Pacific Journal of Mathematics

R-ENDOMORPHISMS OF $R[[X]]$ ARE ESSENTIALLY CONTINUOUS

PAUL M. EAKIN, JR. AND AVINASH MADHAV SATHAYE

Vol. 66, No. 1 November 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 66, No. 1, 1976

R-ENDOMORPHISMS OF $R[[X]]$ ARE ESSENTIALLY CONTINUOUS

PAUL EAKIN AND AVINASH SATHAYE

Let R be a commutative ring with identity, $A = R[[X]]$ and $B = R[[Y]]$ with X and Y finite sets of indeterminates. Consider A and B as topological rings with the respective X and Y-adic topologies. If $\sigma: A \to B$ is any R-homomorphism then there are R-automorphisms s and t of A and B respectively, so that $t \circ \sigma \circ s: A \to B$ is continuous. As a corollary we see that an R-endomorphism of A is surjective only if it is an automorphism.

Let $X = \{X_1, \cdots, X_n\}$ be a set of indeterminates over R. $R[X]$ and $R[[X]]$ denote as usual the polynomial ring and the formal power series ring respectively over R in the variables X. A number of authors have studied and applied automorphisms and endomorphisms of $R[[X]]$ over R; **[3]**, **[4]**, **[5]**, **[6]**, **[7]** and **[1]**. A common feature of many of the arguments seems to be the complexity resulting from the fact that R-endomorphisms of $R[[X]]$ need not be continuous in X-adic topology. In this note we show that they are essentially continuous, i.e. differ from a continuous one by an automorphism. Precisely, we make the following

DEFINITION 1. If A, B are topological rings and $\sigma: A \to B$ is a homomorphism, then σ is said to be *essentially continuous* if, for some automorphisms s and t of A and B respectively, we get that $t \circ \sigma \circ s: A \to B$ is continuous.

With this definition we get the main statement that "*every R-homomorphism between any two formal power series rings over R is essentially continuous.*" (Corollary B)

As a corollary we get an easy proof of the statement that, "*an R-endomorphism of $R[[X]]$ is surjective if and only if it is an R-automorphism of $R[[X]]$.*" (Corollary C)[1]

Finally, we make

DEFINITION 2. If $\mathfrak{A}$ is a finitely generated ideal of R we say that R is *complete in the $\mathfrak{A}$-adic topology* if there is a finite set of indeterminates X and an R-homomorphism $\sigma: R[[X]] \to R$ with $\sigma(XR[[X]]) = \mathfrak{A}$.

[1] O'Malley had done the one variable case of this result in **[3]**. Gilmer and O'Malley have independently given another proof of Corollary C in **[2]**.

Let $I_C(R)$ denote the set of all $a \in R$ such that there is an R-homomorphism $\sigma: R[[X_1]] \to R$ with $\sigma(X_1) = a$.

Using the "essential continuity" we establish that $I_C(R)$ is an *ideal* of R contained in the Jacobson radical of R and containing the nil-radical of R. (Theorem E)

Once $I_C(R)$ is shown to be an ideal it is easy to show that $I_C(R)$ is nothing but the union of all ideals $\mathfrak{A}$ of R such that R is complete in the $\mathfrak{A}$-adic topology. This fact is indeed the reason for the suffix "c" in $I_c(R)$. This fact also answers some questions raised by Gilmer; see remarks at the end.

THEOREM A. *Suppose R is a commutative ring with identity and $X = \{X_i\}_{i=1}^n$ and $Y = \{Y_j\}_{j=1}^m$ are sets of indeterminates over R. Suppose $R[[X]] \xrightarrow{\sigma} R[[Y]]$ is an R-homomorphism and that for each i, $\sigma(X_i) = c_i + f_i$ where $c_i \in R$ and $f_i \in YR[[Y]]$. Then there exists an automorphism t; $R[[X]] \to R[[X]]$ such that $t(X_i) = X_i + c_i$.*[2]

Proof. Let $\beta: R[[Y]] \to R$ be defined by $\beta(Y) = 0$. Then composing β and σ we get a mapping $\sigma^*: R[[X]] \to R$ such that $\sigma^*(X_i) = c_i$. Let $\{Z_i\}_{i=1}^n$ be n additional indeterminates. We extend σ^* to a mapping $\sigma^*: R[[X, Z]] \to R$ by $\sigma^*(Z) = 0$. We now have a sequence

$$R[[Z]] \xrightarrow{\alpha} R[[X, Z]] \xrightarrow{\gamma} R[[Z]]$$

where $\alpha(Z_i) = X_i + Z_i$ and γ is defined by regarding $R[[X, Z]]$ as $R[[X]][[Z]]$ and setting

$$\gamma(\Sigma h_i Z^i) = \Sigma \sigma^*(h_i) Z^i \quad \text{where} \quad h_i \in R[[X]].$$

We define $\tau^* = \gamma \circ \alpha$ and note that $\tau^*(Z_i) = Z_i + c_i$. Since $R[[Z]] \cong R[[Z]]$ by $X \to Z$ there is a mapping $\tau: R[[X]] \to R[[X]]$ such that $\tau(X_i) = X_i + c_i$. We must now see that τ is an automorphism of $R[[X]]$. There is an automorphism δ of $R[[X]]$ which takes X_i to $-X_i$.

The homomorphism $\delta \circ \tau \circ \delta \circ \tau: R[[X]] \to R[[X]]$ is a *continuous* endomorphism carrying X_i to X_i. It is then clear that $\delta \circ \tau \circ \delta \circ \tau$ is the identity map and hence τ is an automorphism.

COROLLARY B. *If R is a commutative ring with 1 and $X = \{X_i\}_{i=1}^n$, $Y = \{Y_j\}_{j=1}^m$ are indeterminates over R, then any R-homomorphism $\sigma: R[[X]] \to R[[Y]]$ is essentially continuous.*

[2] This result in the one-variable case appears in [1].

Proof. Let $\sigma(X_i) = c_i + f_i$ with $c_i \in R$ and $f_i \in YR[[Y]]$. Then by Theorem A, there is an automorphism τ of $R[[X]]$ such that $\tau(X_i) = X_i + c_i$. Thus $\tau^{-1}(X_i) = X_i - c_i$. The mapping $\sigma \circ \tau^{-1}$ is continuous since

$$\sigma \circ \tau^{-1}(X_i) = \sigma(X_i - c_i) = c_i + f_i - c_i = f_i$$

and $f_i \in YR[[Y]]$.

COROLLARY C. *If R is a commutative ring with 1 and $\{X_i\}_{i=1}^n$ are indeterminates, then an R-endomorphism σ: $R[[X]] \to R[[X]]$ is surjective if and only if it is an automorphism.*

Proof. One way is clear. By the proof of Corollary B we may write

$$\sigma(X_i) = l_i + F_i,$$

where l_i is a linear form in X over R and $F_i \in (XR[[X]])^2$.

Using the fact that X_i can be expressed as $\sigma(G_i)$ for some $G_i \in R[[X]]$ and comparing terms of degree one, it is easy to check that if L is the matrix formed by the coefficients of l_i (as the ith row) then L is invertible and hence $\det L$ is a unit in R. Then a standard argument as in Lemma 2, Corollary 2 [ZSII, p. 137] yields that σ is an automorphism.

Now we turn to proving the properties of $I_c(R)$. We will write I_c for $I_c(R)$, whenever there is no confusion.

THEOREM D. *Let*

$I_1 = \{a \in R \mid$ *there exists an R-automorphism σ: $R[[X_1]] \to R[[X_1]]$ with* $\sigma(X_1) = X_1 + a\}$

$I_2 = \{a \in R \mid$ *there exists an R-homomorphism σ: $R[[X]] \to R[[Y]]$ where X, Y are finite sets of indeterminates over R such that $\sigma(X_i) = a + f$ for some $X_i \in X$ and $f \in (YR[[Y]])\}$.*

Then $I_c = I_1 = I_2$.

Proof. $I_1 \subset I_2$ is obvious. If $a \in I_2$ and σ and X_i are as in the definition, let σ^* = the restriction of σ to $R[[X_i]]$ and τ: $R[[Y]] \to R$ the unique R-homomorphism with $\tau(Y_j) = 0$ for all $Y_j \in Y$. Then $\tau \circ \sigma^*$: $R[[X_i]] \to R$ carries X_i to a. Thus $a \in I_c$ and hence $I_2 \subset I_c$. Finally, by Theorem A it is clear that $I_c \subset I_1$.

THEOREM E. *I_c is an ideal contained in the Jacobson radical of R. Moreover, the nil-radical of R is contained in I_c.*

Proof. Let $a \in I_c$. Since X is in the Jacobson radical of $R[[X]]$ and by Theorem A there is an R-automorphism of $R[[X]]$ carrying X to

$X+a$ we get that $X+a$ belongs to the Jacobson radical of $R[[X]]$. Thus a belongs to the Jacobson radical of $R[[X]]$ and hence of R. The last remark is easy to prove, and is left to the reader.

Now let X, Y, Z be indeterminates over R. Let $a, b \in I_c$. Hence by definition we may assume that there exists an R-homomorphism $\sigma: R[[X, Y]] \to R$ with $\sigma(X) = a$ and $\sigma(Y) = b$. Let $r, s \in R$. Let $\tau: R[[Z]] \to R[[X, Y]]$ be the unique R-homomorphism defined by

$$\tau(Z) = rX + xY.$$

Then $\sigma \circ \tau$: $R[[Z]] \to R$ is an R-homomorphism with $\sigma \circ \tau(Z) = ra + sb$. Thus $ra + sb \in I_c$ and hence I_c is an ideal.

REMARKS. (1) The fact that I_c is an ideal shows that Theorem 3.4 of **[1]** is true with no restriction on the element "r". Thus the conjecture which follows that theorem is false.

(2) In his review of **[5]** (MR47 #8532) Gilmer suggests a program for simplifying some of the proofs. This would rest on whether a ring R is a complete *Hausdorff* space in its $(a_1, \cdots, a_n)$-adic topology, if it is a complete Hausdorff space in its (a_i)-adic topology for each i. However, it is easy to give an example where this does not hold. For Gilmer's example in **[1]** is a ring R and an element a such that R is complete, but not Hausdorff in its (a)-adic topology. On the other hand, by Theorem D there is an automorphism of $R[[X]]$ which takes X to $X + a$. Since $R[[X]]$ is a complete Hausdorff space in its X-adic topology, it is also a complete Hausdorff space in its $(X + a)$-adic topology. However, since R is not Hausdorff in its (a)-adic topology, neither is $R[[X]]$. So, since $a \in (X, X + a)R[[X]]$ we see that $R[[X]]$ is not Hausdorff in its $(X, X + a)$-adic topology.

(3) I_c may be properly contained in the Jacobson radical of R and it may properly contain the nil-radical of R. For example if $R' = Z/4[X]$, $\mathcal{M} = (2, X)R'$ and $R = R'_{\mathcal{M}}[[Y]]$. Then the nil-radical of R is $2R$, I_c in this case is $(2, Y)$ and the Jacobson radical is $(2, Y, X)$.

(4) It would be nice to have an intrinsic characterization of the ideal I_c since it allows us to utilize the form of Nakayama's lemma for complete local rings, namely

LEMMA. *Suppose that M is an R-module and $J \subset I_c$ is a finitely generated ideal with $\bigcap J^n M = \{0\}$. If N is a finitely generated submodule of M with $M = N + JM$, then $N = M$.*

The proof would be the same as in the complete local ring case **[8**, Th. 7, p. 259].

REFERENCES

1. R. Gilmer, *R-automorphisms of $R[[X]]$*, Michigan J. Math., **17** (1970), 15–21.
2. R. Gilmer and M. J. O'Malley, *R-endomorphisms of $R[[X_1, \cdots, X_n]]$*, to appear J. Algebra.
3. M. J. O'Malley, *R-automorphisms of $R[[X]]$*, Proc. London Math. Soc., (3) **20** (1970), 60–78.
4. ———, *Isomorphic Power Series Rings*, Pacific J. Math., **41** (1972), 503–512.
5. ———, *Finite Groups of R-automorphisms of $R[[X]]$*, Michigan Math. J., **20** (1973), 277–284.
6. ———, *Some remarks on the formal power series ring*, Bull. Soc. Math. France, **99** (1971), 247–258.
7. M. J. O'Malley and C. Wood, *R-endomorphisms of $R[[X]]$*, J. Algebra, **15** (1970), 314–327.
8. O. Zariski and P. Samuel, *Commutative Algebra*, volume II, Van Nostrand, Princeton, 1960.

Received July 28, 1975.

UNIVERSITY OF KENTUCKY

Pacific Journal of Mathematics

Vol. 66, No. 1 November, 1976