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R -ENDOMORPHISMS OF $R[[X]]$ ARE ESSENTIALLY CONTINUOUS

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Let R be a commutative ring with identity, $A = R[[X]]$ and $B = R[[Y]]$ with X and Y finite sets of indeterminates. Consider A and B as topological rings with the respective X and Y -adic topologies. If $\sigma: A \rightarrow B$ is any R -homomorphism then there are R -automorphisms s and t of A and B respectively, so that $t \circ \sigma \circ s: A \rightarrow B$ is continuous. As a corollary we see that an R -endomorphism of A is surjective only if it is an automorphism.

Let $X = \{X_1, \dots, X_n\}$ be a set of indeterminates over R . $R[X]$ and $R[[X]]$ denote as usual the polynomial ring and the formal power series ring respectively over R in the variables X . A number of authors have studied and applied automorphisms and endomorphisms of $R[[X]]$ over R ; [3], [4], [5], [6], [7] and [1]. A common feature of many of the arguments seems to be the complexity resulting from the fact that R -endomorphisms of $R[[X]]$ need not be continuous in X -adic topology. In this note we show that they are essentially continuous, i.e. differ from a continuous one by an automorphism. Precisely, we make the following

DEFINITION 1. If A, B are topological rings and $\sigma: A \rightarrow B$ is a homomorphism, then σ is said to be *essentially continuous* if, for some automorphisms s and t of A and B respectively, we get that $t \circ \sigma \circ s: A \rightarrow B$ is continuous.

With this definition we get the main statement that “every R -homomorphism between any two formal power series rings over R is essentially continuous.” (Corollary B)

As a corollary we get an easy proof of the statement that, “an R -endomorphism of $R[[X]]$ is surjective if and only if it is an R -automorphism of $R[[X]]$.” (Corollary C)¹

Finally, we make

DEFINITION 2. If \mathfrak{A} is a finitely generated ideal of R we say that R is *complete in the \mathfrak{A} -adic topology* if there is a finite set of indeterminates X and an R -homomorphism $\sigma: R[[X]] \rightarrow R$ with $\sigma(XR[[X]]) = \mathfrak{A}$.

¹ O'Malley had done the one variable case of this result in [3]. Gilmer and O'Malley have independently given another proof of Corollary C in [2].

Let $I_c(R)$ denote the set of all $a \in R$ such that there is an R -homomorphism $\sigma: R[[X_1]] \rightarrow R$ with $\sigma(X_1) = a$.

Using the "essential continuity" we establish that $I_c(R)$ is an ideal of R contained in the Jacobson radical of R and containing the nil-radical of R . (Theorem E)

Once $I_c(R)$ is shown to be an ideal it is easy to show that $I_c(R)$ is nothing but the union of all ideals \mathfrak{A} of R such that R is complete in the \mathfrak{A} -adic topology. This fact is indeed the reason for the suffix "c" in $I_c(R)$. This fact also answers some questions raised by Gilmer; see remarks at the end.

THEOREM A. *Suppose R is a commutative ring with identity and $X = \{X_i\}_{i=1}^n$ and $Y = \{Y_j\}_{j=1}^m$ are sets of indeterminates over R . Suppose $R[[X]] \xrightarrow{\sigma} R[[Y]]$ is an R -homomorphism and that for each i , $\sigma(X_i) = c_i + f_i$, where $c_i \in R$ and $f_i \in YR[[Y]]$. Then there exists an automorphism t ; $R[[X]] \rightarrow R[[X]]$ such that $t(X_i) = X_i + c_i$.²*

Proof. Let $\beta: R[[Y]] \rightarrow R$ be defined by $\beta(Y) = 0$. Then composing β and σ we get a mapping $\sigma^*: R[[X]] \rightarrow R$ such that $\sigma^*(X_i) = c_i$. Let $\{Z_i\}_{i=1}^n$ be n additional indeterminates. We extend σ^* to a mapping $\sigma^*: R[[X, Z]] \rightarrow R$ by $\sigma^*(Z) = 0$. We now have a sequence

$$R[[Z]] \xrightarrow{\alpha} R[[X, Z]] \xrightarrow{\gamma} R[[Z]]$$

where $\alpha(Z_i) = X_i + Z_i$ and γ is defined by regarding $R[[X, Z]]$ as $R[[X]][[Z]]$ and setting

$$\gamma(\sum h_i Z^i) = \sum \sigma^*(h_i) Z^i \quad \text{where } h_i \in R[[X]].$$

We define $\tau^* = \gamma \circ \alpha$ and note that $\tau^*(Z_i) = Z_i + c_i$. Since $R[[Z]] \cong R[[Z]]$ by $X \rightarrow Z$ there is a mapping $\tau: R[[X]] \rightarrow R[[X]]$ such that $\tau(X_i) = X_i + c_i$. We must now see that τ is an automorphism of $R[[X]]$. There is an automorphism δ of $R[[X]]$ which takes X_i to $-X_i$.

The homomorphism $\delta \circ \tau \circ \delta \circ \tau: R[[X]] \rightarrow R[[X]]$ is a continuous endomorphism carrying X_i to X_i . It is then clear that $\delta \circ \tau \circ \delta \circ \tau$ is the identity map and hence τ is an automorphism.

COROLLARY B. *If R is a commutative ring with 1 and $X = \{X_i\}_{i=1}^n$, $Y = \{Y_j\}_{j=1}^m$ are indeterminates over R , then any R -homomorphism $\sigma: R[[X]] \rightarrow R[[Y]]$ is essentially continuous.*

² This result in the one-variable case appears in [1].

Proof. Let $\sigma(X_i) = c_i + f_i$ with $c_i \in R$ and $f_i \in YR[[Y]]$. Then by Theorem A, there is an automorphism τ of $R[[X]]$ such that $\tau(X_i) = X_i + c_i$. Thus $\tau^{-1}(X_i) = X_i - c_i$. The mapping $\sigma \circ \tau^{-1}$ is continuous since

$$\sigma \circ \tau^{-1}(X_i) = \sigma(X_i - c_i) = c_i + f_i - c_i = f_i$$

and $f_i \in YR[[Y]]$.

COROLLARY C. *If R is a commutative ring with 1 and $\{X_i\}_{i=1}^n$ are indeterminates, then an R -endomorphism $\sigma: R[[X]] \rightarrow R[[X]]$ is surjective if and only if it is an automorphism.*

Proof. One way is clear. By the proof of Corollary B we may write

$$\sigma(X_i) = l_i + F_i,$$

where l_i is a linear form in X over R and $F_i \in (XR[[X]])^2$.

Using the fact that X_i can be expressed as $\sigma(G_i)$ for some $G_i \in R[[X]]$ and comparing terms of degree one, it is easy to check that if L is the matrix formed by the coefficients of l_i (as the i th row) then L is invertible and hence $\det L$ is a unit in R . Then a standard argument as in Lemma 2, Corollary 2 [ZSII, p. 137] yields that σ is an automorphism.

Now we turn to proving the properties of $I_c(R)$. We will write I_c for $I_c(R)$, whenever there is no confusion.

THEOREM D. *Let*

$I_1 = \{a \in R \mid \text{there exists an } R\text{-automorphism } \sigma: R[[X_1]] \rightarrow R[[X_1]] \text{ with } \sigma(X_1) = X_1 + a\}$

$I_2 = \{a \in R \mid \text{there exists an } R\text{-homomorphism } \sigma: R[[X]] \rightarrow R[[Y]] \text{ where } X, Y \text{ are finite sets of indeterminates over } R \text{ such that } \sigma(X_i) = a + f \text{ for some } X_i \in X \text{ and } f \in (YR[[Y]])\}$.

Then $I_c = I_1 = I_2$.

Proof. $I_1 \subset I_2$ is obvious. If $a \in I_2$ and σ and X_i are as in the definition, let σ^* = the restriction of σ to $R[[X_i]]$ and $\tau: R[[Y]] \rightarrow R$ the unique R -homomorphism with $\tau(Y_j) = 0$ for all $Y_j \in Y$. Then $\tau \circ \sigma^*: R[[X_i]] \rightarrow R$ carries X_i to a . Thus $a \in I_c$ and hence $I_2 \subset I_c$. Finally, by Theorem A it is clear that $I_c \subset I_1$.

THEOREM E. *I_c is an ideal contained in the Jacobson radical of R . Moreover, the nil-radical of R is contained in I_c .*

Proof. Let $a \in I_c$. Since X is in the Jacobson radical of $R[[X]]$ and by Theorem A there is an R -automorphism of $R[[X]]$ carrying X to

$X + a$ we get that $X + a$ belongs to the Jacobson radical of $R[[X]]$. Thus a belongs to the Jacobson radical of $R[[X]]$ and hence of R . The last remark is easy to prove, and is left to the reader.

Now let X, Y, Z be indeterminates over R . Let $a, b \in I_c$. Hence by definition we may assume that there exists an R -homomorphism $\sigma: R[[X, Y]] \rightarrow R$ with $\sigma(X) = a$ and $\sigma(Y) = b$. Let $r, s \in R$. Let $\tau: R[[Z]] \rightarrow R[[X, Y]]$ be the unique R -homomorphism defined by

$$\tau(Z) = rX + sY.$$

Then $\sigma \circ \tau: R[[Z]] \rightarrow R$ is an R -homomorphism with $\sigma \circ \tau(Z) = ra + sb$. Thus $ra + sb \in I_c$ and hence I_c is an ideal.

REMARKS. (1) The fact that I_c is an ideal shows that Theorem 3.4 of [1] is true with no restriction on the element “ r ”. Thus the conjecture which follows that theorem is false.

(2) In his review of [5] (MR47 # 8532) Gilmer suggests a program for simplifying some of the proofs. This would rest on whether a ring R is a complete Hausdorff space in its (a_1, \dots, a_n) -adic topology, if it is a complete Hausdorff space in its (a_i) -adic topology for each i . However, it is easy to give an example where this does not hold. For Gilmer’s example in [1] is a ring R and an element a such that R is complete, but not Hausdorff in its (a) -adic topology. On the other hand, by Theorem D there is an automorphism of $R[[X]]$ which takes X to $X + a$. Since $R[[X]]$ is a complete Hausdorff space in its X -adic topology, it is also a complete Hausdorff space in its $(X + a)$ -adic topology. However, since R is not Hausdorff in its (a) -adic topology, neither is $R[[X]]$. So, since $a \in (X, X + a)R[[X]]$ we see that $R[[X]]$ is not Hausdorff in its $(X, X + a)$ -adic topology.

(3) I_c may be properly contained in the Jacobson radical of R and it may properly contain the nil-radical of R . For example if $R' = Z/4[X]$, $\mathcal{M} = (2, X)R'$ and $R = R'_{\mathcal{M}}[[Y]]$. Then the nil-radical of R is $2R$, I_c in this case is $(2, Y)$ and the Jacobson radical is $(2, Y, X)$.

(4) It would be nice to have an intrinsic characterization of the ideal I_c since it allows us to utilize the form of Nakayama’s lemma for complete local rings, namely

LEMMA. *Suppose that M is an R -module and $J \subset I_c$ is a finitely generated ideal with $\bigcap J^n M = \{0\}$. If N is a finitely generated submodule of M with $M = N + JM$, then $N = M$.*

The proof would be the same as in the complete local ring case [8, Th. 7, p. 259].

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UNIVERSITY OF KENTUCKY

Helen Elizabeth Adams, <i>Factorization-prime ideals in integral domains</i>	1
Patrick Robert Ahern and Robert Bruce Schneider, <i>The boundary behavior of Henkin's kernel</i>	9
Daniel D. Anderson, Jacob R. Matijevic and Warren Douglas Nichols, <i>The Krull intersection theorem. II</i>	15
Efraim Pacillas Armendariz, <i>On semiprime P.I.-algebras over commutative regular rings</i>	23
Robert H. Bird and Charles John Parry, <i>Integral bases for bicyclic biquadratic fields over quadratic subfields</i>	29
Tae Ho Choe and Young Hee Hong, <i>Extensions of completely regular ordered spaces</i>	37
John Dauns, <i>Generalized monofrom and quasi injective modules</i>	49
F. S. De Blasi, <i>On the differentiability of multifunctions</i>	67
Paul M. Eakin, Jr. and Avinash Madhav Sathaye, <i>R-endomorphisms of $R[[X]]$ are essentially continuous</i>	83
Larry Quin Eifler, <i>Open mapping theorems for probability measures on metric spaces</i>	89
Garret J. Etgen and James Pawlowski, <i>Oscillation criteria for second order self adjoint differential systems</i>	99
Ronald Fintushel, <i>Local S^1 actions on 3-manifolds</i>	111
Kenneth R. Goodearl, <i>Choquet simplexes and σ-convex faces</i>	119
John R. Graef, <i>Some nonoscillation criteria for higher order nonlinear differential equations</i>	125
Charles Henry Heiberg, <i>Norms of powers of absolutely convergent Fourier series: an example</i>	131
Les Andrew Karlovitz, <i>Existence of fixed points of nonexpansive mappings in a space without normal structure</i>	153
Gangaram S. Ladde, <i>Systems of functional differential inequalities and functional differential systems</i>	161
Joseph Michael Lambert, <i>Conditions for simultaneous approximation and interpolation with norm preservation in $C[a, b]$</i>	173
Ernest Paul Lane, <i>Insertion of a continuous function</i>	181
Robert F. Lax, <i>Weierstrass points of products of Riemann surfaces</i>	191
Dan McCord, <i>An estimate of the Nielsen number and an example concerning the Lefschetz fixed point theorem</i>	195
Paul Milnes and John Sydney Pym, <i>Counterexample in the theory of continuous functions on topological groups</i>	205
Peter Johanna I. M. De Paepe, <i>Homomorphism spaces of algebras of holomorphic functions</i>	211
Judith Ann Palagallo, <i>A representation of additive functionals on L^p-spaces, $0 < p < 1$</i>	221
S. M. Patel, <i>On generalized numerical ranges</i>	235
Thomas Thornton Read, <i>A limit-point criterion for expressions with oscillatory coefficients</i>	243
Elemer E. Rosinger, <i>Division of distributions</i>	257
Peter S. Shoenfeld, <i>Highly proximal and generalized almost finite extensions of minimal sets</i>	265
R. Sirois-Dumais and Stephen Willard, <i>Quotient-universal sequential spaces</i>	281
Robert Charles Thompson, <i>Convex and concave functions of singular values of matrix sums</i>	285
Edward D. Tymchatyn, <i>Some n-arc theorems</i>	291
Jang-Mei Gloria Wu, <i>Variation of Green's potential</i>	295