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RONALD FINTUSHEL

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A classification is given for S^1 bundles with structure group $O(2)$ and base space a 2-manifold with nonempty boundary. This result is used to obtain equivariant and topological classification theorems for closed 3-manifolds which admit a local S^1 action; i.e., a decomposition into circles and points such that each decomposition element has an invariant neighborhood admitting an effective circle action with the elements of the decomposition as orbits. This extends certain results of Orlik and Raymond and corrects a theorem of Orlik, Vogt, and Zieschang.

A local S^1 action on a topological space X is a decomposition of X into disjoint points and simple closed curves such that each decomposition element has an invariant neighborhood admitting an effective circle action with the elements of the decomposition as orbits. A 3-manifold with local S^1 action can be described in terms of an S^1 bundle over a 2-manifold with boundary and various building blocks which are attached to this bundle. These building blocks are solid tori, solid Klein bottles, or Möbius band bundles over the circle.

In **[4]** Orlik and Raymond defined a class of 3-manifolds which included the 3-manifolds admitting an S^1 action (**[3]**, **[6]**) and the Seifert manifolds (**[2]**, **[5]**, **[7]**). The manifolds described in **[4]** are those 3-manifolds with local S^1 action which arise from S^1 bundles having no Klein bottle boundary component, and a complete equivariant and topological classification is given in **[4]** for these manifolds. We extend these theorems to the complete class of 3-manifolds with local S^1 action. As a result of the added generality we obtain in Theorem 4 a new class of $K(\pi, 1)$ 3-manifolds which admit unique local S^1 actions. The referee has pointed out that these manifolds are actually special cases of 3-dimensional "Seifert Fiberings", studied by Conner and Raymond **[1]**, where the properly discontinuous group of transformations of the plane, N, contains reflections.

In order to extend the results of **[4]** it is necessary to classify up to weak bundle equivalence S^1 bundles with structure group $O(2)$ and base space a compact 2-manifold with boundary. This is done in §1. In view of the results of this section, the fiber-preserving homeomorphism classification of Seifert manifolds given by Orlik, Vogt, and Zieschang **[5]** must be corrected when considering Seifert manifolds with Klein bottle boundary components ($l \neq 0$ in **[5]**).

We have retained most of the notation of [4], and we refer the reader to that paper for explanation of the terminology and notation not explicitly defined here.

1. Circle bundles over compact 2-manifolds with boundary. Let B be a compact connected 2-manifold with nonempty boundary. The usual equivalence classes of S^1 bundles over B with structure group $\mathbf{O}(2)$ are in $1-1$ correspondence with $\mathrm{Hom}(\pi_1(B), C_2)$, C_2 being the multiplicative group of order 2. If $p\colon E\to B$ is an S^1 bundle, the corresponding homomorphism ω sends an element of $\pi_1(B)$ to $+1$ or -1 according as the orientation on a fiber is preserved or reversed when a representative loop in B is traversed.

We wish to classify S^1 bundles over B up to weak bundle equivalence, meaning that the equivalence is allowed to induce a nontrivial homeomorphism of B. An automorphism of $\pi_1(B)$ is called geometric if it is induced by a homeomorphism of B. The following lemma is easily verified.

LEMMA 1. *The S^1 bundles $p_1\colon E_1\to B$ and $p_2\colon E_2\to B$ are weakly equivalent if and only if there is a geometric automorphism α of $\pi_1(B)$ such that $\omega_{p_2}\circ\alpha = \omega_{p_1}$, where ω_{p_i} is the homomorphism $\pi_1(B)\to C_2$ corresponding to $p_i\colon E_i\to B$.*

Thus it suffices to classify homomorphisms $\pi_1(B)\to C_2$ up to the equivalence given by $\omega_1\sim\omega_2$ if there is a geometric automorphism α of $\pi_1(B)$ such that $\omega_2\circ\alpha=\omega_1$. If B has genus g and m boundary components $\pi_1(B)=(a_j, b_j, s_i \mid s_1\cdots s_m[a_1,b_1]\cdots[a_g,b_g])$ if B is orientable, and $\pi_1(B)=(v_j, s_i \mid s_1\cdots s_m v_1^2\cdots v_g^2)$ if B is nonorientable. Seifert's classification [7] for the case $m=1$ is as follows where the o_i correspond to orientable B and the n_i correspond to nonorientable B:

o_1: represented by $\omega(a_j)=\omega(b_j)=+1$ for all j

o_2: represented by $\omega(a_j)=\omega(b_j)=-1$ for all j $(g\geqq 1)$

n_1: represented by $\omega(v_j)=+1$ for all j

n_2: represented by $\omega(v_j)=-1$ for all j

n_3: represented by $\omega(v_1)=+1$, $\omega(v_j)=-1$ for $j>1$ $(g\geqq 2)$

n_4: represented by $\omega(v_1)=\omega(v_2)=+1$, $\omega(v_j)=-1$ for $j>2$ $(g\geqq 3)$.

The classification is carried out for the general case by Orlik, Vogt, and Zieschang in [5] where it is claimed that Seifert's classification holds for $m>1$. The argument given in [5] shows this to be true if $\omega(s_i)=+1$

for all i. However, in case some $\omega(s_i) = -1$, we show that o_1 and o_2 collapse to a single class o, and n_1, n_2, n_3, n_4 collapse to a single class n.

THEOREM 1. *Let B be a compact connected 2-manifold with nonempty boundary. The weak equivalence classes of S^1 bundles over B with structure group* $\mathbf{O}(2)$ *are in* $1-1$ *correspondence with the pairs* (ϵ, k) *where the even integer k is the number of s_i with $\omega(s_i) = -1$, and if B is orientable $\epsilon = o_1$ or o_2 if $k = 0$ and $\epsilon = o$ if $k \neq 0$; if B is nonorientable $\epsilon = n_1, n_2, n_3,$ or n_4 if $k = 0$ and $\epsilon = n$ if $k \neq 0$.*

We need the following result of Zieschang **[8]** to verify that the automorphisms which we define are geometric:

An automorphism α of $\pi_1(B)$ is geometric if and only if $\alpha(s_i) = l_i s_{r_i}^{\epsilon_i} l_i^{-1}$, $i = 1, \cdots, m$, with $(i \to r_i)$ a permutation and $\zeta(l_i)\epsilon_i = \epsilon = \pm 1$, $l_i \in \pi_1(B)$. [The expression $\zeta(l_i)$ is $+1$ or -1 according as the number of v_i's occurring in an expression for l_i is even or odd.]

Proof of Theorem 1. Consider $\omega \in \mathrm{Hom}(\pi_1(B), C_2)$ with $\omega(s_d) = -1$ for some d. Composing ω with the following geometric automorphism ψ of $\pi_1(B)$ shows that we may assume $\omega(s_m) = -1$.

$$\psi(s_i) = s_i, \qquad i = 1, \cdots, d-1$$
$$\psi(s_d) = (s_d \cdots s_{m-1}) s_m (s_d \cdots s_{m-1})^{-1}$$
$$\psi(s_i) = s_d s_i s_d^{-1}, \qquad i = d+1, \cdots, m-1$$
$$\psi(s_m) = s_d$$
$$\psi(a_j) = a_j, \quad \psi(b_j) = b_j, \quad \psi(v_j) = v_j, \quad j = 1, \cdots, g.$$

If B is orientable and $\omega(a_1) = -1$ then **[2**; Theorem 5.2] exhibits a geometric automorphism γ of $\pi_1(B)$ such that $\omega \circ \gamma(a_j) = \omega \circ \gamma(b_j) = -1$ for all j. Suppose $\omega(a_1) = +1$ and define a geometric automorphism α of $\pi_1(B)$ by

$$\alpha(s_i) = s_i, \qquad i \neq m$$
$$\alpha(s_m) = (s_m a_1 b_1 a_1^{-1}) s_m (s_m a_1 b_1 a_1^{-1})^{-1}$$
$$\alpha(a_1) = s_m a_1$$
$$\alpha(a_j) = a_j, \qquad j \neq 1$$
$$\alpha(b_j) = b_j, \quad \text{for all} \quad j.$$

Then $\omega \circ \alpha(a_1) = -1$. After applying γ if necessary, we may suppose that $\omega(a_j) = \omega(b_j) = -1$ for all j.

If B is nonorientable, the arguments of **[5]** place ω in one of the classes n_1, n_2, n_3, n_4. Define the geometric automorphisms β_r and δ of $\pi_1(B)$ as follows.

$$\begin{aligned}
&\beta_r(s_i) = s_i, && i = 1, \cdots, m \\
&\beta_r(v_r) = v_r v_{r+1}^{-1} v_r^{-1} && \\
&\beta_r(v_{r+1}) = v_r v_{r+1}^2 && \\
&\beta_r(v_j) = v_j, && j \neq r, r+1
\end{aligned}$$

$$\begin{aligned}
&\delta(s_i) = s_i, && i \neq m \\
&\delta(s_m) = (s_m v_1) s_m^{-1} (s_m v_1)^{-1} && \\
&\delta(v_1) = s_m v_1 && \\
&\delta(v_j) = v_j, && j \neq 1.
\end{aligned}$$

P. Orlik [**2**; Theorem 5.2] has defined geometric automorphisms which reduce by two the number of v_j such that $\omega(v_j) = +1$. By applying these together with δ and the β_r we may reduce each of the classes n_1, n_3, n_4 to n_2.

Finally, note that if k is the number of s_i such that $\omega(s_i) = -1$, then k is invariant under geometric automorphism. This follows immediately from the theorem of Zieschang stated above. Since $\omega(s_1 \cdots s_m [a_1, b_1] \cdots [a_g, b_g]) = +1$ (respectively, $\omega(s_1 \cdots s_m v_1^2 \cdots v_g^2) = +1$), k is even.

2. Equivariant classification. Let M be a closed connected 3-manifold with local S^1 action and let $p: M \to M^*$ be the decomposition map. The decomposition elements (fibers) are circles and points. A point fiber has a neighborhood in M which is fiber homeomorphic to the 3-disk with orthogonal S^1 action. A circle fiber has an invariant tubular neighborhood in M fiber homeomorphic to one of the following: (i) $D^2 \times S^1$ with S^1 action by translation in the second factor. (ii) $D^2 \times_{\mathbf{Z}_\alpha} S^1$ with S^1 action where $\mathbf{Z}_\alpha$ acts on D^2 by rotations. The fiber $\{0\} \times_{\mathbf{Z}_\alpha} S^1$ is called an E-fiber and is assigned Seifert invariants (α, β) as in **[4]**. (iii) $D^2 \times_{\mathbf{Z}_2} S^1$ with S^1 action where $\mathbf{Z}_2$ acts on D^2 by reflection. If I is the axis of reflection in D^2 then the fibers of $I \times_{\mathbf{Z}_2} S^1$ in $D^2 \times_{\mathbf{Z}_2} S^1$ are called SE-fibers. Circle fibers which are not E or SE-fibers are called regular fibers. Restricted to the union of the regular fibers of M, p is the projection map of an S^1 bundle with structure group $\mathbf{O}(2)$.

It is now seen that M^* is a 2-manifold whose boundary components are the images of components of SE-fibers or point fibers. The components of point fibers in M are circles which we call F-components. An F-component has an invariant 2-disk bundle neighborhood V, whose boundary, ∂V, is a subbundle of the bundle of regular fibers. If ∂V is an orientable S^1 bundle then V is a solid torus which is one of the building blocks described in **[4]**. If ∂V is fibered as the nonorientable S^1 bundle over S^1 then the building block V is a solid Klein bottle fiber homeomorphic to the space obtained by taking $\partial V \times [0, 1]$ and collapsing the fibers of $\partial V \times \{0\}$ to points. Any cross section of ∂V extends to a cross section of V. We shall distinguish between the above two cases by calling the F-component orientation-preserving in the first case and orientation-reversing in the second case.

A component C of SE-fibers has an invariant tubular neighborhood U which is a Möbius band bundle over the boundary circle $p(C)$ of M^*. The restriction of this bundle to ∂U is a subbundle of the S^1 bundle of regular fibers. If ∂U is an orientable S^1 bundle over S^1 then U is homeomorphic to the product of the Möbius band with the circle, and C is a torus. This neighborhood U is one of the building blocks of **[4]**. If ∂U is a nonorientable S^1 bundle over S^1 then U is the nontrivial Möbius band bundle over $p(C)$, and C is a Klein bottle. This building block U is fiber homeomorphic to the space obtained by taking $\partial U \times [0, 1]$ and collapsing the fibers of $\partial U \times \{0\}$ by the antipodal map. Any cross section of ∂U thus extends to a cross section of U.

Let ϵ take on the values $o, o_1, o_2, n, n_1, n_2, n_3$, or n_4. Let $g, \bar{h}, k_1, t, k_2$ be nonnegative integers such that $k = k_1 + k_2$ is even, $k_1 \leqq \bar{h}$, $k_2 \leqq t$, $g \geqq 1$ for $\epsilon = o_2, n_1, n_2, n$; $g \geqq 2$ for $\epsilon = n_3$, $g \geqq 3$ for $\epsilon = n_4$, and $\epsilon = o$ or n if and only if $k \neq 0$. Let (α_j, β_j) for $j = 1, \cdots, r$ be pairs of relatively prime integers such that $0 < \beta_j < \alpha_j$ if $\epsilon = o_1$ or n_2 and $0 < \beta_j \leqq \alpha_j/2$ otherwise. Let $b = 0$ if $\bar{h} + t > 0$ or if $\epsilon = o_2, n_1, n_3$, or n_4 and for some j, $\alpha_j = 2$. Let $b = 0$ or 1 if $\bar{h} + t = 0$ and $\epsilon = o_2, n_1, n_3$, or n_4 and no $\alpha_j = 2$. Let b be an arbitrary integer if $\bar{h} + t = 0$ and $\epsilon = o_1$ or n_2.

By the 3-manifold $M = \{b; (\epsilon, g, (\bar{h}, k_1), (t, k_2)); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$ we mean the closed connected 3-manifold constructed as follows. If $\bar{h} + t > 0$ let $\bar{M}_0^*$ be a 2-manifold of genus g with $\bar{h} + t + r$ boundary components and which is orientable if $\epsilon = o$ or o_i and nonorientable if $\epsilon = n$ or n_i. Let $\bar{M}_0$ be an S^1 bundle over $\bar{M}_0^*$ determined by (ϵ, k). Standard obstruction theory shows that this bundle admits a cross section. On each of the $\bar{h} + t + r - k$ torus boundary components of $\bar{M}_0$ the structure group of the bundle reduces to $\mathbf{SO}(2)$ and we have an S^1 action. For each $j = 1, \cdots, r$ equivariantly sew a solid torus $D^2 \times_{Z_\alpha} S^1$ of type (α_j, β_j) to a torus boundary component of $\bar{M}_0$. Now equivalently sew $\bar{h} - k_1$ building block neighborhoods of orientation-preserving F-components and $t - k_2$ building block neighborhoods of torus SE-components to the remaining torus boundary components of $\bar{M}_0$. These sewings are described in **[4]**. Sew k_1 building block neighborhoods of orientation-reversing F-components and k_2 building block neighborhoods of Klein bottle SE-components to the Klein bottle boundary components of $\bar{M}_0$ using $\mathbf{O}(2)$ bundle homeomorphisms for the attachings. The resulting manifold is M. If $\bar{h}+t=0$ let M be the Seifert manifold $\{b; (\epsilon, g, 0, 0); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$ of **[4]**.

In any case if $k=0$ the manifold $\{b; (\epsilon, g_1(\bar{h}, 0), (t, 0)); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$ is just the manifold $\{b, (\epsilon, g, \bar{h}, t); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$ of **[4]**. Since Orlik and Raymond have given equivariant and topological classification of these, we shall deal with the case $k \neq 0$.

If M and M' are 3-manifolds with local S^1 actions they are defined to be equivariantly equivalent if there is a fiber-preserving homeomorphism from M to M' which is orientation-preserving on $M - SE$, if $M - SE$ is oriented. When $k \neq 0$ equivariant equivalence reduces to fiber-preserving homeomorphism.

THEOREM 2. *If M is a closed connected 3-manifold with local S^1 action, it is determined up to equivariant equivalence by the orbit invariants* $\{b; (\epsilon, g, (\bar{h}, k_1), (t, k_2)); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$.

Proof. Given M_1, the invariants are assigned in the obvious fashion. When $\epsilon \neq o_1$ or n_2 the Seifert invariants (α_j, β_j) must be reduced to $0 < \beta_j \leqq \alpha_j/2$ since $M - SE$ is nonorientable. (See, for example, **[7**; p. 189].) If $k = k_1 + k_2 = 0$ the result is Theorem 0 of **[4]**. The general case is an easy extension using the fact that cross sections on the boundary of building block neighborhoods of F or SE-components extend to cross sections of the whole neighborhood.

3. Topological classification. Again we discuss only the case $k \neq 0$. See **[4]** Theorems 1, 2, and 3 where the case $k=0$ is completed. We first deal with $F \neq \emptyset$.

THEOREM 3. *Let* $M = \{0; (\epsilon, g, (\bar{h}, k_1), (t, k_2)); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$ *with $k \neq 0$ and $\bar{h} \neq 0$. Then M is homeomorphic to $N \# (S^2 \times S^1)^{\bar{g}} \# (\mathbf{P}^2 \times S^1)^t \# L(\alpha_1, \beta_1) \# \cdots \# L(\alpha_r, \beta_r)$ where $\bar{g} = 2g + \bar{h} - 2$ if $\epsilon = o$ and $\bar{g} = g + \bar{h} - 2$ if $\epsilon = n$, where N is replaced by $S^2 \times S^1$ if $t > 0$.*

Proof. The proof is the same as that given in **[3]**, **[4]**, and **[6]** once it is noted that $\{0; (o, 0, (2, 2), (0, 0))\} = N$ and $\{0; (o, 0, (1, 1), (1, 1))\} = \mathbf{P}^2 \times S^1$.

LEMMA 2. *Let $M = \{0; (\epsilon, g, (0, 0), (t, k)); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$ with $k \neq 0$. The orientable double cover of M is*

$$\hat{M} = \{-r; (n_2, \hat{g}, (0,0), (0,0)); (\alpha_1, \beta_1), (\alpha_1, \alpha_1 - \beta_1), \cdots, (\alpha_r, \beta_r), (\alpha_r, \alpha_r - \beta_r)\}$$

where $\hat{g} = 4g + 2t - 2$ *if* $\epsilon = o$ *and* $\hat{g} = 2g + 2t - 2$ *if* $\epsilon = n$.

Proof. A building block neighborhood of a Klein bottle SE-component has orientable double cover an annulus bundle over S^1 with boundary a torus. It is a Seifert manifold whose decomposition space is the Möbius band. It follows that $\hat{M}$ is a Seifert manifold. The computation of $\hat{M}$ may now be carried out in the spirit of [7; §9].

THEOREM 4. *Suppose* $M = \{0; (\epsilon, g, (0,0), (t, k)); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$ *with* $k \neq 0$. *Then*

(1) $\{0; (o, 0, (0,0), (2,2))\} = S_*K$ (*see* [4])

(2) *All other manifolds admit only the local* S^1 *action given. They are* $K(\pi, 1)$*'s and do not fiber over* S^1.

Proof. The only manifold of this class whose orientable double cover is not a large Seifert manifold (see [2]) is (1). That (1) is S_*K is seen geometrically. Consider an M not (1). If $\epsilon = o$

$$\pi_1(M) = (a_i, b_i, q_j, h, r_l, h_l, s_p, h'_p \mid \pi_* r_1 \cdots r_{t-k} s_1 \cdots s_k, a_i h a_i^{-1} h, b_i h b_i^{-1} h, [q_j, h], q_j^{\alpha_j} h^{\beta_j}, [r_l, h_l], s_p h'_p s_p^{-1} h'_p, h_l^2 h^{-1}, h_p'^2 h^{-1})$$

where $i = 1, \cdots, g$; $j = 1, \cdots, r$; $p = 1, \cdots, k$; $l = 1, \cdots, t-k$, and $\pi_* = q_1 \cdots q_r [a_1, b_1] \cdots [a_g, b_g]$. If $\epsilon = n$,

$$\pi_1(M) = (v_i, q_j, h, r_l, h_l, s_p, h'_p \mid \pi_* r_1 \cdots r_{t-k} s_1 \cdots s_k, v_i h v_i^{-1} h, [q_j, h], q_j^{\alpha_j} h^{\beta_j}, [r_l, h_l], s_p h'_p s_p^{-1} h'_p, h_l^2 h^{-1}, h_p'^2 h^{-1}),$$

where $\pi_* = q_1 \cdots q_r v_1^2 \cdots v_g^2$.

The subgroup (h) is infinite and is the unique maximal cyclic normal subgroup of $\pi_1(M)$.

$$\pi_1(M)/(h) = (\bar{a}_i, \bar{b}_i, \bar{q}_j, \bar{r}_l, \bar{h}_l, \bar{s}_p, \bar{h}'_p \mid \bar{\pi}_* \bar{r}_1 \cdots \bar{r}_{t-k} \bar{s}_1 \cdots \bar{s}_k, \bar{q}_j^{\alpha_j}, [\bar{r}_l, \bar{h}_l], [\bar{s}_p, \bar{h}'_p], \bar{h}_l^2, \bar{h}_p'^2)$$

or

$$(\bar{v}_i, \bar{q}_j, \bar{r}_l, \bar{h}_l, \bar{s}_p, \bar{h}'_p \mid \bar{\pi}_* \bar{r}_1 \cdots \bar{r}_{t-k} \bar{s}_1 \cdots \bar{s}_k, \bar{q}_j^{\alpha_j}, [\bar{r}_l, \bar{h}_l], [\bar{s}_p, \bar{h}'_p], \bar{h}_l^2, \bar{h}_p'^2).$$

By [8; Theorem IV.11] $\pi_1(M)/(h)$ is a planar discontinuous group. Thus by [8; Theorem IV.9] ϵ, g, and t are determined by $\pi_1(M)/(h)$. The number of conjugacy classes of elements of finite order in $\pi_1(M)/(h)$ which have noncyclic centralizer is t.

Let $[\bar{x}]$ denote the conjugacy class of $\bar{x} \in \pi_1(M)/(h)$. The integer k is the number of conjugacy classes $[\bar{x}]$ of elements of finite order in $\pi_1(M)/(h)$ such that there is a $z \in \pi_1(M)$ which is not in the centralizer of h such that $\bar{z}$ is in the centralizer of $\bar{y}$ for some $\bar{y} \in [\bar{x}]$. Thus k is determined by $\pi_1(M)$. The orientable double cover M is the large Seifert manifold given by Lemma 3. Thus $\alpha_1, \cdots, \alpha_r$ are determined, and $\beta_1, \cdots, \beta_r$ are determined by the normalization $0 < \beta_j \leqq \alpha_j/2$.

If M admits a different local S^1 action, say

$$\{b'; (\epsilon', g', (\bar{h}', k_1'), (t', k_2')); (\alpha_1', \beta_1'), \cdots, (\alpha_r', \beta_r')\}$$

then $\bar{h}' = 0$, for otherwise $\pi_1(M)$ is cyclic or a nontrivial free product. But $\pi_1(M)$ is not cyclic, and $\pi_1(M)$ is not a nontrivial free product since it contains a nontrivial cyclic normal subgroup. If $b' \neq 0$ then each element of $\pi_1(M)/(h)$ has cyclic centralizer (see **[4]**), so $b' = 0$. Now the above arguments show that the rest of the invariants must agree with the original ones. So M admits a unique local S^1 action.

The manifold M is a $K(\pi, 1)$ because it is covered by the $K(\hat{\pi}, 1), \hat{M}$. Because $2h = 0$ in $H_1(M)$, it follows from arguments similar to those of **[5]**, §5] that M does not fiber over S^1.

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Tulane University

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