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Let S and T denote complete separable metric spaces. Let $P(S)$ denote the collection of probability measures on S and equip $P(S)$ with the weak topology. If $\varphi: S \to T$ is continuous and onto, then φ induces a weakly continuous mapping φ^0 of $P(S)$ onto $P(T)$. We show that φ^0 is open in the weak topology if and only if φ is open. However, φ^0 is always open in the norm topology. Let K be a totally disconnected compact metric space and let S^K denote the set of continuous mappings of K into S. Then there exists a natural mapping π of $P(S^K)$ into $P(S)^K$. Blumenthal and Corson have shown that π is onto. We establish that π is an open mapping in the weak topology.

1. Introduction. Let S be a complete separable metric space and let $C(S)$ denote the algebra of bounded continuous real-valued functions on S. Let $M(S)$ denote the collection of Borel measures on S which have finite total variation $\|\mu\|$. Given $f \in C(S)$ and $\mu \in M(s)$, set $\mu(f) = \int f(s)d\mu(s)$. The weak topology on $M(S)$ is the topology on $M(S)$ induced by $C(S)$. Thus, a neighborhood system at μ in $M(S)$ is given by sets of the form

$$N_\epsilon(\mu; f_1, \cdots, f_n) = \{\nu \in M(S): |(\mu - \nu)f_i| < \epsilon \text{ for } i = 1, \cdots, n\}$$
$$\text{where } \epsilon > 0 \text{ and } f_1, \cdots, f_n \in C(S).$$

Let $M^+(S)$ denote the non-negative measures and let $P(S)$ denote the probability measures in $M(S)$.

Our goal is to establish open mapping theorems for some naturally induced mappings between sets of probability measures. Let φ be a continuous map of S onto T where S and T are complete separable metric spaces. Define φ^0: $M(S) \to M(T)$ by

$$\varphi^0\mu(g) = \mu(g \circ \varphi) \text{ for each } g \in C(T).$$

A result of P. A. Meyer [**9**, p. 126] shows that φ^0 maps $P(S)$ onto $P(T)$. We show that φ^0 is open in the weak topology if and only if φ is open.

Let K be a totally disconnected compact metric space and let S^K

denote the collection of continuous maps of K into S. Given $f, g \in S^K$, set $D(f, g) = \max\{d(f(x), g(x)) : x \in K\}$ where d is the metric on S. Then S^K is a complete separable metric space with respect to D. Given $f \in C(S)$ and $x \in K$, we may define a mapping $f_x : S^K \to \mathbf{R}$ by $f_x(g) = f(g(x))$ for each $g \in S^K$. Now define a mapping π: $P(S^K) \to P(S)^K$ by

$$(\pi\mu)_x(f) = \mu(f_x) \text{ for each } f \in C(S).$$

One easily checks that $x \to (\pi\mu)_x$ is continuous in the weak topology and so one may consider the family $(\pi\mu)_x$ as a continuous family of marginals associated with μ. Blumenthal and Corson [**1**] have shown that π maps $P(S^K)$ *onto* $P(S)^K$. We show that π is open in the weak topology.

2. The mapping φ^0: $P(S) \to P(T)$. Other than the interior mapping principle for F-spaces [**6**, p. 55] and its generalizations, there are few results in functional analysis on openness of mappings. For example, P. Cohen [**4**] has shown that if $T: \ell_1 \times \ell_1 \to \ell_1$ is a continuous bilinear mapping which is onto, then T need *not* be open at $(0,0)$. If Ω is a compact subset of a Banach space B and if the mapping $(x, y) \to \frac{1}{2}(x, y)$ is open on $\Omega \times \Omega$, then the set $\mathrm{ex}(\Omega)$ of extreme points of Ω is closed. Our example below shows that the converse, which was left unresolved by Vesterstrom [**10**, p. 293], is false. However, convex averaging is open on $P(S)$ and this plays a crucial role in our results.

EXAMPLE 2.1. There exists a compact convex subset Ω of $\mathbf{R}^4$ such that the extreme points of Ω are closed and the midpoint mapping $(x, y) \to \frac{1}{2}(x, y)$ is not open on $\Omega \times \Omega$. Let Ω be the convex hull of $(0, 1, 0, 0)$ and $(0, -1, 0, 0)$ and $(x, 0, 1, x^2)$ and $(x, 0, -1, x^2)$ for $0 \leqq x \leqq 1$. The extreme points of Ω are the two points and two arcs described above. But, the midpoint mapping is not open since $(0, 1, 0, 0) + (0, -1, 0, 0) = (0, 0, 0, 0)$ and $u, v \in \Omega$ with $\frac{1}{2}(u + v) = (x, 0, 0, x^2)$ where $x \neq 0$ implies u and v are of the form $(x, 0, \lambda, x^2)$ where $-1 \leqq \lambda \leqq 1$.

Let S be a complete separable metric space. We recall here some topological properties of $P(S)$ and $M^+(S)$. Every measure μ in $P(S)$ is tight [**8**, p. 32], i.e., given $\epsilon > 0$, there is a compact subset F of S such that $\mu(S \backslash F) < \epsilon$. The weak topology on $M^+(S)$ is topologically complete. Thus, we may consider $M^+(S)$ and $P(S)$ as complete separable metric spaces. By embedding S is a countable product of unit intervals and using the fact that the unit ball in space of uniformly continuous functions on a totally bounded metric space is separable, we have the following result [**8**, p. 47].

LEMMA 2.2. *Let S be a complete separable metric space. There exist continuous real-valued functions $g_1, g_2, \cdots$ on S such that $\|g_n\|_\infty \leqq 1$ for $n = 1, 2, \cdots$ and such that the metric ρ defined on $M^+(S)$ by*

$$\rho(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n} |(\mu - \nu)g_n|$$

is equivalent to the weak topology on $M^+(S)$.

We now show that convex averaging is open on $M^+(S)$. But, first we establish a result on selecting weakly convergent measures. We write $\mu_n \to \mu$ if $(\mu_n)_{n=1}^{\infty}$ converges to μ in the weak topology.

PROPOSITION 2.3. *Let $\mu_n, \mu \in M^+(S)$ where $\mu_n \to \mu$. Assume $0 \leqq \nu \leqq \mu$. Then there exists $0 \leqq \nu_n \leqq \mu_n$ for $n = 1, 2, \cdots$ such that $\nu_n \to \nu$.*

Proof. Given $\epsilon > 0$, there exists g continuous on S such that $0 \leqq g \leqq 1$ and $\rho(g\mu, \nu) < \epsilon$. Hence, we may choose f_n continuous on S such that $0 \leqq f_n \leqq 1$ and $f_n\mu \to \nu$. But $f_n\mu_k \to f_n\mu$ as $k \to \infty$. So there exist $n_1 \leqq n_2 \leqq \cdots$ such that $n_k \to \infty$ and $\nu_k = f_{n_k}\mu_k \to \nu$.

THEOREM 2.4. *Let S be a complete separable metric space. Let $0 < \lambda < 1$. The mapping $(\mu, \nu) \to \lambda\mu + (1 - \lambda)\nu$ is open on $M^+(S) \times M^+(S)$ and is open on $P(S) \times P(S)$.*

Proof. Fix μ, $\nu \in M^+(S)$ and set $\omega = \lambda\mu + (1 - \lambda)\nu$. Assume $\omega_n \to \omega$ where $\omega_n \in M^+(S)$. Since $\lambda\mu \leqq \omega$, there exist $\mu_n \in M^+(S)$ such that $\mu_n \to \lambda\mu$ and $0 \leqq \mu_n \leqq \omega_n$. Hence,

$$\frac{1}{\lambda}\mu_n \to \mu \quad \text{and} \quad \frac{1}{1-\lambda}(\omega_n - \mu_n) \to \nu.$$

Thus, the mapping $(\mu, \nu) \to \lambda\mu + (1 - \lambda)\nu$ is an open map of $M^+(S) \times M^+(S)$ onto $M^+(S)$. One readily obtains that convex averaging is an open map of $P(S) \times P(S)$ onto $P(S)$.

Let S and T be complete separable metric spaces and let $\varphi: S \to T$ be continuous and onto. Then φ induces a mapping $\varphi^0: M(S) \to M(T)$ defined by $\varphi^0\mu(g) = \mu(g \circ \varphi)$ for each $g \in C(T)$. As noted in §1, φ^0 maps $P(S)$ onto $P(T)$. We examine the openness of φ^0 on $P(S)$ with respect to the weak topology and the norm topology.

THEOREM 2.5. *Let S and T be complete separable metric spaces and* *let $\varphi: S \to T$ be continuous and onto. Then φ is open if and only if φ^0: $P(S) \to P(T)$ is open with respect to the weak topology.*

Proof. Assume φ^0: $P(S) \to P(T)$ is open in the weak topology. Fix $s_0 \in S$ and set $t_0 = \varphi(s_0)$. Assume φ is not open at s_0. Then there exist $t_n \to t_0$ and $\epsilon > 0$ such that $d(s_0, \varphi^{-1}(t_n)) \geqq \epsilon$ for $n = 1, 2, \cdots$. Choose $f \in C(S)$ such that $f(s_0) = 1$ and $f = 0$ on $\{s \in S: d(s, s_0) \geqq \epsilon\}$. Since $\mathcal{U} = \{\mu \in P(S): |(\mu - \delta_{s_0})f| < \epsilon\}$ is a weak neighborhood of δ_{s_0}, there exist N and $\mu_n \in \mathcal{U}$ such that $\varphi^0\mu_n = \delta_{t_n}$ for $n \geqq N$. But $\mu_n(f) = 0$ since $\varphi^{-1}(t_n)$ supports μ_n and so $\mu_n \notin \mathcal{U}$, a contradiction.

Assume $\varphi: S \to T$ is open. Fix $\mu \in P(S)$. Let $\epsilon > 0$ and let $f_1, \cdots, f_n$: $S \to [0, 1]$ be continuous. Set $\mathcal{V} = \{\nu \in P(S): |(\mu - \nu)f_i| < \epsilon$ for $i = 1, \cdots, n\}$. We must show that $\varphi^0\mathcal{V}$ is a neighborhood of $\varphi^0\mu$ in $P(T)$. Choose $\mu_0, \mu_1, \cdots, \mu_m \in P(S)$ and $\lambda_0, \lambda_1, \cdots, \lambda_m > 0$ such that

(1) $\mu = \Sigma\, \lambda_j\mu_j$

(2) $\lambda_0 < \epsilon$ and each of $\mu_1, \cdots, \mu_m$ has compact support

(3) the oscillation of f_i on the support of μ_j is less than $\epsilon/2$ for each $i = 1, \cdots, n$ and $j = 1, \cdots, m$.

Set $\mathcal{V}_j = \{\nu \in P(S): |(\nu - \mu_j)f_i| < \epsilon$ for $i = 1, \cdots, n\}$. Clearly, we have $\lambda_0 P(S) + \lambda_1\mathcal{V}_1 + \cdots + \lambda_m\mathcal{V}_m \subseteq \mathcal{V}$. We claim that $\varphi^0\mathcal{V}_j$ is a weak neighborhood of $\varphi^0\mu_j$. For each $j = 1, \cdots, m$ choose an open subset U_j of S containing the support of μ_j such that the oscillations of $f_1, \cdots, f_n$ on U_j are less than $\epsilon/2$. Then $V_j = \varphi(U_j)$ is an open subset of T containing the support of $\nu_j = \varphi^0\mu_j$. It suffices to show that $\nu \in \varphi^0(\mathcal{V}_j)$ if $\nu(V_j) > 1 - \epsilon/2$ and $\nu \in P(T)$. Choose $\beta_0 \in P(T)$ and $\beta \in P(V_j)$ such that

$$\nu = \frac{\epsilon}{2}\,\beta_0 + \left(1 - \frac{\epsilon}{2}\right)\beta.$$

Choose $\alpha_0 \in P(S)$ and $\alpha \in P(U_j)$ such that $\varphi^0\alpha_0 = \beta_0$ and $\varphi^0\alpha = \beta$. We have

$$\varphi^0\left[\frac{\epsilon}{2}\,\alpha_0 + \left(1 - \frac{\epsilon}{2}\right)\alpha\right] = \nu$$

and for $i = 1, \cdots, n$

$$\left|\left[\mu_j - \frac{\epsilon}{2}\,\alpha_0 - \left(1 - \frac{\epsilon}{2}\right)\alpha\right] f_i\right| \leqq \frac{\epsilon}{2}\,|(\mu_j - \alpha_0)f_i| + |(\mu_j - \alpha)f_i| < \epsilon.$$

But $\varphi^0\mathcal{V} \supset \lambda_0 P(T) + \lambda_1\varphi^0\mathcal{V}_1 + \cdots + \lambda_m\varphi^0\mathcal{V}_m$ and so by Theorem 2.4, $\varphi^0\mathcal{V}$ is a weak neighborhood of $\varphi^0\mu$.

We next show that the mapping φ^0 is open in the norm topology.

THEOREM 2.6. *Let S and T be complete separable metric spaces and let $\varphi: S \to T$ be continuous and onto. Then φ^0: $M^+(S) \to M^+(T)$ is norm open and hence, φ^0: $P(S) \to P(T)$ is norm open.*

Proof. Fix $\mu \in M^+(S)$ and set $\nu = \varphi^0\mu$. Assume $\nu_n \to \nu$ in norm where $\nu_n \in M^+(T)$. Choose compact subsets $K_1 \subset K_2 \subset \cdots$ of S such that $\mu(K_n) \to \mu(S)$. Set $\alpha_n = \mu \mid K_n$ and $\beta_n = \varphi^0\alpha_n$. Then β_n has compact support and $\beta_n \to \nu$. Also, $\nu_k \wedge \beta_n \to \beta_n$ as $k \to \infty$. Hence, there exist $1 = n_1 \leqq n_2 \leqq \cdots$ such that $n_k \to \infty$ and $\nu_k \wedge \beta_{n_k} \to \nu$. As shown in [**5**, Lemma 2.2], there exist $0 \leqq \mu_k \leqq \alpha_{n_k}$ satisfying $\rho^0\mu_k = \nu_k \wedge \beta_{n_k}$. Then $\mu_k \to \mu$ in norm. Choose $\gamma_k \in M^+(S)$ such that $\varphi^0\gamma_k = \nu_k - (\nu_k \wedge \beta_{n_k})$. Then $\|\gamma_k\| \to 0$ and so $\mu_k + \gamma_k \to \mu$. Hence, φ^0 is open in the norm topology at μ.

REMARK 2.7. The proof of the openness of φ^0 in the weak topology seems to break into the two parts (1) φ^0 is open at the extreme points of $P(S)$ and (2) convex averaging is open on $P(T)$. There should be a general theorem on the openness of affine maps between convex subsets equipped with a metric which would yield Theorem 2.5.

CONJECTURE. Let E and F be Banach spaces and let $(E)_1$ and $(F)_1$ denote the closed unit ball in E and F respectively. Let T: $E \to F$ be continuous and linear. If T maps $(E)_1$ onto $(F)_1$ and if $(E)_1$ is strictly convex, then T is an open map of $(E)_1$ onto $(F)_1$.

Note. Example 2.1 resolves a conjecture of Clausing and Magerl in [**3**, p. 76]. S. M. Chang [**2**] has extended Theorem 2.4 to averaging of continuous collections of probability measures.

3. The mapping π: $P(S^K) \to P(S)^K$. Let S be a complete separable metric space and let K be a totally disconnected compact metric space. Let S^K denote the collection of continuous maps of K into S. We equip S^K with the metric $D(f, g) = \max\{d(f(x), g(x)):$ $x \in K\}$ where d is the metric on S. Thus S^K is a complete separable metric space. The space $P(S)$ can be equipped with a metric which is equivalent to the weak topology and with respect to which $P(S)$ is complete and separable. Thus, the space $P(S)^K$ denotes the continuous maps of K into $P(S)$ and $P(S)^K$ is equipped with the topology of uniform convergence in the weak topology. There is a natural mapping of $P(S^K)$ into $P(S)^K$. Let $\mu \in P(S^K)$ and $x \in K$. If U is a Borel subset of S, then $\mu_x(U) = \mu(\{g \in S^K$: $g(x) \in U\})$ defines a probability measure μ_x on S. One recognizes the family $(\mu_x)_{x \in K}$ as a family of marginals associated with μ. The measure μ_x may alternately be defined as follows. Given $f \in C(S)$ and $x \in K$, define f_x: $S^K \to \mathbf{R}$ by $f_x(g) = f(g(x))$. If $\mu \in P(S^K)$ and $x \in K$, then $\mu_x(f) = \mu(f_x)$. This latter equation shows that the mapping $x \to \mu_x$ is continuous in the weak topology. We set $\pi\mu(x) = \mu_x$. Blumenthal and Corson **[1]** have shown that π maps $P(S^K)$ onto $P(S)^K$. Although there is no natural way of pulling back elements of $P(S)^K$ to $P(S^K)$, we shall prove that π is an open mapping. We begin by extending Prop. 2.3 to continuous collections of nonnegative measures.

LEMMA 3.1. *Let S be a complete separable metric space and let X be a compact Hausdorff space. Let $0 < \lambda < 1$ and let Φ, Ψ: $X \to P(S)$ be continuous. Assume $\Phi_x \geqq \lambda\Psi_x$ for each $x \in X$. If Φ_n: $X \to P(S)$ and $\Phi_n \to \Phi$ uniformly in the weak topology, then there exist continuous maps Ψ_n: $X \to P(S)$ such that $\Phi_n \geqq \lambda\Psi_n$ for $n = 1, 2, \cdots$ and $\Psi_n \to \Psi$ uniformly in the weak topology.*

Proof. By Lemma 2.2, we may choose continuous maps $g_1, g_2, \cdots$ of S into $[0,1]$ such that the metric ρ on $P(S)$ defined by $\rho(\mu, \nu) = \Sigma 2^{-n} |(\mu - \nu)g_n|$ is equivalent to the weak topology on $P(S)$. If $f \in C^+(S)$ and if $\mu \in P(S)$, then we define a nonnegative measure $f \cdot \mu$ on S by $(f \cdot \mu)g = \mu(fg)$ for each $g \in C(S)$. For each $p = 1, 2, \cdots$, choose a partition of unity $f_1^p, \cdots, f_{n_p}^p$ for S such that each of $g_1, \cdots, g_p$ has oscillation less than $1/p$ on the support of f_i^p for $i = 1, \cdots, n_p$. Pick $\epsilon_p > 0$ satisfying $p\epsilon_p n_p = 1$. Given Λ: $X \to P(S)$, define $\pi_p(\Lambda)$: $X \to M^+(S)$ by

$$\pi_p(\Lambda)_x = \sum \frac{\Psi_x(f_i^p)}{\Phi_x(f_i^p + \epsilon_p)} f_i^p \cdot \Lambda_x.$$

Recall that $f_i^p \cdot \Lambda_x(g) = \Lambda_x(f_i^p g)$ for each $g \in C(S)$.

Setting $f_i = f_i^p$ and $\epsilon = \epsilon_p$, we have

$$\pi_p(\Phi_m)_x\,(g_k) = \sum \frac{\Psi_x(f_i)}{\Phi_x(f_i + \epsilon)} (\Phi_m)_x\,(f_i g_k)$$

where $x \in X$ and $1 \leqq k \leqq p$. Let $\alpha_i^k(\beta_i^k)$ denote the minimum (maximum) of g_k over the support of f_i. Then $\beta_i^k - \alpha_i^k < 1/p$. Also,

$$\sum \alpha_i^k \Psi_x(f_i) \leqq \Psi_x(g_k) \leqq \sum \beta_i^k \Psi_x(f_i).$$

Choose M such that

$$1-\frac{1}{p}<\frac{(\Phi_m)_x\,(f_i+\epsilon)}{\Phi_x(f_i+\epsilon)}<1+\frac{1}{p}\quad\text{for}\quad m\geqq M.$$

For $m \geqq M$ and $1 \leqq k \leqq p$, we have

$$\begin{aligned}
&\pi_p(\Phi_m)_x\,(g_k)-\Psi_x(g_k)\\
&\quad\leqq \sum \frac{\Psi_x(f_i)}{\Phi_x(f_i+\epsilon)}\,\beta_i^k(\Phi_m)_x\,(f_i)-\sum \alpha_i^k\Psi_x(f_i)\\
&\quad\leqq \sum \left(\frac{1}{p}+\beta_i^k-\alpha_i^k\right)\Psi_x(f_i)\\
&\quad<\frac{2}{p}.
\end{aligned}$$

On the other hand, for $m \geqq M$ and $1 \leqq k \leqq p$, we have

$$\begin{aligned}
&\pi_p(\Phi_m)_x\,(g_k)-\Psi_x(g_k)\\
&\quad\geqq \sum \frac{\Psi_x(f_i)}{\Phi_x(f_i+\epsilon)}\,\alpha_i^k(\Phi_m)_x\,(f_i)-\sum \beta_i^k\Psi_x(f_i)\\
&\quad\geqq \sum \frac{\Psi_x(f_i)}{\Phi_x(f_i+\epsilon)}\,\alpha_i^k(\Phi_m)_x\,(f_i+\epsilon)-\sum \beta_i^k\Psi_x(f_i)-\frac{1}{\lambda p}\\
&\quad\geqq \sum \Psi_x(f_i)\alpha_i^k\left(1-\frac{1}{p}\right)-\sum \beta_i^k\Psi_x(f_i)-\frac{1}{\lambda p}\\
&\quad\geqq -\frac{2}{p}-\frac{1}{\lambda p}=-\frac{1}{p}\left(2+\frac{1}{\lambda}\right).
\end{aligned}$$

Hence, for $m \geqq M$, $\|[\pi_p(\Phi_m)-\Psi](g_k)\|_X \leqq (2+1/\lambda)/p$ if $1 \leqq k \leqq p$. Thus, we may choose $m_1 < m_2 < \cdots$ such that $\|[\pi_p(\Phi_m)-\Psi](g_k)\| \leqq (2+1/\lambda)/p$ if $k \leqq p$ and $m \geqq m_p$. Setting $\Psi_m = \pi_p(\Phi_m)$ if $m_p \leqq m < m_{p+1}$ and $\Psi_m = \Phi_m$ if $m < m_1$, we have $\Psi_m \to \Psi$ uniformly in the weak topology and also, $\lambda\Psi_m \leqq \Phi_m$. One may now modify the Ψ_m so that $\Psi_m : X \to P(S)$ and at the same time preserve the uniform convergence to Ψ and the inequality $\lambda\Psi_m \leqq \Phi_m$.

We next show that convex averaging is open on $P(S)^X$.

LEMMA 3.2. *Let X be a compact Hausdorff space and assume $0 < \lambda < 1$. Let Φ, Ψ: $X \to P(S)$ be continuous. If $\mathcal{U}$ and $\mathcal{V}$ are neighborhoods of Φ and Ψ in $P(S)^X$ respectively, then $\lambda\mathcal{U}+(1-\lambda)\mathcal{V}$ is a neighborhood of $\lambda\Phi+(1-\lambda)\Psi$.*

Proof. Let $\Lambda_n \to \lambda\Phi+(1-\lambda)\Psi$ where Λ_n: $X \to P(S)$ is continuous. Then there exist Φ_n: $X \to P(S)$ such that $\Phi_n \to \Phi$ and $\lambda\Phi_n \leqq$

Λ_n. Then $1/(1-\lambda)(\Lambda_n - \lambda\Phi_n) \to \Psi$. Hence, $\lambda\mathcal{U} + (1-\lambda)\mathcal{V}$ is a neighborhood of $\lambda\Phi + (1-\lambda)\Psi$.

We are now prepared to show that the "marginal" mapping π of $P(S^K)$ onto $P(S)^K$ is an open map. In **[5]**, this result was proved for the case S is compact and K is a two point space.

THEOREM 3.3. *Let S be a complete separable metric space and let K be a totally disconnected compact metric space. Then $\pi: P(S^K) \to P(S)^K$ is open in the weak topology.*

Proof. Let $\mu \in P(S^K)$. Fix continuous maps $G_1, \cdots, G_m$ of S^K into $[0,\infty)$. Set $\mathcal{U} = \{\nu \in P(S^k): |(\nu - \mu)G_j| < 1 \text{ for } j = 1, \cdots, m\}$. We need to show that $\pi\mathcal{U}$ is a neighborhood of $\pi\mu$. There exist $\mu_0, \mu_1, \cdots, \mu_n \in P(S^K)$, $\lambda_0, \lambda_1, \cdots, \lambda_n > 0$, $\delta > 0$ and $f_1, \cdots, f_n \in S^K$ such that $\mu = \Sigma\, \lambda_i\mu_i$ and (1) the support of μ_i is a compact subset of $N_\delta(f_i) = \{f \in S^K: D(f, f_i) < \delta\}$ and (2) the oscillation of G_j is less than 1/2 over $N_{2\delta}(f_i)$ for each $i = 1, \cdots, n$ and $j = 1, \cdots, m$. Now set $\mathcal{U}_i = \{\nu \in P(S^K): |(\nu - \mu_i)G_j| < 1 \text{ for } j = 1, \cdots, m\}$ for $i = 1, \cdots, n$. Then $\lambda_0 P(S^K) + \lambda_1\mathcal{U}_1 + \cdots + \lambda_n\mathcal{U}_n \subseteq \mathcal{U}$. By Lemma 3.2, it remains to verify that $\pi\mathcal{U}_i$ is a neighborhood of $\pi\mu_i$. Let M be an upper bound for $G_1, \cdots, G_m$. Choose $x_1, \cdots, x_p$ and compact subsets $K_1, \cdots, K_p$ of K such that K is the disjoint union of $K_1, \cdots, K_p$ and $x_j \in K_j$ and $K_j \subseteq N_\delta(x_j) = \{x: d(x, x_j) < \delta\}$ and such that $f_i(K_j) \subseteq N_\delta(f_i(x_j))$ for each $i = 1, \cdots, n$ and $j = 1, \cdots, p$. Now the support of $\pi\mu_i(x)$ is contained in $N_{2\delta}(f_i(x_j))$ when $x \in K_j$. Choose $0 < \lambda < 1$ such that $(1-\lambda)M < 1/2$. Consider the set $\mathcal{V}_i = \{\Phi \in P(S)^K: \exists \Psi \in P(S)^K \text{ such that } \Phi \geqq \lambda\Psi$ and the support of Ψ_x is contained in $N_\delta(f_i(x_j))$ whenever $x \in K_j\}$. Then $\mathcal{V}_i$ is a neighborhood of $\pi\mu_i$. We claim that $\pi\mathcal{U}_i \supset \mathcal{V}_i$. Fix $\Phi \in \mathcal{V}_i$ and choose $\Psi \in P(S)^K$ such that $\Phi \geqq \lambda\Psi$ and the support of Ψ_x is contained in $N_\delta(f_i(x_j))$ whenever $x \in K_j$. Then $\Psi | K_j$ is a continuous mapping of K_j into $P(N_\delta(f_i(x_j)))$. By the result of Blumenthal and Corson **[1]**, we can choose $\nu_j \in P(N_\delta(f_i(x_j))^{K_j})$ such that $\pi\nu_j = \Psi | K_j$. Set $\nu = \nu_1 \times \cdots \times \nu_p$. Then ν is a probability measure on S^K and satisfies $\pi\nu = \Psi$. Now choose $\omega \in P(S^K)$ such that $\pi\omega = (\Phi - \lambda\Psi)/\lambda$. Then $\pi[\lambda\nu + (1-\lambda)\omega] = \Phi$. Finally, we check that $\lambda\nu + (1-\lambda)\omega$ belongs to $\mathcal{U}_i$. If $1 \leqq j \leqq m$, then

$$\begin{aligned} &|(\lambda\nu + (1-\lambda)\omega - \mu_i)G_j| \\ &\quad \leqq \lambda\, |(\nu - \mu_i)G_j| + (1-\lambda)|(\omega - \mu_i)G_j| \\ &\quad \leqq \lambda/2 + (1-\lambda)M < 1. \end{aligned}$$

Thus, $\pi\mathcal{U}_i$ is a neighborhood of $\pi\mu_i$.

4. Marginals for $P(\Pi X_\lambda)$. Let X_λ be a compact Hausdorff space for each $\lambda \in \Lambda$ and let π_λ denote the projection of ΠX_λ onto X_λ. If μ is a probability measure on ΠX_λ, then the family of probability measures $(\mu_\lambda)_{\lambda\in\Lambda}$, defined by $\mu_\lambda(E) = \mu(\pi_\lambda^{-1}(E))$ for each Borel subset E of X_λ, is the family of marginals associated with μ. We next give an open mapping result for the mapping $\mu \to (\mu_\lambda)_{\lambda\in\Lambda}$ with respect to the norm topology.

THEOREM 4.1. *Suppose X_λ is a compact Hausdorff space for each $\lambda \in \Lambda$. Let $\alpha \in P(\Pi X_\lambda)$ and let $(\alpha_\lambda)_{\lambda\in\Lambda}$ be the family of marginals associated with α. Assume $(\beta_\lambda)_{\lambda\in\Lambda}$ is a family of probability measures where $\beta_\lambda \in P(X_\lambda)$. Then there exists $\beta \in P(\Pi X_\lambda)$ such that $(\beta_\lambda)_{\lambda\in\Lambda}$ is the family of marginals associated with β and $\|\alpha - \beta\| \leqq \Sigma\|\alpha_\lambda - \beta_\lambda\|$.*

Proof. Let $\alpha \in P(\Pi X_\lambda)$ and let $(\alpha_\lambda)_{\lambda\in\Lambda}$ be the family of marginals associated with α. Fix $(\beta_\lambda)_{\lambda\in\Lambda}$ in $\Pi P(X_\lambda)$. Choose $x_\lambda \in X_\lambda$ for each $\lambda \in \Lambda$. Given a finite subset $F = \{\lambda_1, \cdots, \lambda_n\}$ of Λ, let α_F denote the probability measure obtained from α by the natural projection of ΠX_λ onto $\Pi_{i=1}^n X_{\lambda_i}$. The associated marginals of α_F are $\alpha_{\lambda_1}, \cdots, \alpha_{\lambda_n}$. By applying a result in [**5**, Thm. 2.2], there exists a probability measure β_F on ΠX_{λ_i} with associated marginals $\beta_{\lambda_1}, \cdots, \beta_{\lambda_n}$ satisfying $\|\alpha_F - \beta_F\| \leqq \Sigma\|\alpha_{\lambda_i} - \beta_{\lambda_i}\|$. Let δ_F denote the point mass measure at $(x_\lambda)_{\lambda\in\Lambda\setminus F}$ in $\Pi_{\lambda\in\Lambda\setminus F} X_\lambda$. Then $\delta_F \times \alpha_F$ and $\delta_F \times \beta_F$ are probability measures on ΠX_λ. The net $\delta_F \times \alpha_F$ converges to α in the weak* topology. Let β be a weak* limit point of the net $\delta_F \times \beta_F$ in $P(\Pi X_\lambda)$. Then, β has associated marginals $(\beta_\lambda)_{\lambda\in\Lambda}$. Also, $\|\alpha - \beta\| \leqq \sup_F\|\alpha_F - \beta_F\| \leqq \Sigma\|\alpha_\lambda - \beta_\lambda\|$.

REFERENCES

1. R. M. Blumenthal and H. H. Corson, *On continuous collections of measures*, Ann. Inst. Fourier, Grenoble, **20** (1970), 193–199.
2. S. M. Chang, *On continuous image averaging of probability measures*, (submitted to Pacific J. Math.).
3. A. Clausing and G. Magerl, *Generalized Dirichlet problems and continuous selections of representing measures*, Math. Ann., **216** (1975), 71–78.
4. P. Cohen, *A counter example to the closed graph theorem for bilinear maps*, J. Functional Analysis, **16** (1974), 235–240.
5. S. Z. Ditor and L. Q. Eifler, *Some open mapping theorems for measures*, Trans. Amer. Math. Soc., **164** (1972), 287–293.
6. N. Dunford and J. T. Schwartz, *Linear operators. Part I*, Interscience, 1958.
7. L. Q. Eifler, *Some open mapping theorems for marginals*, Trans. Amer. Math. Soc., **211** (1975).
8. K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, 1967.
9. L. Schwartz, *Radon Measures*, Oxford University Press, 1973.
10. J. Vesterstrom, *On open maps, compact convex sets, and operator algebras*, J. London Math. Soc., (2), **6** (1973), 289–297.

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