

# Pacific Journal of Mathematics

***R*-ENDOMORPHISMS OF  $R[[X]]$  ARE ESSENTIALLY  
CONTINUOUS**

PAUL M. EAKIN, JR. AND AVINASH MADHAV SATHAYE

## R-ENDOMORPHISMS OF $R[[X]]$ ARE ESSENTIALLY CONTINUOUS

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**Let  $R$  be a commutative ring with identity,  $A = R[[X]]$  and  $B = R[[Y]]$  with  $X$  and  $Y$  finite sets of indeterminates. Consider  $A$  and  $B$  as topological rings with the respective  $X$  and  $Y$ -adic topologies. If  $\sigma: A \rightarrow B$  is any  $R$ -homomorphism then there are  $R$ -automorphisms  $s$  and  $t$  of  $A$  and  $B$  respectively, so that  $t \circ \sigma \circ s: A \rightarrow B$  is continuous. As a corollary we see that an  $R$ -endomorphism of  $A$  is surjective only if it is an automorphism.**

Let  $X = \{X_1, \dots, X_n\}$  be a set of indeterminates over  $R$ .  $R[X]$  and  $R[[X]]$  denote as usual the polynomial ring and the formal power series ring respectively over  $R$  in the variables  $X$ . A number of authors have studied and applied automorphisms and endomorphisms of  $R[[X]]$  over  $R$ ; [3], [4], [5], [6], [7] and [1]. A common feature of many of the arguments seems to be the complexity resulting from the fact that  $R$ -endomorphisms of  $R[[X]]$  need not be continuous in  $X$ -adic topology. In this note we show that they are essentially continuous, i.e. differ from a continuous one by an automorphism. Precisely, we make the following

DEFINITION 1. If  $A, B$  are topological rings and  $\sigma: A \rightarrow B$  is a homomorphism, then  $\sigma$  is said to be *essentially continuous* if, for some automorphisms  $s$  and  $t$  of  $A$  and  $B$  respectively, we get that  $t \circ \sigma \circ s: A \rightarrow B$  is continuous.

With this definition we get the main statement that “*every  $R$ -homomorphism between any two formal power series rings over  $R$  is essentially continuous.*” (Corollary B)

As a corollary we get an easy proof of the statement that, “*an  $R$ -endomorphism of  $R[[X]]$  is surjective if and only if it is an  $R$ -automorphism of  $R[[X]]$ .*” (Corollary C)<sup>1</sup>

Finally, we make

DEFINITION 2. If  $\mathfrak{A}$  is a finitely generated ideal of  $R$  we say that  $R$  is *complete in the  $\mathfrak{A}$ -adic topology* if there is a finite set of indeterminates  $X$  and an  $R$ -homomorphism  $\sigma: R[[X]] \rightarrow R$  with  $\sigma(XR[[X]]) = \mathfrak{A}$ .

<sup>1</sup> O'Malley had done the one variable case of this result in [3]. Gilmer and O'Malley have independently given another proof of Corollary C in [2].

Let  $I_c(R)$  denote the set of all  $a \in R$  such that there is an  $R$ -homomorphism  $\sigma: R[[X_1]] \rightarrow R$  with  $\sigma(X_1) = a$ .

Using the "essential continuity" we establish that  $I_c(R)$  is an *ideal* of  $R$  contained in the Jacobson radical of  $R$  and containing the nil-radical of  $R$ . (Theorem E)

Once  $I_c(R)$  is shown to be an ideal it is easy to show that  $I_c(R)$  is nothing but the union of all ideals  $\mathfrak{A}$  of  $R$  such that  $R$  is complete in the  $\mathfrak{A}$ -adic topology. This fact is indeed the reason for the suffix "c" in  $I_c(R)$ . This fact also answers some questions raised by Gilmer; see remarks at the end.

**THEOREM A.** *Suppose  $R$  is a commutative ring with identity and  $X = \{X_i\}_{i=1}^n$  and  $Y = \{Y_j\}_{j=1}^m$  are sets of indeterminates over  $R$ . Suppose  $R[[X]] \xrightarrow{\sigma} R[[Y]]$  is an  $R$ -homomorphism and that for each  $i$ ,  $\sigma(X_i) = c_i + f_i$  where  $c_i \in R$  and  $f_i \in YR[[Y]]$ . Then there exists an automorphism  $t; R[[X]] \rightarrow R[[X]]$  such that  $t(X_i) = X_i + c_i$ .<sup>2</sup>*

*Proof.* Let  $\beta: R[[Y]] \rightarrow R$  be defined by  $\beta(Y) = 0$ . Then composing  $\beta$  and  $\sigma$  we get a mapping  $\sigma^*: R[[X]] \rightarrow R$  such that  $\sigma^*(X_i) = c_i$ . Let  $\{Z_i\}_{i=1}^n$  be  $n$  additional indeterminates. We extend  $\sigma^*$  to a mapping  $\sigma^*: R[[X, Z]] \rightarrow R$  by  $\sigma^*(Z) = 0$ . We now have a sequence

$$R[[Z]] \xrightarrow{\alpha} R[[X, Z]] \xrightarrow{\gamma} R[[Z]]$$

where  $\alpha(Z_i) = X_i + Z_i$  and  $\gamma$  is defined by regarding  $R[[X, Z]]$  as  $R[[X]][[Z]]$  and setting

$$\gamma(\sum h_i Z^i) = \sum \sigma^*(h_i) Z^i \quad \text{where } h_i \in R[[X]].$$

We define  $\tau^* = \gamma \circ \alpha$  and note that  $\tau^*(Z_i) = Z_i + c_i$ . Since  $R[[Z]] \cong R[[Z]]$  by  $X \rightarrow Z$  there is a mapping  $\tau: R[[X]] \rightarrow R[[X]]$  such that  $\tau(X_i) = X_i + c_i$ . We must now see that  $\tau$  is an automorphism of  $R[[X]]$ . There is an automorphism  $\delta$  of  $R[[X]]$  which takes  $X_i$  to  $-X_i$ .

The homomorphism  $\delta \circ \tau \circ \delta \circ \tau: R[[X]] \rightarrow R[[X]]$  is a *continuous* endomorphism carrying  $X_i$  to  $X_i$ . It is then clear that  $\delta \circ \tau \circ \delta \circ \tau$  is the identity map and hence  $\tau$  is an automorphism.

**COROLLARY B.** *If  $R$  is a commutative ring with 1 and  $X = \{X_i\}_{i=1}^n$ ,  $Y = \{Y_j\}_{j=1}^m$  are indeterminates over  $R$ , then any  $R$ -homomorphism  $\sigma: R[[X]] \rightarrow R[[Y]]$  is essentially continuous.*

<sup>2</sup> This result in the one-variable case appears in [1].

*Proof.* Let  $\sigma(X_i) = c_i + f_i$  with  $c_i \in R$  and  $f_i \in YR[[Y]]$ . Then by Theorem A, there is an automorphism  $\tau$  of  $R[[X]]$  such that  $\tau(X_i) = X_i + c_i$ . Thus  $\tau^{-1}(X_i) = X_i - c_i$ . The mapping  $\sigma \circ \tau^{-1}$  is continuous since

$$\sigma \circ \tau^{-1}(X_i) = \sigma(X_i - c_i) = c_i + f_i - c_i = f_i$$

and  $f_i \in YR[[Y]]$ .

**COROLLARY C.** *If  $R$  is a commutative ring with 1 and  $\{X_i\}_{i=1}^n$  are indeterminates, then an  $R$ -endomorphism  $\sigma: R[[X]] \rightarrow R[[X]]$  is surjective if and only if it is an automorphism.*

*Proof.* One way is clear. By the proof of Corollary B we may write

$$\sigma(X_i) = l_i + F_i,$$

where  $l_i$  is a linear form in  $X$  over  $R$  and  $F_i \in (XR[[X]])^2$ .

Using the fact that  $X_i$  can be expressed as  $\sigma(G_i)$  for some  $G_i \in R[[X]]$  and comparing terms of degree one, it is easy to check that if  $L$  is the matrix formed by the coefficients of  $l_i$  (as the  $i$ th row) then  $L$  is invertible and hence  $\det L$  is a unit in  $R$ . Then a standard argument as in Lemma 2, Corollary 2 [ZSII, p. 137] yields that  $\sigma$  is an automorphism.

Now we turn to proving the properties of  $I_c(R)$ . We will write  $I_c$  for  $I_c(R)$ , whenever there is no confusion.

**THEOREM D.** *Let*

$I_1 = \{a \in R \mid \text{there exists an } R\text{-automorphism } \sigma: R[[X_1]] \rightarrow R[[X_1]] \text{ with } \sigma(X_1) = X_1 + a\}$

$I_2 = \{a \in R \mid \text{there exists an } R\text{-homomorphism } \sigma: R[[X]] \rightarrow R[[Y]] \text{ where } X, Y \text{ are finite sets of indeterminates over } R \text{ such that } \sigma(X_i) = a + f \text{ for some } X_i \in X \text{ and } f \in (YR[[Y]])\}$ .

*Then  $I_c = I_1 = I_2$ .*

*Proof.*  $I_1 \subset I_2$  is obvious. If  $a \in I_2$  and  $\sigma$  and  $X_i$  are as in the definition, let  $\sigma^*$  = the restriction of  $\sigma$  to  $R[[X_i]]$  and  $\tau: R[[Y]] \rightarrow R$  the unique  $R$ -homomorphism with  $\tau(Y_j) = 0$  for all  $Y_j \in Y$ . Then  $\tau \circ \sigma^*: R[[X_i]] \rightarrow R$  carries  $X_i$  to  $a$ . Thus  $a \in I_c$  and hence  $I_2 \subset I_c$ . Finally, by Theorem A it is clear that  $I_c \subset I_1$ .

**THEOREM E.**  *$I_c$  is an ideal contained in the Jacobson radical of  $R$ . Moreover, the nil-radical of  $R$  is contained in  $I_c$ .*

*Proof.* Let  $a \in I_c$ . Since  $X$  is in the Jacobson radical of  $R[[X]]$  and by Theorem A there is an  $R$ -automorphism of  $R[[X]]$  carrying  $X$  to

$X + a$  we get that  $X + a$  belongs to the Jacobson radical of  $R[[X]]$ . Thus  $a$  belongs to the Jacobson radical of  $R[[X]]$  and hence of  $R$ . The last remark is easy to prove, and is left to the reader.

Now let  $X, Y, Z$  be indeterminates over  $R$ . Let  $a, b \in I_c$ . Hence by definition we may assume that there exists an  $R$ -homomorphism  $\sigma: R[[X, Y]] \rightarrow R$  with  $\sigma(X) = a$  and  $\sigma(Y) = b$ . Let  $r, s \in R$ . Let  $\tau: R[[Z]] \rightarrow R[[X, Y]]$  be the unique  $R$ -homomorphism defined by

$$\tau(Z) = rX + sY.$$

Then  $\sigma \circ \tau: R[[Z]] \rightarrow R$  is an  $R$ -homomorphism with  $\sigma \circ \tau(Z) = ra + sb$ . Thus  $ra + sb \in I_c$  and hence  $I_c$  is an ideal.

REMARKS. (1) The fact that  $I_c$  is an ideal shows that Theorem 3.4 of [1] is true with no restriction on the element “ $r$ ”. Thus the conjecture which follows that theorem is false.

(2) In his review of [5] (MR47 # 8532) Gilmer suggests a program for simplifying some of the proofs. This would rest on whether a ring  $R$  is a complete Hausdorff space in its  $(a_1, \dots, a_n)$ -adic topology, if it is a complete Hausdorff space in its  $(a_i)$ -adic topology for each  $i$ . However, it is easy to give an example where this does not hold. For Gilmer’s example in [1] is a ring  $R$  and an element  $a$  such that  $R$  is complete, but not Hausdorff in its  $(a)$ -adic topology. On the other hand, by Theorem D there is an automorphism of  $R[[X]]$  which takes  $X$  to  $X + a$ . Since  $R[[X]]$  is a complete Hausdorff space in its  $X$ -adic topology, it is also a complete Hausdorff space in its  $(X + a)$ -adic topology. However, since  $R$  is not Hausdorff in its  $(a)$ -adic topology, neither is  $R[[X]]$ . So, since  $a \in (X, X + a)R[[X]]$  we see that  $R[[X]]$  is not Hausdorff in its  $(X, X + a)$ -adic topology.

(3)  $I_c$  may be properly contained in the Jacobson radical of  $R$  and it may properly contain the nil-radical of  $R$ . For example if  $R' = Z/4[X]$ ,  $\mathcal{M} = (2, X)R'$  and  $R = R'_{\mathcal{M}}[[Y]]$ . Then the nil-radical of  $R$  is  $2R$ ,  $I_c$  in this case is  $(2, Y)$  and the Jacobson radical is  $(2, Y, X)$ .

(4) It would be nice to have an intrinsic characterization of the ideal  $I_c$  since it allows us to utilize the form of Nakayama’s lemma for complete local rings, namely

LEMMA. *Suppose that  $M$  is an  $R$ -module and  $J \subset I_c$  is a finitely generated ideal with  $\bigcap J^n M = \{0\}$ . If  $N$  is a finitely generated submodule of  $M$  with  $M = N + JM$ , then  $N = M$ .*

The proof would be the same as in the complete local ring case [8, Th. 7, p. 259].

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Received July 28, 1975.

UNIVERSITY OF KENTUCKY







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