OPEN MAPPING THEOREMS FOR PROBABILITY MEASURES ON METRIC SPACES

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Let $S$ and $T$ denote complete separable metric spaces. Let $P(S)$ denote the collection of probability measures on $S$ and equip $P(S)$ with the weak topology. If $\varphi : S \to T$ is continuous and onto, then $\varphi$ induces a weakly continuous mapping $\varphi^0$ of $P(S)$ onto $P(T)$. We show that $\varphi^0$ is open in the weak topology if and only if $\varphi$ is open. However, $\varphi^0$ is always open in the norm topology. Let $K$ be a totally disconnected compact metric space and let $S^K$ denote the set of continuous mappings of $K$ into $S$. Then there exists a natural mapping $\pi$ of $P(S^K)$ into $P(S^K)$. Blumenthal and Corson have shown that $\pi$ is onto. We establish that $\pi$ is an open mapping in the weak topology.

1. Introduction. Let $S$ be a complete separable metric space and let $C(S)$ denote the algebra of bounded continuous real-valued functions on $S$. Let $M(S)$ denote the collection of Borel measures on $S$ which have finite total variation $\|\mu\|$. Given $f \in C(S)$ and $\mu \in M(S)$, set $\mu(f) = \int f(s) d\mu(s)$. The weak topology on $M(S)$ is the topology on $M(S)$ induced by $C(S)$. Thus, a neighborhood system at $\mu$ in $M(S)$ is given by sets of the form

$$N_\varepsilon(\mu; f_1, \cdots, f_n) = \{\nu \in M(S) : |(\mu - \nu)f_i| < \varepsilon \text{ for } i = 1, \cdots, n\}$$

where $\varepsilon > 0$ and $f_1, \cdots, f_n \in C(S)$.

Let $M^+(S)$ denote the non-negative measures and let $P(S)$ denote the probability measures in $M(S)$.

Our goal is to establish open mapping theorems for some naturally induced mappings between sets of probability measures. Let $\varphi$ be a continuous map of $S$ onto $T$ where $S$ and $T$ are complete separable metric spaces. Define $\varphi^0 : M(S) \to M(T)$ by

$$\varphi^0 \mu(g) = \mu(g \circ \varphi) \text{ for each } g \in C(T).$$

A result of P. A. Meyer [9, p. 126] shows that $\varphi^0$ maps $P(S)$ onto $P(T)$. We show that $\varphi^0$ is open in the weak topology if and only if $\varphi$ is open.

Let $K$ be a totally disconnected compact metric space and let $S^K$
denote the collection of continuous maps of \( K \) into \( S \). Given \( f, g \in S^K \), set \( D(f, g) = \max \{ d(f(x), g(x)) : x \in K \} \) where \( d \) is the metric on \( S \). Then \( S^K \) is a complete separable metric space with respect to \( D \). Given \( f \in C(S) \) and \( x \in K \), we may define a mapping \( f_x : S^K \to \mathbb{R} \) by \( f_x(g) = f(g(x)) \) for each \( g \in S^K \). Now define a mapping \( \pi : P(S^K) \to P(S)^K \) by

\[
(\pi \mu)_x(f) = \mu(f_x) \text{ for each } f \in C(S).
\]

One easily checks that \( x \to (\pi \mu)_x \) is continuous in the weak topology and so one may consider the family \( (\pi \mu)_x \) as a continuous family of marginals associated with \( \mu \). Blumenthal and Corson [1] have shown that \( \pi \) maps \( P(S^K) \) onto \( P(S)^K \). We show that \( \pi \) is open in the weak topology.

2. The mapping \( \varphi^0 : P(S) \to P(T) \). Other than the interior mapping principle for F-spaces [6, p. 55] and its generalizations, there are few results in functional analysis on openness of mappings. For example, P. Cohen [4] has shown that if \( T : \ell_1 \times \ell_1 \to \ell_1 \) is a continuous bilinear mapping which is onto, then \( T \) need not be open at \((0, 0)\). If \( \Omega \) is a compact subset of a Banach space \( B \) and if the mapping \( (x, y) \to \frac{1}{2}(x, y) \) is open on \( \Omega \times \Omega \), then the set \( \text{ex} (\Omega) \) of extreme points of \( \Omega \) is closed. Our example below shows that the converse, which was left unresolved by Vesterstrom [10, p. 293], is false. However, convex averaging is open on \( P(S) \) and this plays a crucial role in our results.

Example 2.1. There exists a compact convex subset \( \Omega \) of \( \mathbb{R}^4 \) such that the extreme points of \( \Omega \) are closed and the midpoint mapping \( (x, y) \to \frac{1}{2}(x, y) \) is not open on \( \Omega \times \Omega \). Let \( \Omega \) be the convex hull of \((0, 1, 0, 0)\) and \((0, -1, 0, 0)\) and \((x, 0, 1, x^2)\) and \((x, 0, -1, x^2)\) for \( 0 \leq x \leq 1 \). The extreme points of \( \Omega \) are the two points and two arcs described above. But, the midpoint mapping is not open since \((0, 1, 0, 0) + (0, -1, 0, 0) = (0, 0, 0, 0)\) and \( u, v \in \Omega \) with \( \frac{1}{2}(u + v) = (x, 0, 0, x^2) \) where \( x \neq 0 \) implies \( u \) and \( v \) are of the form \((x, 0, \lambda, x^2)\) where \(-1 \leq \lambda \leq 1 \).

Let \( S \) be a complete separable metric space. We recall here some topological properties of \( P(S) \) and \( M^+ (S) \). Every measure \( \mu \) in \( P(S) \) is tight [8, p. 32], i.e., given \( \epsilon > 0 \), there is a compact subset \( F \) of \( S \) such that \( \mu (S \setminus F) < \epsilon \). The weak topology on \( M^+ (S) \) is topologically complete. Thus, we may consider \( M^+ (S) \) and \( P(S) \) as complete separable metric spaces. By embedding \( S \) is a countable product of unit intervals and using the fact that the unit ball in space of uniformly continuous functions on a totally bounded metric space is separable, we have the following result [8, p. 47].
Lemma 2.2. Let $S$ be a complete separable metric space. There exists continuous real-valued functions $g_1, g_2, \cdots$ on $S$ such that $\|g_n\|_\infty \leq 1$ for $n = 1, 2, \cdots$ and such that the metric $\rho$ defined on $M^+(S)$ by

$$\rho(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n} |(\mu - \nu)g_n|$$

is equivalent to the weak topology on $M^+(S)$.

We now show that convex averaging is open on $M^+(S)$. But, first we establish a result on selecting weakly convergent measures. We write $\mu_n \to \mu$ if $(\mu_n)_{n=1}^\infty$ converges to $\mu$ in the weak topology.

Proposition 2.3. Let $\mu_n, \mu \in M^+(S)$ where $\mu_n \to \mu$. Assume $0 \leq \nu \leq \mu$. Then there exists $0 \leq \nu_n \leq \mu_n$ for $n = 1, 2, \cdots$ such that $\nu_n \to \nu$.

Proof. Given $\epsilon > 0$, there exists $g$ continuous on $S$ such that $0 \leq g \leq 1$ and $\rho(g\mu, \nu) < \epsilon$. Hence, we may choose $f_n$ continuous on $S$ such that $0 \leq f_n \leq 1$ and $f_n\mu \to \nu$. But $f_n\mu_k \to f_n\mu$ as $k \to \infty$. So there exist $n_1 \leq n_2 \leq \cdots$ such that $n_k \to \infty$ and $\nu_k = f_n\mu_k \to \nu$.

Theorem 2.4. Let $S$ be a complete separable metric space. Let $0 < \lambda < 1$. The mapping $(\mu, \nu) \mapsto \lambda \mu + (1 - \lambda)\nu$ is open on $M^+(S) \times M^+(S)$ and is open on $P(S) \times P(S)$.

Proof. Fix $\mu, \nu \in M^+(S)$ and set $\omega = \lambda \mu + (1 - \lambda)\nu$. Assume $\omega_n \to \omega$ where $\omega_n \in M^+(S)$. Since $\lambda \mu \leq \omega$, there exist $\mu_n \in M^+(S)$ such that $\mu_n \to \lambda \mu$ and $0 \leq \mu_n \leq \omega_n$. Hence,

$$\frac{1}{\lambda} \mu_n \to \mu \quad \text{and} \quad \frac{1}{1-\lambda} (\omega_n - \mu_n) \to \nu.$$

Thus, the mapping $(\mu, \nu) \mapsto \lambda \mu + (1 - \lambda)\nu$ is an open map of $M^+(S) \times M^+(S)$ onto $M^+(S)$. One readily obtains that convex averaging is an open map of $P(S) \times P(S)$ onto $P(S)$.

Let $S$ and $T$ be complete separable metric spaces and let $\varphi: S \to T$ be continuous and onto. Then $\varphi$ induces a mapping $\varphi^0: M(S) \to M(T)$ defined by $\varphi^0(g)(\mu) = \mu(g \circ \varphi)$ for each $g \in C(T)$. As noted in §1, $\varphi^0$ maps $P(S)$ onto $P(T)$. We examine the openness of $\varphi^0$ on $P(S)$ with respect to the weak topology and the norm topology.

Theorem 2.5. Let $S$ and $T$ be complete separable metric spaces and...
let \( \varphi: S \to T \) be continuous and onto. Then \( \varphi \) is open if and only if \( \varphi^0: P(S) \to P(T) \) is open with respect to the weak topology.

**Proof.** Assume \( \varphi^0: P(S) \to P(T) \) is open in the weak topology. Fix \( s_0 \in S \) and set \( t_0 = \varphi(s_0) \). Assume \( \varphi \) is not open at \( s_0 \). Then there exist \( t_n \to t_0 \) and \( \epsilon > 0 \) such that \( d(s_0, \varphi^{-1}(t_n)) \geq \epsilon \) for \( n = 1, 2, \ldots \). Choose \( f \in C(S) \) such that \( f(s_0) = 1 \) and \( f = 0 \) on \( \{s \in S: d(s, s_0) \geq \epsilon\} \). Since \( \mathcal{U} = \{\mu \in P(S): |(\mu - \delta_{t_0})f| < \epsilon\} \) is a weak neighborhood of \( \delta_{t_0} \), there exist \( N \) and \( \mu_n \in \mathcal{U} \) such that \( \varphi^0\mu_n = \delta_{t_n} \) for \( n \geq N \). But \( \mu_n(f) = 0 \) since \( \varphi^{-1}(t_n) \) supports \( \mu_n \) and so \( \mu_n \not\in \mathcal{U} \), a contradiction.

Assume \( \varphi: S \to T \) is open. Fix \( \mu \in P(S) \). Let \( \epsilon > 0 \) and let \( f_1, \ldots, f_n: S \to [0, 1] \) be continuous. Set \( \mathcal{V} = \{\nu \in P(S): |(\mu - \nu)f_i| < \epsilon \text{ for } i = 1, \ldots, n\} \). We must show that \( \varphi^0\mathcal{V} \) is a neighborhood of \( \varphi^0\mu \) in \( P(T) \). Choose \( \mu_0, \mu_1, \ldots, \mu_m \in P(S) \) and \( \lambda_0, \lambda_1, \ldots, \lambda_m > 0 \) such that

1. \( \mu = \sum \lambda_i \mu_i \)
2. \( \lambda_0 < \epsilon \) and each of \( \mu_1, \ldots, \mu_m \) has compact support
3. the oscillation of \( f_i \) on the support of \( \mu_i \) is less than \( \epsilon/2 \) for each \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \).

Set \( \mathcal{V}_j = \{\nu \in P(S): |(\nu - \mu_j)f_i| < \epsilon \text{ for } i = 1, \ldots, n\} \). Clearly, we have \( \lambda_0 P(S) + \lambda_1 \mathcal{V}_1 + \cdots + \lambda_m \mathcal{V}_m \subseteq \mathcal{V} \). We claim that \( \varphi^0\mathcal{V}_j \) is a weak neighborhood of \( \varphi^0\mu_j \). For each \( j = 1, \ldots, m \) choose an open subset \( U_j \) of \( S \) containing the support of \( \mu_j \) such that the oscillations of \( f_1, \ldots, f_n \) on \( U_j \) are less than \( \epsilon/2 \). Then \( \mathcal{V}_j = \varphi(U_j) \) is an open subset of \( T \) containing the support of \( \nu_j = \varphi^0\mu_j \). It suffices to show that \( \nu \in \varphi^0(\mathcal{V}_j) \) if \( \nu(V_j) > 1 - \epsilon/2 \) and \( \nu \in P(T) \). Choose \( \beta_0 \in P(T) \) and \( \beta \in P(\mathcal{V}_j) \) such that

\[
\nu = \epsilon \beta_0 + \left(1 - \frac{\epsilon}{2}\right) \beta.
\]

Choose \( \alpha_0 \in P(S) \) and \( \alpha \in P(U_j) \) such that \( \varphi^0\alpha_0 = \beta_0 \) and \( \varphi^0\alpha = \beta \). We have

\[
\varphi^0 \left[ \frac{\epsilon}{2} \alpha_0 + \left(1 - \frac{\epsilon}{2}\right) \alpha \right] = \nu
\]

and for \( i = 1, \ldots, n \)

\[
\left| \left[ \mu_i - \frac{\epsilon}{2} \alpha_0 - \left(1 - \frac{\epsilon}{2}\right) \alpha \right] f_i \right| \leq \frac{\epsilon}{2} |(\mu_i - \alpha_0)f_i| + |(\mu_i - \alpha)f_i| < \epsilon.
\]

But \( \varphi^0\mathcal{V} \supset \lambda_0 P(T) + \lambda_1 \varphi^0\mathcal{V}_1 + \cdots + \lambda_m \varphi^0\mathcal{V}_m \) and so by Theorem 2.4, \( \varphi^0\mathcal{V} \) is a weak neighborhood of \( \varphi^0\mu \).
We next show that the mapping $\varphi^0$ is open in the norm topology.

**Theorem 2.6.** Let $S$ and $T$ be complete separable metric spaces and let $\varphi: S \to T$ be continuous and onto. Then $\varphi^0: M^+(S) \to M^+(T)$ is norm open and hence, $\varphi^0: P(S) \to P(T)$ is norm open.

**Proof.** Fix $\mu \in M^+(S)$ and set $v = \varphi^0 \mu$. Assume $v_n \to v$ in norm where $v_n \in M^+(T)$. Choose compact subsets $K_1 \subset K_2 \subset \cdots$ of $S$ such that $\mu(K_n) \to \mu(S)$. Set $\alpha_n = \mu | K_n$ and $\beta_n = \varphi^0 \alpha_n$. Then $\beta_n$ has compact support and $\beta_n \to v$. Also, $v_k \wedge \beta_n \to \beta_n$ as $k \to \infty$. Hence, there exist $1 = n_1 \leq n_2 \leq \cdots$ such that $n_k \to \infty$ and $v_k \wedge \beta_n \to v$. As shown in [5, Lemma 2.2], there exist $0 \leq \mu_k \leq \alpha_n$ satisfying $p^0 \mu_k = v_k \wedge \beta_n$. Then $\mu_k \to \mu$ in norm. Choose $\gamma_k \in M^+(S)$ such that $\varphi^0 \gamma_k = v_k - (v_k \wedge \beta_n)$. Then $\| \gamma_k \| \to 0$ and so $\mu_k + \gamma_k \to \mu$. Hence, $\varphi^0$ is open in the norm topology at $\mu$.

**Remark 2.7.** The proof of the openness of $\varphi^0$ in the weak topology seems to break into the two parts (1) $\varphi^0$ is open at the extreme points of $P(S)$ and (2) convex averaging is open on $P(T)$. There should be a general theorem on the openness of affine maps between convex subsets equipped with a metric which would yield Theorem 2.5.

**Conjecture.** Let $E$ and $F$ be Banach spaces and let $(E)_c$ and $(F)_c$ denote the closed unit ball in $E$ and $F$ respectively. Let $T: E \to F$ be continuous and linear. If $T$ maps $(E)$ onto $(F)$, and if $(E)_c$ is strictly convex, then $T$ is an open map of $(E)$ onto $(F)_c$.

**Note.** Example 2.1 resolves a conjecture of Clausing and Magerl in [3, p. 76]. S. M. Chang [2] has extended Theorem 2.4 to averaging of continuous collections of probability measures.

**3. The mapping $\pi: P(S^K) \to P(S)^K$.** Let $S$ be a complete separable metric space and let $K$ be a totally disconnected compact metric space. Let $S^K$ denote the collection of continuous maps of $K$ into $S$. We equip $S^K$ with the metric $D(f, g) = \max \{d(f(x), g(x)) : x \in K\}$ where $d$ is the metric on $S$. Thus $S^K$ is a complete separable metric space. The space $P(S)$ can be equipped with a metric which is equivalent to the weak topology and with respect to which $P(S)$ is complete and separable. Thus, the space $P(S^K)$ denotes the continuous maps of $K$ into $P(S)$ and $P(S)^K$ is equipped with the topology of uniform convergence in the weak topology. There is a natural mapping of $P(S^K)$ into $P(S)^K$. Let $\mu \in P(S^K)$ and $x \in K$. If $U$ is a Borel subset of $S$, then $\mu_x(U) = \mu(\{g \in S^K : g(x) \in U\})$ defines a probability measure $\mu_x$ on $S$. One recognizes the family $(\mu_x)_{x \in K}$ as a family of marginals...
associated with \( \mu \). The measure \( \mu_x \) may alternately be defined as follows. Given \( f \in C(S) \) and \( x \in \mathcal{K} \), define \( f_x : S^x \to \mathbb{R} \) by \( f_x(g) = f(g(x)) \). If \( \mu \in \mathcal{P}(S^x) \) and \( x \in \mathcal{K} \), then \( \mu_x(f) = \mu(f_x) \). This latter equation shows that the mapping \( x \to \mu_x \) is continuous in the weak topology. We set \( \pi \mu(x) = \mu_x \). Blumenthal and Corson [1] have shown that \( \pi \) maps \( \mathcal{P}(S^x) \) onto \( \mathcal{P}(S^x) \). Although there is no natural way of pulling back elements of \( \mathcal{P}(S^x) \) to \( \mathcal{P}(S^x) \), we shall prove that \( \pi \) is an open mapping. We begin by extending Prop. 2.3 to continuous collections of nonnegative measures.

**Lemma 3.1.** Let \( S \) be a complete separable metric space and let \( X \) be a compact Hausdorff space. Let \( 0 < \lambda < 1 \) and let \( \Phi, \Psi : X \to \mathcal{P}(S) \) be continuous. Assume \( \Phi_x \geq \lambda \Psi_x \) for each \( x \in X \). If \( \Phi_n : X \to \mathcal{P}(S) \) and \( \Phi_n \to \Phi \) uniformly in the weak topology, then there exist continuous maps \( \Psi_n : X \to \mathcal{P}(S) \) such that \( \Phi_n \geq \lambda \Psi_n \) for \( n = 1, 2, \cdots \) and \( \Psi_n \to \Psi \) uniformly in the weak topology.

**Proof.** By Lemma 2.2, we may choose continuous maps \( g_1, g_2, \cdots \) of \( S \) into \([0,1]\) such that the metric \( \rho \) on \( \mathcal{P}(S) \) defined by \( \rho(\mu, \nu) = \sum 2^{-n} |(\mu - \nu)g_n| \) is equivalent to the weak topology on \( \mathcal{P}(S) \). If \( f \in C(S) \) and \( \mu \in \mathcal{P}(S) \), then we define a nonnegative measure \( f \cdot \mu \) on \( S \) by \( (f \cdot \mu)g = \mu(fg) \) for each \( g \in C(S) \). For each \( p = 1, 2, \cdots \), choose a partition of unity \( g_1, \cdots, g_p \) for \( S \) such that each of \( g_1, \cdots, g_p \) has oscillation less than \( 1/p \) on the support of \( f_i \) for \( i = 1, \cdots, n_p \). Pick \( \varepsilon_p > 0 \) satisfying \( p\varepsilon_p n_p = 1 \). Given \( \Lambda : X \to \mathcal{P}(S) \), define \( \pi_p(\Lambda) : X \to M^+(S) \) by

\[
\pi_p(\Lambda)_x = \sum \frac{\Psi_x(f^p)}{\Phi_x(f^p + \varepsilon_p)} f^p \cdot \Lambda_x.
\]

Recall that \( f^p \cdot \Lambda_x(g) = \Lambda_x(f^p g) \) for each \( g \in C(S) \).

Setting \( f_i = f^p \) and \( \varepsilon = \varepsilon_p \), we have

\[
\pi_p(\Phi_m)_x(g_k) = \sum \frac{\Psi_x(f_i)}{\Phi_x(f_i + \varepsilon)} (\Phi_m)_x(f_i g_k)
\]

where \( x \in X \) and \( 1 \leq k \leq p \). Let \( \alpha^+_i(\beta^+_i) \) denote the minimum (maximum) of \( g_k \) over the support of \( f_i \). Then \( \beta^+_i - \alpha^+_i < 1/p \). Also,

\[
\sum \alpha^+_i \Psi_x(f_i) \leq \Psi_x(g_k) \leq \sum \beta^+_i \Psi_x(f_i).
\]

Choose \( M \) such that
For $m \geq M$ and $1 \leq k \leq p$, we have
\[
\pi_p(\Phi_m)_x(g_k) - \Psi_x(g_k)
\leq \sum \frac{\Psi_x(f_i)}{\Phi_x(f_i + \epsilon)} \beta^k(\Phi_m)_x(f_i) - \sum \alpha^k \Psi_x(f_i)
\leq \sum \left(\frac{1}{p} + \beta^k - \alpha^k\right) \Psi_x(f_i)
< \frac{2}{p}.
\]

On the other hand, for $m \geq M$ and $1 \leq k \leq p$, we have
\[
\pi_p(\Phi_m)_x(g_k) - \Psi_x(g_k)
\leq \sum \frac{\Psi_x(f_i)}{\Phi_x(f_i + \epsilon)} \alpha^k(\Phi_m)_x(f_i) - \sum \beta^k \Psi_x(f_i)
\leq \sum \frac{\Psi_x(f_i)}{\Phi_x(f_i + \epsilon)} \alpha^k(\Phi_m)_x(f_i + \epsilon) - \sum \beta^k \Psi_x(f_i) - \frac{1}{\lambda p}
\leq \sum \Psi_x(f_i) \alpha^k \left(1 - \frac{1}{p}\right) - \sum \beta^k \Psi_x(f_i) - \frac{1}{\lambda p}
\leq - \frac{2}{p} - \frac{1}{\lambda p} = - \frac{1}{p} \left(2 + \frac{1}{\lambda}\right).
\]

Hence, for $m \geq M$, \[\|\pi_p(\Phi_m) - \Psi\|_\infty \leq (2 + 1/\lambda)/p \text{ if } 1 \leq k \leq p.\] Thus, we may choose $m_1 < m_2 < \cdots$ such that \[\|\pi_p(\Phi_m) - \Psi\|_\infty \leq (2 + 1/\lambda)/p \text{ if } k \leq p \text{ and } m \geq m_p.\] Setting $\Psi_m = \pi_p(\Phi_m)$ if $m_p \leq m < m_{p+1}$ and $\Psi_m = \Phi_m$ if $m < m_1$, we have $\Psi_m \to \Psi$ uniformly in the weak topology and also, $\lambda \Psi_m \leq \Phi_m$. One may now modify the $\Psi_m$ so that $\Psi_m : X \to P(S)$ and at the same time preserve the uniform convergence to $\Psi$ and the inequality $\lambda \Psi_m \leq \Phi_m$.

We next show that convex averaging is open on $P(S)^x$.

**Lemma 3.2.** Let $X$ be a compact Hausdorff space and assume $0 < \lambda < 1$. Let $\Phi, \Psi : X \to P(S)$ be continuous. If $U$ and $V$ are neighborhoods of $\Phi$ and $\Psi$ in $P(S)^x$ respectively, then $\lambda U + (1 - \lambda) V$ is a neighborhood of $\lambda \Phi + (1 - \lambda) \Psi$.

**Proof.** Let $\Lambda_n \to \lambda \Phi + (1 - \lambda) \Psi$ where $\Lambda_n : X \to P(S)$ is continuous. Then there exist $\Phi_n : X \to P(S)$ such that $\Phi_n \to \Phi$ and $\lambda \Phi_n \leq
\[
\Lambda_n. \text{ Then } 1/(1 - \lambda)(\Lambda_n - \lambda \Phi_n) \to \Psi. \text{ Hence, } \lambda \mathcal{U} + (1 - \lambda)\mathcal{V} \text{ is a neighborhood of } \lambda \Phi + (1 - \lambda)\Psi.
\]

We are now prepared to show that the "marginal" mapping \(\pi\) of \(P(S^K)\) onto \(P(S)^K\) is an open map. In [5], this result was proved for the case \(S\) is compact and \(K\) is a two point space.

**Theorem 3.3.** Let \(S\) be a complete separable metric space and let \(K\) be a totally disconnected compact metric space. Then \(\pi: P(S^K) \to P(S)^K\) is open in the weak topology.

**Proof.** Let \(\mu \in P(S^K)\). Fix continuous maps \(G_1, \ldots, G_m\) of \(S^K\) into \([0, \infty)\). Set \(\mathcal{U} = \{\nu \in P(S^K): |(\nu - \mu)G_i| < 1 \text{ for } j = 1, \ldots, m\}\). We need to show that \(\pi \mathcal{U}\) is a neighborhood of \(\pi \mu\). There exist \(\mu_0, \mu_1, \ldots, \mu_n \in P(S^K), \lambda_0, \lambda_1, \ldots, \lambda_n > 0, \delta > 0\) and \(f_1, \ldots, f_n \in S^K\) such that \(\mu = \sum \lambda_i \mu_i\) and (1) the support of \(\mu_i\) is a compact subset of \(N_{\delta}(f_i) = \{f \in S^K: D(f, f_i) < \delta\}\) and (2) the oscillation of \(G_i\) is less than \(1/2\) over \(N_{\delta}(f_i)\) for each \(i = 1, \ldots, n\) and \(j = 1, \ldots, m\). Now set \(\mathcal{U}_i = \{\nu \in P(S^K): |(\nu - \mu_i)G_i| < 1 \text{ for } j = 1, \ldots, m\}\) for \(i = 1, \ldots, n\). Then \(\lambda_0 \mathcal{P}(S^K) + \lambda_1 \mathcal{U}_1 + \cdots + \lambda_n \mathcal{U}_n \subseteq \mathcal{U}\). By Lemma 3.2, it remains to verify that \(\pi \mathcal{U}\) is a neighborhood of \(\pi \mu\). Let \(M\) be an upper bound for \(G_1, \ldots, G_m\). Choose \(x_1, \ldots, x_p\) and compact subsets \(K_1, \ldots, K_p\) of \(K\) such that \(K\) is the disjoint union of \(K_1, \ldots, K_p\) and \(x \in K_i\) and \(K_i \subseteq N_{\delta}(x_i) = \{x: d(x, x_i) < \delta\}\) and such that \(f_i(K_i) \subseteq N_{\delta}(f_i(x_i))\) for each \(i = 1, \ldots, n\) and \(j = 1, \ldots, m\). Now the support of \(\pi \mu_i(x)\) is contained in \(N_{\delta}(f_i(x_i))\) whenever \(x \in K_i\). Choose \(0 < \lambda < 1\) such that \((1 - \lambda)M < 1/2\). Consider the set \(\mathcal{V}_i = \{\Phi \in P(S^K): \exists \Psi \in P(S)^K \text{ such that } \Phi \supseteq \lambda \Psi \text{ and the support of } \Psi \text{ is contained in } N_{\delta}(f_i(x_i))\}\) whenever \(x \in K_i\). Then \(\mathcal{V}_i\) is a neighborhood of \(\pi \mu_i\). We claim that \(\pi \mathcal{U} \supseteq \mathcal{V}_i\). Fix \(\Phi \in \mathcal{V}_i\) and choose \(\Psi \in P(S)^K\) such that \(\Phi \supseteq \lambda \Psi\) and the support of \(\Psi\) is contained in \(N_{\delta}(f_i(x_i))\) whenever \(x \in K_i\). Then \(\Psi|K_i\) is a continuous mapping of \(K_i\) into \(P(N_{\delta}(f_i(x_i)))\). By the result of Blumenthal and Corson [1], we can choose \(\nu_i \in P(N_{\delta}(f_i(x_i)))\) such that \(\pi \nu_i = \Psi|K_i\). Set \(\nu = \nu_1 \times \cdots \times \nu_p\). Then \(\nu\) is a probability measure on \(S^K\) and satisfies \(\pi \nu = \Psi\). Now choose \(\omega \in P(S^K)\) such that \(\pi \omega = (\Phi - \lambda \Psi)/\lambda\). Then \(\pi((\lambda \nu + (1 - \lambda) \omega) - \mu)G_i \approx \lambda \nu + (1 - \lambda) \omega \in \mathcal{U}\). Finally, we check that \(\lambda \nu + (1 - \lambda) \omega\) belongs to \(\mathcal{U}\). If \(1 \equiv j \equiv m\), then

\[
|((\lambda \nu + (1 - \lambda) \omega - \mu)G_i| \\
\leq \lambda |(\nu - \mu)G_i| + (1 - \lambda)|((\omega - \mu)G_i| \\
\leq \lambda/2 + (1 - \lambda)M < 1.
\]

Thus, \(\pi \mathcal{U}\) is a neighborhood of \(\pi \mu\).
4. Marginals for \( P(\Pi X_\lambda) \). Let \( X_\lambda \) be a compact Hausdorff space for each \( \lambda \in \Lambda \) and let \( \pi_\lambda \) denote the projection of \( \Pi X_\lambda \) onto \( X_\lambda \). If \( \mu \) is a probability measure on \( \Pi X_\lambda \), then the family of probability measures \( (\mu_\lambda)_{\lambda \in \Lambda} \), defined by \( \mu_\lambda(E) = \mu(\pi_\lambda^{-1}(E)) \) for each Borel subset \( E \) of \( X_\lambda \), is the family of marginals associated with \( \mu \). We next give an open mapping result for the mapping \( \mu \to (\mu_\lambda)_{\lambda \in \Lambda} \) with respect to the norm topology.

**Theorem 4.1.** Suppose \( X_\lambda \) is a compact Hausdorff space for each \( \lambda \in \Lambda \). Let \( \alpha \in P(\Pi X_\lambda) \) and let \( (\alpha_\lambda)_{\lambda \in \Lambda} \) be the family of marginals associated with \( \alpha \). Assume \( (\beta_\lambda)_{\lambda \in \Lambda} \) is a family of probability measures where \( \beta_\lambda \in P(X_\lambda) \). Then there exists \( \beta \in P(\Pi X_\lambda) \) such that \( (\beta_\lambda)_{\lambda \in \Lambda} \) is the family of marginals associated with \( \beta \) and \( \| \alpha - \beta \| \leq \Sigma \| \alpha_\lambda - \beta_\lambda \| \).

**Proof.** Let \( \alpha \in P(\Pi X_\lambda) \) and let \( (\alpha_\lambda)_{\lambda \in \Lambda} \) be the family of marginals associated with \( \alpha \). Fix \( (\beta_\lambda)_{\lambda \in \Lambda} \) in \( \Pi P(X_\lambda) \). Choose \( \lambda \in \Lambda \). Given a finite subset \( F = \{\lambda_1, \ldots, \lambda_n\} \) of \( \Lambda \), let \( \alpha_F \) denote the probability measure obtained from \( \alpha \) by the natural projection of \( \Pi X_\lambda \) onto \( \Pi_{\lambda \in F} X_\lambda \). The associated marginals of \( \alpha_F \) are \( \alpha_{\lambda_1}, \ldots, \alpha_{\lambda_n} \). By applying a result in [5, Thm. 2.2], there exists a probability measure \( \beta_F \) on \( \Pi X_\lambda \) with associated marginals \( \beta_{\lambda_1}, \ldots, \beta_{\lambda_n} \) satisfying \( \| \alpha_F - \beta_F \| \leq \Sigma \| \alpha_{\lambda_1} - \beta_{\lambda_1} \| \). Let \( \delta_F \) denote the point mass measure at \( (x_{\lambda_1})_{\lambda \in F} \) in \( \Pi_{\lambda \in \Lambda \setminus F} X_\lambda \). Then \( \delta_F \times \alpha_F \) and \( \delta_F \times \beta_F \) are probability measures on \( \Pi X_\lambda \). The net \( \delta_F \times \alpha_F \) converges to \( \alpha \) in the weak* topology. Let \( \beta \) be a weak* limit point of the net \( \delta_F \times \beta_F \) in \( P(\Pi X_\lambda) \). Then, \( \beta \) has associated marginals \( (\beta_\lambda)_{\lambda \in \Lambda} \). Also, \( \| \alpha - \beta \| \leq \sup_F \| \alpha_F - \beta_F \| \leq \Sigma \| \alpha_\lambda - \beta_\lambda \| \).

**References**


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UNIVERSITY OF MISSOURI AT KANSAS CITY