$R$-ENDOMORPHISMS OF $R[[X]]$ ARE ESSENTIALLY CONTINUOUS

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Let $R$ be a commutative ring with identity, $A = R[[X]]$ and $B = R[[Y]]$ with $X$ and $Y$ finite sets of indeterminates. Consider $A$ and $B$ as topological rings with the respective $X$ and $Y$-adic topologies. If $\sigma: A \to B$ is any $R$-homomorphism then there are $R$-automorphisms $s$ and $t$ of $A$ and $B$ respectively, so that $t \circ \sigma \circ s: A \to B$ is continuous. As a corollary we see that an $R$-endomorphism of $A$ is surjective only if it is an automorphism.

Let $X = \{X_1, \ldots, X_n\}$ be a set of indeterminates over $R$. $R[X]$ and $R[[X]]$ denote as usual the polynomial ring and the formal power series ring respectively over $R$ in the variables $X$. A number of authors have studied and applied automorphisms and endomorphisms of $R[[X]]$ over $R$; [3], [4], [5], [6], [7] and [1]. A common feature of many of the arguments seems to be the complexity resulting from the fact that $R$-endomorphisms of $R[[X]]$ need not be continuous in $X$-adic topology. In this note we show that they are essentially continuous, i.e. differ from a continuous one by an automorphism. Precisely, we make the following

**Definition 1.** If $A$, $B$ are topological rings and $\sigma: A \to B$ is a homomorphism, then $\sigma$ is said to be **essentially continuous** if, for some automorphisms $s$ and $t$ of $A$ and $B$ respectively, we get that $t \circ \sigma \circ s: A \to B$ is continuous.

With this definition we get the main statement that "every $R$-homomorphism between any two formal power series rings over $R$ is essentially continuous." (Corollary B)

As a corollary we get an easy proof of the statement that, "an $R$-endomorphism of $R[[X]]$ is surjective if and only if it is an $R$-automorphism of $R[[X]]". (Corollary C)\footnote{O'Malley had done the one variable case of this result in [3]. Gilmer and O'Malley have independently given another proof of Corollary C in [2].}

Finally, we make

**Definition 2.** If $\mathfrak{I}$ is a finitely generated ideal of $R$ we say that $R$ is **complete in the $\mathfrak{I}$-adic topology** if there is a finite set of indeterminates $X$ and an $R$-homomorphism $\sigma: R[[X]] \to R$ with $\sigma(XR[[X]]) = \mathfrak{I}$. 
Let $I_c(R)$ denote the set of all $a \in R$ such that there is an $R$-homomorphism $\sigma: R[[X_i]] \to R$ with $\sigma(X_i) = a$.

Using the "essential continuity" we establish that $I_c(R)$ is an ideal of $R$ contained in the Jacobson radical of $R$ and containing the nil-radical of $R$. (Theorem E)

Once $I_c(R)$ is shown to be an ideal it is easy to show that $I_c(R)$ is nothing but the union of all ideals $\mathfrak{I}$ of $R$ such that $R$ is complete in the $\mathfrak{I}$-adic topology. This fact is indeed the reason for the suffix "c" in $I_c(R)$. This fact also answers some questions raised by Gilmer; see remarks at the end.

**Theorem A.** Suppose $R$ is a commutative ring with identity and $X = \{X_i\}_{i=1}^n$ and $Y = \{Y_i\}_{i=1}^m$ are sets of indeterminates over $R$. Suppose $R[[X]] \to R[[Y]]$ is an $R$-homomorphism and that for each $i$, $\sigma(X_i) = c_i + f_i$, where $c_i \in R$ and $f_i \in YR[[Y]]$. Then there exists an automorphism $t: R[[X]] \to R[[X]]$ such that $t(X_i) = X_i + c_i$.

*Proof.* Let $\beta: R[[Y]] \to R$ be defined by $\beta(Y) = 0$. Then composing $\beta$ and $\sigma$ we get a mapping $\sigma^*: R[[X]] \to R$ such that $\sigma^*(X_i) = c_i$. Let $\{Z_i\}_{i=1}^n$ be $n$ additional indeterminates. We extend $\sigma^*$ to a mapping $\sigma^*: R[[X,Z]] \to R$ by $\sigma^*(Z) = 0$. We now have a sequence

$$R[[Z]] \to R[[X,Z]] \to R[[Z]]$$

where $\alpha(Z_i) = X_i + Z_i$ and $\gamma$ is defined by regarding $R[[X,Z]]$ as $R[[X]][[Z]]$ and setting

$$\gamma(\Sigma h_iZ^i) = \Sigma \sigma^*(h_i)Z^i \quad \text{where} \quad h_i \in R[[X]].$$

We define $\tau^* = \gamma \circ \alpha$ and note that $\tau^*(Z_i) = Z_i + c_i$. Since $R[[Z]] \cong R[[X]]$ by $X \to Z$ there is a mapping $\tau: R[[X]] \to R[[X]]$ such that $\tau(X_i) = X_i + c_i$. We must now see that $\tau$ is an automorphism of $R[[X]]$. There is an automorphism $\delta$ of $R[[X]]$ which takes $X_i$ to $-X_i$.

The homomorphism $\delta \circ \tau \circ \delta \circ \tau: R[[X]] \to R[[X]]$ is a continuous endomorphism carrying $X_i$ to $X_i$. It is then clear that $\delta \circ \tau \circ \delta \circ \tau$ is the identity map and hence $\tau$ is an automorphism.

**Corollary B.** If $R$ is a commutative ring with 1 and $X = \{X_i\}_{i=1}^n$, $Y = \{Y_i\}_{i=1}^m$ are indeterminates over $R$, then any $R$-homomorphism $\sigma: R[[X]] \to R[[Y]]$ is essentially continuous.

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2 This result in the one-variable case appears in [1].
Proof. Let $\sigma(X_t) = c_t + f_t$ with $c_t \in R$ and $f_t \in YR[[Y]]$. Then by Theorem A, there is an automorphism $\tau$ of $R[[X]]$ such that $\tau(X_t) = X_t + c_t$. Thus $\tau^{-1}(X_t) = X_t - c_t$. The mapping $\sigma \circ \tau^{-1}$ is continuous since

$$\sigma \circ \tau^{-1}(X_t) = \sigma(X_t - c_t) = c_t + f_t - c_t = f_t$$

and $f_t \in YR[[Y]]$.

Corollary C. If $R$ is a commutative ring with 1 and $\{X_t\}_{t=1}^r$ are indeterminates, then an $R$-endomorphism $\sigma: R[[X]] \to R[[X]]$ is surjective if and only if it is an automorphism.

Proof. One way is clear. By the proof of Corollary B we may write

$$\sigma(X_t) = l_t + F_t,$$

where $l_t$ is a linear form in $X$ over $R$ and $F_t \in (XR[[X]])^2$.

Using the fact that $X_t$ can be expressed as $\sigma(G_t)$ for some $G_t \in R[[X]]$ and comparing terms of degree one, it is easy to check that if $L$ is the matrix formed by the coefficients of $l_t$ (as the $i$th row) then $L$ is invertible and hence $\det L$ is a unit in $R$. Then a standard argument as in Lemma 2, Corollary 2 [ZSII, p. 137] yields that $\sigma$ is an automorphism.

Now we turn to proving the properties of $I_c(R)$. We will write $I_c$ for $I_c(R)$, whenever there is no confusion.

Theorem D. Let

$I_1 = \{a \in R \mid \text{there exists an } R\text{-automorphism } \sigma: R[[X]] \to R[[X]] \text{ with } \sigma(X_t) = X_t + a\}$

$I_2 = \{a \in R \mid \text{there exists an } R\text{-homomorphism } \sigma: R[[X]] \to R[[Y]] \text{ where } X, Y \text{ are finite sets of indeterminates over } R \text{ such that } \sigma(X_t) = a + f \text{ for some } X_t \in X \text{ and } f \in (YR[[Y]])\}$.

Then $I_c = I_1 = I_2$.

Proof. $I_1 \subseteq I_2$ is obvious. If $a \in I_2$ and $\sigma$ and $X_t$ are as in the definition, let $\sigma^* = \text{the restriction of } \sigma \text{ to } R[[X]]$ and $\tau: R[[Y]] \to R$ the unique $R$-homomorphism with $\tau(Y_t) = 0$ for all $Y_t \in Y$. Then $\tau \circ \sigma^*: R[[X]] \to R$ carries $X_t$ to $a$. Thus $a \in I_c$ and hence $I_2 \subseteq I_c$. Finally, by Theorem A it is clear that $I_c \subseteq I_1$.

Theorem E. $I_c$ is an ideal contained in the Jacobson radical of $R$. Moreover, the nil-radical of $R$ is contained in $I_c$.

Proof. Let $a \in I_c$. Since $X$ is in the Jacobson radical of $R [[X]]$ and by Theorem A there is an $R$-automorphism of $R[[X]]$ carrying $X$ to
X + a we get that X + a belongs to the Jacobson radical of R[[X]]. Thus a belongs to the Jacobson radical of R[[X]] and hence of R. The last remark is easy to prove, and is left to the reader.

Now let X, Y, Z be indeterminates over R. Let a, b ∈ I. Hence by definition we may assume that there exists an R-homomorphism σ: R[[X, Y]] → R with σ(X) = a and σ(Y) = b. Let r, s ∈ R. Let τ: R[[Z]] → R[[X, Y]] be the unique R-homomorphism defined by

\[ τ(Z) = rX + xY. \]

Then σ ◦ τ: R[[Z]] → R is an R-homomorphism with σ ◦ τ(Z) = ra + sb. Thus ra + sb ∈ I and hence I is an ideal.

**Remarks.**

(1) The fact that I is an ideal shows that Theorem 3.4 of [1] is true with no restriction on the element “r”. Thus the conjecture which follows that theorem is false.

(2) In his review of [5] (MR47 # 8532) Gilmer suggests a program for simplifying some of the proofs. This would rest on whether a ring R is a complete Hausdorff space in its \( (a, \cdots, a_n) \)-adic topology, if it is a complete Hausdorff space in its \( (a_i) \)-adic topology for each i. However, it is easy to give an example where this does not hold. For Gilmer’s example in [1] is a ring R and an element a such that R is complete, but not Hausdorff in its \( (a) \)-adic topology. On the other hand, by Theorem D there is an automorphism of R[[X]] which takes X to X + a. Since R[[X]] is a complete Hausdorff space in its X-adic topology, it is also a complete Hausdorff space in its \( (X + a) \)-adic topology. However, since R is not Hausdorff in its \( (a) \)-adic topology, neither is R[[X]]. So, since \( a \in (X, X + a)R[[X]] \) we see that R[[X]] is not Hausdorff in its \( (X, X + a) \)-adic topology.

(3) I may be properly contained in the Jacobson radical of R and it may properly contain the nil-radical of R. For example if \( R' = \mathbb{Z}/4[X], M = (2, X)R' \) and \( R = R_{a}[Y] \). Then the nil-radical of R is \( 2R \), I in this case is \( (2, Y) \) and the Jacobson radical is \( (2, Y, X) \).

(4) It would be nice to have an intrinsic characterization of the ideal I since it allows us to utilize the form of Nakayama’s lemma for complete local rings, namely

**Lemma.** Suppose that M is an R-module and J ⊂ I is a finitely generated ideal with \( \cap J^n M = \{0\} \). If N is a finitely generated submodule of M with \( M = N + JM \), then \( N = M \).

The proof would be the same as in the complete local ring case [8, Th. 7, p. 259].
REFERENCES

2. R. Gilmer and M. J. O’Malley, $R$-endomorphisms of $R[[X_1, \ldots, X_n]]$, to appear J. Algebra.

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