mathcal{B}(Y)$ is said to be *upper semicontinuous* (= u.s.c.) at $x \in U$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $F(x + h) \subset F(x) + \epsilon S$, when $\|h\| < \delta$. F is said to be *continuous* at x if for every $\epsilon > 0$ there exists $\delta > 0$ such that $F(x + h) \subset F(x) + \epsilon S$ and $F(x) \subset F(x + h) + \epsilon S$, when $\|h\| < \delta$.

DEFINITION 2.4. $F: X \rightarrow \mathcal{B}(Y)$ is said to be *homogeneous* if $F(tx) = tF(x)$, $t \geq 0$, $x \in X$.

The following definition of differentiability is suggested by an idea due to Bridgland [3].

DEFINITION 2.5. $F: U \rightarrow \mathcal{B}(Y)$ is said to be *differentiable* at $x \in U$ if there exist a map $D_x: X \rightarrow \mathcal{C}_0(Y)$, which is u.s.c. and homogeneous, and a number $\delta > 0$ such that

$$d(F(x + h), F(x) + D_x(h)) = o(h) \quad \text{when} \quad \|h\| < \delta.$$

(Here $o(h)$ denotes a nonnegative function such that $\lim_{h \rightarrow 0} o(h)/\|h\| = 0$.) D_x is called the (*multivalued*) *differential* of F at x .

REMARK 2.6. Let F be a map from U to $\mathcal{H}_0(Y)$. In [18] Martelli and Vignoli define F to be differentiable at $x \in U$ if there exists a map $S_x: X \rightarrow \mathcal{H}_0(Y)$, which is u.s.c. and homogeneous, and a number $\delta > 0$ such that

$$F(x+h) = F(x) + S_x(h) + R(h), \quad \text{when } \|h\| < \delta,$$

and $\lim_{h \rightarrow 0} \|R(h)\|/\|h\| = 0$. We shall see later that the existence of S_x implies that of D_x and $S_x = D_x$. The converse is false. To see this define $F: (-\pi/4, \pi/4) \rightarrow \mathcal{H}_0(Y)$, $Y = \mathbb{R}^n$, by

$$F(t) \begin{cases} = \bar{S} & \text{if } t = 0 \\ = (1 + t^2 \sin 1/t) \bar{S} & \text{if } 0 < |t| < \pi/4. \end{cases}$$

Then F is differentiable at 0 and $D_0 = 0$. But S_0 does not exist for the existence of S_0 implies that, in a neighborhood of 0 the diameter of $F(t)$ is not less than the diameter of $F(0)$, which is clearly impossible.

3. Properties of differentiable multifunctions. In this section several elementary properties of differentiable multifunctions are reviewed. Let U be a non void open subset of X . The following theorem shows that the differential D_x is well defined.

THEOREM 3.1. *The multivalued differential D_x of $F: U \rightarrow \mathcal{B}(Y)$ at $x \in U$ if it exists is unique.*

Proof. Let δ correspond to D_x . Let there exist D_x^1 and $\delta_1 > 0$ such that $d(F(x+h), F(x) + D_x^1(h)) = o^1(h)$, when $\|h\| < \delta_1$. Trivially $D_x(0) = D_x^1(0) = 0$, being both D_x and D_x^1 homogeneous. Let $u \neq 0$. Let $t > 0$ be such that $t\|u\| < \delta, \delta_1$. Then, by Lemma 2.2 (iii),

$$\begin{aligned} d(D_x(tu), D_x^1(tu)) &= d(D_x(tu) + F(x), D_x^1(tu) + F(x)) \\ &\leq d(D_x(tu) + F(x), F(x+tu)) \\ &\quad + d(F(x+tu), D_x^1(tu) + F(x)) \\ &\leq o(tu) + o^1(tu). \end{aligned}$$

Thus $d(D_x(u), D_x^1(u)) \leq o(tu)/t + o^1(tu)/t$ and, letting $t \rightarrow 0$, $d(D_x(u), D_x^1(u)) = 0$. Since $D_x(u), D_x^1(u)$ are bounded closed we have $D_x(u) = D_x^1(u)$.

REMARK 3.2. Suppose $F: U \rightarrow \mathcal{H}_0(Y)$ has the differential S_x . Then D_x exists and $S_x = D_x$. In fact, if $\|h\| < \delta$,

$$\begin{aligned} d(F(x+h), F(x) + S_x(h)) &\leq d(F(x+h), F(x) + S_x(h) + R(h)) \\ &\quad + d(F(x) + S_x(h) + R(h), F(x) + S_x(h)) \\ &\leq \|R(h)\| \end{aligned}$$

and, since $\lim_{h \rightarrow 0} \|R(h)\|/\|h\| = 0$, we have $d(F(x+h), F(x) + S_x(h)) = o(h)$. By the uniqueness of D_x it follows $D_x = S_x$.

THEOREM 3.3. If $F: U \rightarrow \mathcal{B}(Y)$ is differentiable at x it is there continuous.

Proof. Let $\epsilon > 0$. Since F is differentiable at x there exists $\delta > 0$ such that $d(F(x+h), F(x) + D_x(h)) = o(h)$, when $\|h\| < \delta$. Furthermore, since D_x is u.s.c. at the origin and $D_x(0) = 0$, there exists $0 < \delta_1 < \delta$ such that $D_x(h) \subset \epsilon S$ if $\|h\| < \delta_1$. For $\|h\| < \delta_1$ we have

$$\begin{aligned} d(F(x+h), F(x)) &\leq d(F(x+h), F(x) + D_x(h)) + d(F(x) + D_x(h), F(x)) \\ &\leq o(h) + \|D_x(h)\| \\ &\leq o(h) + \epsilon \end{aligned}$$

and F is continuous at x .

THEOREM 3.4. Let U be a non void open and convex subset of X . The multifunction $F: U \rightarrow \mathcal{C}(Y)$ is constant if and only if, for every $x \in U$, $D_x = 0$.

Proof. Let us prove the sufficiency of the condition (the necessity is trivial). For every $x \in U$ there exists $\delta > 0$ such that $d(F(x+h), F(x)) = o(h)$ if $\|h\| < \delta$. Let $x, x_1 \in U$. We have

$$\begin{aligned} |d(F(x_1), F(x+h)) - d(F(x_1), F(x))| &\leq d(F(x+h), F(x)) \\ |d(F(x_1), F(x+h)) - d(F(x_1), F(x))|/\|h\| &\leq o(h)/\|h\|, \quad 0 < \|h\| < \delta. \end{aligned}$$

Let $h \rightarrow 0$. Then the real valued functional $x \mapsto d(F(x_1), F(x))$, having zero Fréchet differential for every $x \in U$, must be constant. Since it vanishes for $x = x_1$ it is identically zero.

For $A, B \in \mathcal{C}(Y)$ define $d^*(A, B) = \inf\{t > 0: A \subset B + tS\}$. We have:

$$\begin{aligned}
d^*(A, B) &\geq 0, & d^*(A, B) &= 0 \text{ if and only if } A \subset B \\
d^*(A, B) &\leq d^*(A, C) + d^*(C, B) \\
d^*(A, B) &\leq d(A, B).
\end{aligned}$$

If $A = \{a\}$, $B = \{b\}$ then $d^*(A, B) = d(A, B) = \|a - b\|$.

Given a map $F: U \rightarrow \mathcal{C}(Y)$, a single valued function $f: U \rightarrow Y$ satisfying $f(x) \in F(x)$, $x \in U$, is called a *selection* of F .

THEOREM 3.5. *Let $F: X \rightarrow \mathcal{C}(Y)$, $F(0) = 0$, be differentiable at the origin with differential D_0 . Let f be a selection of F in a neighborhood $S(0, \delta_1)$ of the origin of X . If f has Fréchet differential f'_0 at the origin then f'_0 is a selection of D_0 .*

Proof. There exists $0 < \delta < \delta_1$ such that

$$d(F(h), D_0(h)) = o(h), \quad \|f(h) - f'_0(h)\| = o^1(h) \quad \text{if } \|h\| < \delta.$$

Trivially D_0 and f'_0 are equal for $u = 0$. Let $u \neq 0$. Let $t > 0$ be such that $t\|h\| < \delta$. Then we have

$$\begin{aligned}
d^*(f'_0(tu), D_0(tu)) &\leq d^*(f'_0(tu), f(tu)) + d^*(f(tu), F(tu)) \\
&\quad + d^*(F(tu), D_0(tu))
\end{aligned}$$

and

$$\begin{aligned}
d^*(f'_0(u), D_0(u)) &\leq t^{-1}\|f'_0(tu) - f(tu)\| + t^{-1}d(F(tu), D_0(tu)) \\
&\leq o^1(tu)/t + o(tu)/t.
\end{aligned}$$

Letting $t \rightarrow 0$ we obtain $d^*(f'_0(u), D_0(u)) = 0$.

4. Comparison with another definition of differential.

In [17] Lasota and Strauss gave the definition of a multivalued differential Δ_x for a single-valued map $F: R^n \rightarrow R^n$ and used such definition to prove a perturbation theorem for ordinary differential equations in R^n . Further results along this same direction were established in [10] and, for difference equations, in [9]. In this section the definition of Δ_x is extended to maps $F: X \rightarrow \mathcal{K}(Y)$, where X, Y are Banach spaces. Furthermore the relationship between the multivalued differential D_x and the Lasota–Strauss differential Δ_x is considered.

Let X, Y be Banach spaces, $U \subset X$ be open and non void.

DEFINITION 4.1. $F: U \rightarrow \mathcal{K}(Y)$ is said to be *Lipschitzian* at $x \in U$ if $F(x)$ is singleton and there exist constants $L \geq 0$ and $\delta > 0$ such that $d(F(x+h), F(x)) \leq L\|h\|$ if $\|h\| < \delta$.

DEFINITION 4.2. Let $F: U \rightarrow \mathcal{K}(Y)$ be Lipschitzian at $x \in U$. A map $\varphi: X \rightarrow \mathcal{K}_0(Y)$ is said to be an *upper differential* of F at x if φ is u.s.c., homogeneous and there exists $\delta > 0$ such that

$$F(x + h) \subset F(x) + \varphi(h) \quad \text{if} \quad \|h\| < \delta.$$

Denote by \mathcal{F} the set of all upper differentials of F at x . \mathcal{F} may be empty. However, if $\dim(Y) < \infty$, F has at least one upper differential, namely $\varphi(h) = L \|h\| \bar{S}$.

DEFINITION 4.3. Let $F: U \rightarrow \mathcal{K}(Y)$ be Lipschitzian at $x \in U$. Suppose that $\mathcal{F} \neq \emptyset$ and, for each $h \in X$, $\bigcap_{\varphi \in \mathcal{F}} \varphi(h) \neq \emptyset$. Define the L. S. differential $\Delta_x: X \rightarrow \mathcal{K}_0(Y)$ by

$$\Delta_x(h) = \bigcap_{\varphi \in \mathcal{F}} \varphi(h) \quad h \in X.$$

The above definition reduces to that given by Lasota and Strauss [17] for single-valued maps $F: R^n \rightarrow R^n$.

REMARK 4.4. Berge [2] (p. 114) defines a map $F: U \rightarrow \mathcal{K}(Y)$ to be u.s.c. at $x \in U$ if for every open set $G \supset F(x)$ there exists $\delta > 0$ such that $F(x + h) \subset G$ if $\|h\| < \delta$. If F is u.s.c. in this sense it is also u.s.c. according to Definition 2.3. Conversely let F be u.s.c. at x . To prove that F is u.s.c. according to Berge's definition it is sufficient to show the existence of a positive integer n such that $F(x) + 2^{-n}S \subset G$. Indeed, in the contrary case, for every $n \in \mathbb{N}$, we have $(F(x) + 2^{-n}S) \cap (Y \setminus G) \neq \emptyset$. This implies the existence of a sequence $\{y_n + s_n\}$, $y_n \in F(x)$, $s_n \in 2^{-n}S$ such that $y_n + s_n \in Y \setminus G$. By the compactness of $F(x)$ we can and do assume, without loss of generality, $y_n \rightarrow y \in F(x)$. Since $y_n + s_n \rightarrow y$ and $Y \setminus G$ is closed, $y \in Y \setminus G$. From the contradiction the claim follows. Since ([2] p. 119) the intersection of any family of u.s.c. (homogeneous) mappings is u.s.c. (homogeneous) it remains proved that Δ_x if it exists is u.s.c. (homogeneous). Clearly for any $h \in X$, $\Omega(h) = \bigcap_{\varphi \in \mathcal{F}} \varphi(h)$ belongs to $\mathcal{K}_0(Y)$, provided it is non void. Thus the existence of Δ_x is finally established if we show that, for every $h \in X$, $\Omega(h) \neq \emptyset$.

THEOREM 4.5. Let Y be reflexive. Let $F: U \rightarrow \mathcal{K}(Y)$ be Lipschitzian at $x \in U$. If $\mathcal{F} \neq \emptyset$ the L. S. differential Δ_x of F at x exists. Moreover Δ_x is u.s.c. and homogeneous.

Proof. After Remark 4.4 the only fact which requires a proof is that $\Omega(h) \neq \emptyset$, $h \in X$. Let $h \neq 0$ (the case $h = 0$ is trivial). There exists a positive integer k such that

$$\frac{d(F(x + h/n), F(x))}{\|h/n\|} \leq L \quad \text{if } n \geq k.$$

Choose $y_n \in F(x + h/n)$. Since the sequence $\{(y_n - F(x))\|h/n\|^{-1}\}_{n \geq k}$ is bounded in the reflexive Banach space Y we assume, without loss of generality that it converges weakly to some element $z \in Y$. Then

$$z \in \overline{\text{co}} \left\{ \frac{y_n - F(x)}{\|h/n\|} \right\}_{n \geq k} \subset \overline{\text{co}} \left\{ \frac{F(x + h/n) - F(x)}{\|h/n\|} \right\}_{n \geq k}.$$

Let φ be any upper differential of F at x and let $\delta > 0$ correspond. There exists $k_1 \geq k$ such that $n \geq k_1$ implies $\|h/n\| < \delta$. For $n \geq k_1$ we have $F(x + h/n) \subset F(x) + \varphi(h/n)$. Therefore

$$z \in \overline{\text{co}} \left\{ \frac{\varphi(h/n)}{\|h/n\|} \right\}_{n \geq k_1} = \varphi \left(\frac{h}{\|h\|} \right)$$

and $\|h\|z \in \varphi(h)$. Since φ is arbitrary $\|h\|z \in \Omega(h)$.

If $\dim(Y) < \infty$ the hypothesis $\mathcal{F} \neq \emptyset$ in the above theorem can be omitted.

LEMMA 4.6. *Let X and Y be separable Banach spaces. Let $F: U \rightarrow \mathcal{K}(Y)$ be Lipschitzian at x with L . S . differential Δ_x . Then there exists a sequence $\{\varphi_n\}$ of upper differentials of F at x such that*

$$(4.1) \quad \varphi_n(h) \supset \varphi_{n+1}(h), \quad \Delta_x(h) = \bigcap_{n=1}^{\infty} \varphi_n(h) \quad h \in X.$$

Proof. Let Ψ be any upper differential of F at x . The graph G_Ψ of Ψ is closed for Ψ is u.s.c. (Berge [2] p. 117). Since X and Y have countable bases, $X \times Y$ has the same property and, by Lindelöf theorem (Dunford and Schwartz [12] p. 12) there exists a sequence $\{\Psi_n\}$ of upper differentials such that $\bigcap_{\varphi \in \mathcal{F}} G_\varphi = \bigcap_{n=1}^{\infty} G_{\Psi_n}$. Then

$$G_{\Delta_x} = \bigcap_{\varphi \in \mathcal{F}} G_\varphi = \bigcap_{n=1}^{\infty} G_{\Psi_n} = G_{\bigcap_{n=1}^{\infty} \Psi_n}$$

implies $\Delta_x = \bigcap_{n=1}^{\infty} \Psi_n$. Since a finite intersection of u.s.c. (homogeneous) maps is u.s.c. (homogeneous) the sequence $\{\varphi_n\}$, $\varphi_n = \bigcap_{k=1}^n \Psi_k$ consists of upper differentials which satisfy the conclusions of the lemma.

The following result is useful in perturbation theory [10].

THEOREM 4.7. *Let X, Y be finite dimensional Banach spaces. Let $F: U \rightarrow \mathcal{H}(Y)$ be Lipschitzian at $x \in U$, with constant L . Let $\Delta_x: X \rightarrow \mathcal{H}_0(Y)$ be continuous. Then the map $V_\epsilon: h \mapsto \Delta_x(h) + \epsilon \|h\| \bar{S}$, $\epsilon > 0$, is an upper differential of F at x .*

Proof. The map V_ϵ from X to $\mathcal{H}_0(Y)$ is continuous and homogeneous. To conclude that V_ϵ is an upper differential of F at x we need to show that there exists $\delta > 0$ such that $F(x+h) \subset F(x) + V_\epsilon(h)$ if $\|h\| < \delta$. Suppose the contrary. There exists a sequence $\{h_n\}$, $h_n \neq 0$, $h_n \rightarrow 0$ such that $F(x+h_n) \not\subset F(x) + V_\epsilon(h_n)$. Thus there exists a sequence $\{y_n\}$, $y_n \in F(x+h_n)$, satisfying $y_n - F(x) \notin V_\epsilon(h_n)$ or, equivalently,

$$(y_n - F(x)) / \|h_n\| \notin V_\epsilon(h_n / \|h_n\|) \quad n \in N.$$

Since $\{h_n / \|h_n\|\}$ and $\{(y_n - F(x)) / \|h_n\|\}$ are bounded and X, Y are finite dimensional, we can and do assume (without loss of generality)

$$(4.2) \quad h_n / \|h_n\| \rightarrow h \in X, \quad (y_n - F(x)) / \|h_n\| \rightarrow y \in Y.$$

Suppose $y \in \Delta_x(h) + (\epsilon/2)\bar{S}$. This implies $y + (\epsilon/4)\bar{S} \subset \Delta_x(h) + (3/4)\epsilon\bar{S}$ and for n sufficiently large, say $n \geq k$, $(y_n - F(x)) / \|h_n\| \in \Delta_x(h) + (3/4)\epsilon\bar{S}$. Since Δ_x is continuous at h there exists $k_1 \geq k$ such that $\Delta_x(h) \subset \Delta_x(h_n / \|h_n\|) + (\epsilon/4)\bar{S}$ if $n \geq k_1$. Thus

$$(y_n - F(x)) / \|h_n\| \in \Delta_x(h_n / \|h_n\|) + \epsilon\bar{S} = V_\epsilon(h_n / \|h_n\|)$$

if $n \geq k_1$, a contradiction.

Suppose $y \notin \Delta_x(h) + (\epsilon/2)\bar{S}$. Then if ϵ_1 is such that $0 < \epsilon_1 < \epsilon/2$ we have $\bar{S}(y, \epsilon_1) \cap \Delta_x(h) = \emptyset$. By Lemma 4.6 there exists a sequence $\{\varphi_m\}$ of upper differentials of F at x satisfying (4.1).

We claim that there is $k \in N$ such that for all $m \geq k$ we have $\bar{S}(y, \epsilon_1) \cap \varphi_m(h) = \emptyset$. Let the claim be false. Since $\varphi_1(h) \supset \varphi_2(h) \supset \dots$, for every $m \in N$ there exists z_m in both sets $\bar{S}(y, \epsilon_1)$ and $\varphi_m(h)$. Without loss of generality we assume $z_m \rightarrow z$. Then $z \in \bar{S}(y, \epsilon_1)$ and $z \in \varphi_m(h)$ for every $m \in N$, thus $z \in \bar{S}(y, \epsilon_1) \cap \Delta_x(h)$, a contradiction. The claim is true. This implies

$$(4.3) \quad \bar{S}\left(y, \frac{\epsilon_1}{2}\right) \cap \left(\varphi_m(h) + \frac{\epsilon_1}{2} \bar{S}\right) = \emptyset, \quad m \geq k.$$

By (4.2), for all n sufficiently large say $n \geq r$ we have

$$(y_n - F(x)) / \|h_n\| \in \bar{S}\left(y, \frac{\epsilon_1}{2}\right), \quad \varphi_m(h_n / \|h_n\|) \subset \varphi_m(h) + \frac{\epsilon_1}{2} \bar{S}$$

and so, by virtue of (4.3), $(y_n - F(x))/\|h_n\| \notin \varphi_m(h_n/\|h_n\|)$ i.e. $y_n \notin F(x) + \varphi_m(h_n)$ for all $n \geq r$. This implies $F(x + h_n) \not\subset F(x) + \varphi_m(h_n)$, $n \geq r$, a contradiction since φ_m is an upper differential of F .

Next theorem shows that $D_x = \Delta_x$ if both exist.

THEOREM 4.8. *Let X, Y be Banach spaces. Let $F: U \rightarrow \mathcal{K}_0(Y)$ be Lipschitzian at $x \in U \subset X$. Then $\Delta_x = D_x$ if both exist.*

Proof. By hypothesis there exists $\delta > 0$ such that $d(F(x + h), F(x) + D_x(h)) = o(h)$ if $\|h\| < \delta$. Let $\epsilon > 0$. Then there exists $0 < \delta_1 < \delta$ such that

$$(4.4) \quad F(x + h) \subset F(x) + D_x(h) + \epsilon \|h\| \bar{S}$$

$$(4.5) \quad F(x) + D_x(h) \subset F(x + h) + \epsilon \|h\| \bar{S} \quad \text{if } \|h\| < \delta_1.$$

Let φ be any upper differential of F . This implies the existence of $0 < \delta_2 < \delta_1$ such that $F(x + h) \subset F(x) + \varphi(h)$, if $\|h\| < \delta_2$. Then $F(x + h) + \epsilon \|h\| \bar{S} \subset F(x) + \varphi(h) + \epsilon \|h\| \bar{S}$ and, by (4.5), $F(x) + D_x(h) \subset F(x) + \varphi(h) + \epsilon \|h\| \bar{S}$, if $\|h\| < \delta_2$. Thus, $D_x(h) \subset \varphi(h) + \epsilon \|h\| \bar{S}$ from which one easily obtains $D_x(h) \subset \varphi(h)$, if $\|h\| < \delta_2$. Since φ is any upper differential of F we have $D_x(h) \subset \Delta_x(h)$ and, by the homogeneity of D_x and Δ_x , the inclusion holds for all $h \in X$.

Next let us show the reverse inclusion. Let φ be any upper differential of F . Define $\varphi_1(h) = \varphi(h) \cap (D_x(h) + \epsilon \|h\| \bar{S})$, $h \in X$. We claim that φ_1 is an upper differential of F . From (4.4) and $F(x + h) \subset F(x) + \varphi(h)$, which hold for $\|h\|$ small enough, it follows that $\varphi_1(h) \neq \emptyset$ in a neighborhood of the origin and, by homogeneity, for all $h \in X$. Trivially $\varphi_1(h)$ is convex, for every $h \in X$. Furthermore, for each $h \in X$, $D_x(h)$ is compact, for it is contained in $\Delta_x(h)$, and so $\varphi_1(h)$ is compact being the intersection of $\varphi(h)$ compact, and $D_x(h) + \epsilon \|h\| \bar{S}$ closed. Thus φ_1 maps X into $\mathcal{K}_0(Y)$. Clearly φ_1 is homogeneous and satisfies $F(x + h) \subset F(x) + \varphi_1(h)$, for $\|h\|$ sufficiently small. So to conclude that φ_1 is an upper differential of F it remains to be shown that it is u.s.c. But this follows at once from a result of Berge ([2] p. 117) because the map $h \mapsto D_x(h) + \epsilon \|h\| \bar{S}$ from X to $\mathcal{C}_0(Y)$ is closed and $\varphi: X \rightarrow \mathcal{K}_0(Y)$ is u.s.c. Then there exists $\delta_3 > 0$ such that $\Delta_x(h) \subset \varphi_1(h) \subset D_x(h) + \epsilon \|h\| \bar{S}$, if $\|h\| < \delta_3$, which implies $\Delta_x(h/\|h\|) \subset D_x(h/\|h\|)$, $0 < \|h\| < \delta_3$. By homogeneity, $\Delta_x(h) \subset D_x(h)$ for every $h \in X$.

5. The differential of a γ -Lipschitz function. In this section it is shown that the differential D_x of a multifunction which is γ -Lipschitz with constant k possesses this same property. Let us introduce the following

DEFINITION 5.1. Let $A \in \mathcal{B}(Y)$. The *measure* $\gamma(A)$ of *non-compactness* of A is defined by

$$\gamma(A) = \inf\{t > 0: \text{there exists } C \in \mathcal{K}(Y) \text{ such that } A \subset C + t\bar{S}\}.$$

There are alternative (non equivalent) definitions of measures of noncompactness ([5], [14], [16], [22]). That which we use seems to be flexible enough to be adapted for the measure of noncompactness in the weak topology as well [6]. The following theorem is well known. However we include the proofs of those statements which are proved in a different, perhaps simpler, way (see (f)–(i)).

THEOREM 5.2. *The functional γ has the properties:*

- (a) $A \subset B$ implies $\gamma(A) \leq \gamma(B)$
- (b) $\gamma(A) = \gamma(\bar{A})$
- (c) $\gamma(A) = 0$ if and only if \bar{A} is compact
- (d) $\gamma(A + B) \leq \gamma(A) + \gamma(B)$
- (e) $\gamma(sA) = s\gamma(A) \quad s \geq 0$
- (f) $\gamma(A) = \gamma(\overline{\text{co}} A)$
- (g) $\gamma(\bigcup_{u \in [0, s]} uA) = s\gamma(A)$
- (h) $\gamma(S) = 1$ if $\dim(Y) = \infty$
- (i) $\gamma(A + B) = \gamma(A)$ if $\gamma(B) = 0$.

Proof. (a)–(e) follow easily from the Definition 5.1. (f) Let $\epsilon > 0$. There exist $\gamma(A) < t < \gamma(A) + \epsilon$ and $C \in \mathcal{K}(Y)$ such that $A \subset C + t\bar{S}$. This implies $A \subset \overline{\text{co}} C + t\bar{S}$ where, by Mazur theorem (Dunford and Schwartz [12] p. 416), $\overline{\text{co}} C$ is compact. Thus $\overline{\text{co}} A \subset \overline{\text{co}} C + t\bar{S}$, being the second member convex and closed. The last inclusion shows $\gamma(\overline{\text{co}} A) \leq t$ and $\gamma(\overline{\text{co}} A) \leq \gamma(A)$. The reverse inequality is trivial.

(g) Let $\epsilon > 0$. There exist $\gamma(A) < t < \gamma(A) + \epsilon$ and $C \in \mathcal{K}(Y)$ such that $A \subset C + t\bar{S}$. This implies $A \subset (C \cup \{0\}) + t\bar{S} \subset C_1 + t\bar{S}$ where $C_1 = \overline{\text{co}}(C \cup \{0\})$ is compact. Thus, for every $u \in [0, s]$, $uA \subset uC_1 + ut\bar{S} \subset sC_1 + st\bar{S}$ (since C_1 is convex and contains the origin) and we have $\bigcup_{u \in [0, s]} uA \subset sC_1 + st\bar{S}$. This implies $\gamma(\bigcup_{u \in [0, s]} uA) \leq st$ and $\gamma(\bigcup_{u \in [0, s]} uA) \leq s\gamma(A)$. The reverse inequality is obvious.

(h) Since $\bar{S} = \{0\} + 1\bar{S}$ we have $\gamma(\bar{S}) \leq 1$. Suppose $\gamma(\bar{S}) < 1$. Then there exist $\gamma(\bar{S}) < t < 1$ and $C \in \mathcal{K}(Y)$ such that $\bar{S} \subset C + t\bar{S}$. From this

$$\bar{S} \subset \overline{\text{co}} C + t\bar{S}$$

$$(1 - t)\bar{S} + t\bar{S} \subset \overline{\text{co}} C + t\bar{S}$$

and, by Lemma 2.1, $(1 - t)\bar{S} \subset \overline{\text{co}} C$. Thus $\bar{S} \subset (1 - t)^{-1}\overline{\text{co}} C$ and since the

set on the right is compact such must be \bar{S} . This is a contradiction since $\dim(Y) = \infty$.

(i) Let $b \in B$. Then $A \subset A + B + \{-b\}$ implies

$$\gamma(A) \leq \gamma(A + B + \{-b\}) = \gamma(A + B) \leq \gamma(A) + \gamma(B) = \gamma(A)$$

and (i) is true.

Denote by U a non void open subset of Y .

DEFINITION 5.3. $F: U \rightarrow \mathcal{H}(Y)$ is said to be γ -Lipschitz, with constant $k \geq 0$, if for every $A \in \mathcal{B}(Y)$, $A \subset U$, we have $\gamma(F(A)) \leq k\gamma(A)$.

Now we want to show that the multivalued differential of a γ -Lipschitz map is γ -Lipschitz, with the same constant.

THEOREM 5.4. Let $F: U \rightarrow \mathcal{H}(Y)$ be γ -Lipschitz with constant k . Let D_x be the differential of F at $x \in U$. Then D_x is γ -Lipschitz with the same constant k .

Proof. There exists $\delta > 0$ such that $d(F(x+h), F(x) + D_x(h)) = o(h)$ if $\|h\| < \delta$. This implies

$$F(x) + D_x(h) \subset F(x+h) + (o(h) + \|h\|^2)S \quad \text{if } \|h\| < \delta.$$

Let $A \in \mathcal{B}(Y)$, $A \subset U$. Let $t > 0$ be such that $t\|A\| < \delta$. Let $\sigma(t) = \sup\{o(h) : h \in tA\}$. It is easy to see that $\lim_{t \rightarrow 0} \sigma(t)/t = 0$. Let $h \in tA$. We have

$$\begin{aligned} F(x) + D_x(h) &\subset F(x+tA) + [\sigma(t) + t^2\|A\|^2]S \\ F(x) + D_x(tA) &\subset F(x+tA) + [\sigma(t) + t^2\|A\|^2]S. \end{aligned}$$

Using the properties of γ

$$\begin{aligned} \gamma(D_x(tA)) &= \gamma(F(x) + D_x(tA)) \\ &\leq \gamma(F(x+tA)) + \sigma(t) + t^2\|A\|^2 \\ &\leq k\gamma(tA) + \sigma(t) + t^2\|A\|^2. \end{aligned}$$

Thus

$$\gamma(D_x(A)) \leq k\gamma(A) + \sigma(t)/t + t\|A\|^2$$

and, letting $t \rightarrow 0$, the desired result follows.

COROLLARY 5.5 (Daneš [4], Nussbaum [19], Sadovskii [22]). *Let $F: U \rightarrow Y$ be a single valued γ -Lipschitz map with constant k . Let F'_x be the Fréchet differential of F at $x \in U$. Then F'_x is γ -Lipschitz with the same constant k .*

DEFINITION 5.6. Let $U = \{x \in X: \|x\| > r\}$, $r > 0$. $F: U \rightarrow \mathcal{B}(Y)$ is said to be *differentiable at infinity* if there exist a map $D_\infty: X \rightarrow \mathcal{C}_0(Y)$, which is u.s.c. and homogeneous, and a number $\delta > r$ such that

$$d(F(x), D_\infty(x)) = o(x) \quad \text{when} \quad \|x\| > \delta,$$

and $\lim_{x \rightarrow \infty} o(x)/\|x\| = 0$. D_∞ is called the *asymptotic differential* of F .

DEFINITION 5.7. (Krasnosel'skii [15] p. 207). Let $F: U \rightarrow Y$ be a continuous single valued map, U being as in the above definition. Let there exist a linear map F'_∞ and a number $\delta > r$ such that $F(x) = F'_\infty(x) + z(x)$, if $\|x\| > \delta$, and $\lim_{x \rightarrow \infty} z(x)/\|x\| = 0$. Then F is said to be *asymptotically linear* and F'_∞ is called the *asymptotic derivative* of F .

THEOREM 5.8. *The asymptotic differential D_∞ of $F: U \rightarrow \mathcal{B}(Y)$ if exists is unique.*

Proof. Similar to that of Theorem 3.1.

THEOREM 5.9. *Let $U = \{y \in Y: \|y\| > r\}$, $r > 0$. Let $F: U \rightarrow \mathcal{H}(Y)$ be γ -Lipschitz with constant k . Let D_∞ be the asymptotic differential of F . Then D_∞ is γ -Lipschitz with the same constant k .*

Proof. Similar to that of Theorem 5.4.

Since a single valued continuous map is completely continuous if and only if it is γ -Lipschitz with constant $k = 0$, we have

COROLLARY 5.10 (Krasnosel'skii [15] p. 207). *The asymptotic derivative F'_∞ of a completely continuous single valued map $F: U \rightarrow Y$ is completely continuous.*

REFERENCES

1. H. T. Banks and M. Q. Jacobs, *A differential calculus for multifunctions*, J. Math. Anal. Appl., **29** (1970), 246–272.
2. C. Berge, *Espaces topologiques et fonctions multivoques*, Dunod, Paris, 1966.
3. T. F. Bridgland, *Trajectory integrals of set valued functions*, Pacific J. Math., **33** (1970), 43–67.
4. J. Daneš, *Some fixed point theorems in metric and Banach spaces*, Comment. Math. Univ. Carolinae, **12** (1971), 37–51.

5. ———, *On the Istrăţescu's measure of noncompactness*, Bull. Math. Soc. Sci. Math. R. S. Roumanie, **16** (64) (1972), 403–406.
6. F. S. De Blasi, *On a property of the unit sphere in a Banach space* (submitted).
7. ———, *Existence and stability of solutions for autonomous multivalued differential equations in Banach space*, Atti Acc. Naz. Lincei, Rend. Cl. Sc. Math. Nat. (to appear).
8. F. S. De Blasi and F. Iervolino, *Equazioni differenziali con soluzioni a valore compatto convesso*, Boll. Un. Mat. Ital., **4** (1969), 491–501.
9. F. S. de Blasi and J. Schinas, *Exponential stability of difference equations which cannot be linearized*, Atti Acc. Naz. Lincei, Rend. Cl. Sc. Mat. Nat., **54** (1973), 16–21.
10. ———, *The Lyapunov theorem on the behavior of solutions of a non autonomous differential system in a neighborhood of a periodic solution with corresponding multivalued variational equation*, J. Differential Equations, (to appear).
11. G. Debreu, *Integration of correspondences*, in Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability, Vol. II, part 1, p. 351–372, University of California Press, Los Angeles, 1967.
12. N. Dunford and J. T. Schwartz, *Linear Operators, part I*, Wiley (Interscience), New York, 1959.
13. M. Hukuhara, *Intégration des applications mesurables dont la valeur est un compact convexe*, Funkcial. Ekvac., **10** (1967), 205–223.
14. V. I. Istrăţescu, *On a measure of noncompactness*, Bull. Math. Soc. Sci. Math. R. S. Roumanie, **16** (64) (1972), 195–197.
15. M. Krasnosel'skiĭ, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, London, 1963.
16. C. Kuratowski, *Sur les espaces complets*, Fund. Math., **15** (1930), 301–309.
17. A. Lasota and A. Strauss, *Asymptotic behavior for differential equations which cannot be locally linearized*, J. Differential Equations, **10** (1971), 152–172.
18. M. Martelli and A. Vignoli, *On differentiability of multi-valued maps*, Boll. Un. Mat. Ital., (4) **10** (1974), 701–712.
19. R. D. Nussbaum, *The fixed point index and fixed point theorems for k -set contractions*, Ph. D. Thesis, University of Chicago, 1969.
20. L. Pontryagin, *Équations différentielles ordinaires*, Éditions MIR, Moscow, 1969.
21. H. Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc., **3** (1952), 165–169.
22. B. N. Sadovskii, *Limit-compact and condensing operators*, Uspehi Math. Nauk, **271** (1972), (in Russian). Eng. Trans., Russian Math. Survey, p. 85–155.

Received July 17, 1975 and in revised form February 12, 1976.

ISTITUTO MATEMATICO "U. DINI"
VIALE MORGAGNI 67/A
ITALIA

Pacific Journal of Mathematics

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