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**CHOQUET SIMPLEXES AND  $\sigma$ -CONVEX FACES**

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## CHOQUET SIMPLEXES AND $\sigma$ -CONVEX FACES

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**The purpose of this paper is to present a simple characterization of the split-faces in a Choquet simplex  $K$ , i.e., those faces  $F$  such that  $K$  is a direct convex sum of  $F$  and its complementary face. It is shown that a face  $F$  is a split-face if and only if it is  $\sigma$ -convex, i.e., closed under infinite convex combinations. This is proved by means of a measure-theoretic characterization of the  $\sigma$ -convex faces of  $K$ . As a consequence, it is shown that the lattice of  $\sigma$ -convex faces of a Choquet simplex forms a complete Boolean algebra.**

It is well-known that infinite convex combinations are always available in a compact convex subset  $K$  of a locally convex, Hausdorff, linear topological space: given points  $x_1, x_2, \dots$  in  $K$  and nonnegative real numbers  $\alpha_1, \alpha_2, \dots$  such that  $\sum \alpha_k = 1$ , the series  $\sum \alpha_k x_k$  must converge to some point in  $K$ . We refer to  $\sum \alpha_k x_k$  as a  $\sigma$ -convex combination of the  $x_k$ . We define  $\sigma$ -convex subsets of  $K$  and  $\sigma$ -convex hulls in  $K$  in the obvious manner. A face of  $K$  which happens to be  $\sigma$ -convex is called a  $\sigma$ -convex face, and we note that given any subset  $X \subseteq K$ , there is a smallest  $\sigma$ -convex face containing  $X$ , called the  $\sigma$ -convex face generated by  $X$ . We also note that if  $F$  is any face of  $K$  ( $\sigma$ -convex or not), and if we have a  $\sigma$ -convex combination  $\sum \alpha_k x_k \in F$  with all  $\alpha_k > 0$ , then all  $x_k \in F$ . (This follows from the defining property of a face, since

$$\alpha_j x_j + (1 - \alpha_j) \left[ \sum_{k \neq j} \alpha_k x_k / (1 - \alpha_j) \right] = \sum \alpha_k x_k \in F$$

for all  $j$ .)

**DEFINITION.** Let  $F$  be a face of a compact convex set  $K$ . Then  $F'$  is defined to be the union of those faces of  $K$  which are disjoint from  $F$ . We say that  $F$  is a *split-face* of  $K$  [2] provided  $F'$  is a face and for each  $x \in K - (F \cup F')$  there is a unique convex combination  $x = \alpha y + (1 - \alpha)z$  with  $y \in F$ ,  $z \in F'$ . In this event,  $F'$  is a complement for  $F$  in the lattice of faces of  $K$ .

**THEOREM 1.** *Let  $F$  be a face of a Choquet simplex  $K$ . Then  $F'$  is a face of  $K$ . For any  $x \in K - (F \cup F')$ , there is at most one convex combination  $x = \alpha y + (1 - \alpha)z$  with  $y \in F$  and  $z \in F'$ .*

*Proof.* [1, Theorem 1 and Proposition 2].

**COROLLARY 2.** *Let  $F$  be a face of a Choquet simplex  $K$ . Then  $F$  is a split-face of  $K$  if and only if the convex hull of  $F \cup F'$  is  $K$ .*

**DEFINITION.** For any compact Hausdorff space  $K$ , we use  $M_1^+(K)$  to denote the set of all probability measures on  $K$ , equipped with the vague (weak\*) topology. There is also a natural norm topology on  $M_1^+(K)$ , obtained from the identification of the space  $M(K)$  [of all signed regular Borel measures on  $K$ ] with the dual Banach space  $C(K)^*$ . If  $K$  is a compact convex set, then for any  $\mu \in M_1^+(K)$  we use  $x_\mu$  to denote the resultant (barycenter) of  $\mu$  in  $K$ . If  $K$  is a Choquet simplex, then for any  $x \in K$  we use  $\mu_x$  to denote the unique maximal measure in  $M_1^+(K)$  whose resultant is  $x$ .

The existence of  $\sigma$ -convex combinations is more straightforward in  $M_1^+(K)$  than in general compact convex sets. Given measures  $\mu_1, \mu_2, \dots$  in  $M_1^+(K)$  and nonnegative real numbers  $\alpha_1, \alpha_2, \dots$  such that  $\sum \alpha_k = 1$ , we may define a Borel measure  $\mu$  on  $K$  by setting  $\mu(A) = \sum \alpha_k \mu_k(A)$  for all Borel sets  $A \subseteq K$ . It is clear that  $\mu \in M_1^+(K)$ . Observing that  $\mu(f) = \sum \alpha_k \mu_k(f)$  for all  $f \in C(K)$ , we see that  $\sum \alpha_k \mu_k$  converges to  $\mu$  in the vague topology. Therefore  $\mu$  coincides with the  $\sigma$ -convex combination  $\sum \alpha_k \mu_k$  in  $M_1^+(K)$ .

**PROPOSITION 3.** *Let  $K$  be a compact Hausdorff space,  $\mu \in M_1^+(K)$ ,  $X \subseteq M_1^+(K)$ . Then  $\mu$  lies in the face generated by  $X$  if and only if there exists  $\nu$  in the convex hull of  $X$  such that  $\mu \leq \alpha \nu$  for some  $\alpha > 0$ .*

*Proof.* Analogous to [4, Proposition 1.2].

**THEOREM 4.** *Let  $K$  be a compact Hausdorff space,  $\mu \in M_1^+(K)$ ,  $X \subseteq M_1^+(K)$ . Then the following conditions are equivalent:*

- (a)  $\mu$  lies in the  $\sigma$ -convex face generated by  $X$ .
- (b)  $\mu$  lies in the  $\sigma$ -convex hull of the face generated by  $X$ .
- (c) There is some  $\nu$  in the  $\sigma$ -convex hull of  $X$  for which  $\mu \ll \nu$ .

*Proof.* (a)  $\Rightarrow$  (c): Let  $F$  denote the set of those measures  $\mu' \in M_1^+(K)$  which are absolutely continuous with respect to some  $\nu'$  (depending on  $\mu'$ ) in the  $\sigma$ -convex hull of  $X$ . We claim that  $F$  is a  $\sigma$ -convex face of  $M_1^+(K)$ .

First consider a  $\sigma$ -convex combination  $\mu' = \sum \alpha_k \mu_k$  in  $M_1^+(K)$  such that each  $\mu_k \in F$ . For each  $k$ , there is some  $\nu_k$  in the  $\sigma$ -convex hull of  $X$  such that  $\mu_k \ll \nu_k$ . Then  $\nu' = \sum \alpha_k \nu_k$  lies in the  $\sigma$ -convex hull of  $X$ , and we infer that  $\mu' \ll \nu'$ , whence  $\mu' \in F$ . Thus  $F$  is  $\sigma$ -convex.

Next consider a proper convex combination  $\mu' = \alpha \mu_1 + (1 - \alpha) \mu_2$  in  $M_1^+(K)$  such that  $\mu' \in F$ . There is some  $\nu'$  in the  $\sigma$ -convex hull of  $X$  such

that  $\mu' \ll \nu'$ . Since  $\mu_1 \leq \alpha^{-1}\mu'$  and  $\mu_2 \leq (1 - \alpha)^{-1}\mu'$ , we find that  $\mu_1, \mu_2 \ll \nu'$ , and consequently  $\mu_1, \mu_2 \in F$ . Thus  $F$  is a face of  $M_1^+(K)$ .

Clearly  $X \subseteq F$ , hence  $F$  must contain the  $\sigma$ -convex face generated by  $X$ . Therefore  $\mu \in F$ .

(c)  $\Rightarrow$  (b): There is a  $\sigma$ -convex combination  $\nu = \sum \alpha_k \nu_k$  with each  $\nu_k \in X$ . Renumbering if necessary, we may assume that  $\alpha_1 > 0$ . For each  $k$ , set  $\alpha'_k = \alpha_1 + \dots + \alpha_k$  and  $\nu'_k = (\alpha_1 \nu_1 + \dots + \alpha_k \nu_k) / \alpha'_k$ , and note that  $\nu'_k$  is a measure in  $M_1^+(K)$  which lies in the convex hull of  $X$ .

For each positive integer  $n$ , take a Hahn Decomposition of the signed measure  $n\alpha'_n \nu'_n - \mu$ . This gives us a Borel set  $K_n \subseteq K$  such that  $\mu(A) \leq n\alpha'_n \nu'_n(A)$  for all Borel sets  $A \subseteq K_n$  and  $n\alpha'_n \nu'_n(A) \leq \mu(A)$  for all Borel sets  $A \subseteq K - K_n$ .

Since  $2\alpha'_2 \nu'_2(K_1 - K_2) \leq \mu(K_1 - K_2) \leq \alpha'_1 \nu'_1(K_1 - K_2) \leq \alpha'_2 \nu'_2(K_1 - K_2)$ , we see that  $\mu(K_1 - K_2) = \nu'_2(K_1 - K_2) = 0$ . Thus we may replace  $K_2$  by  $K_1 \cup K_2$ , so that now  $K_1 \subseteq K_2$ . Continuing in this manner, we see that we may assume that  $K_n \subseteq K_{n+1}$  for all  $n$ .

Set  $J = K - (\cup K_n)$ , and note that  $\alpha'_n \nu'_n(J) \leq \mu(J)/n$  for all  $n$ . For all  $k \geq n$ ,  $\alpha'_k \nu'_k(J) \leq \mu(J)/k \leq \mu(J)/n$ , hence  $\nu(J) = \lim_{k \rightarrow \infty} \alpha'_k \nu'_k(J) \leq \mu(J)/n$ . Since this holds for all  $n$ , we obtain  $\nu(J) = 0$ . Since  $\nu'_n \leq \nu / \alpha'_n$  and  $\mu \ll \nu$ , it follows that  $\nu'_n(J) = 0$  for all  $n$  and  $\mu(J) = 0$ . Thus we may replace each  $K_n$  by  $K_n \cup J$ , without affecting the properties obtained above. As a result, we now have  $\cup K_n = K$ .

Now set  $L_1 = K_1$  and  $L_n = K_n - K_{n-1}$  for all  $n > 1$ , so that  $L_1, L_2, \dots$  are pairwise disjoint Borel sets whose union is  $K$ . Set  $I = \{n \mid \mu(L_n) > 0\}$ . For  $n \in I$ , define  $\mu_n \in M_1^+(K)$  by setting  $\mu_n(A) = \mu(A \cap L_n) / \mu(L_n)$  for all Borel sets  $A \subseteq K$ . For such  $A$ , we have  $A \cap L_n \subseteq L_n \subseteq K_n$  and so

$$\mu_n(A) = \mu(A \cap L_n) / \mu(L_n) \leq n\alpha'_n \nu'_n(A \cap L_n) / \mu(L_n) \leq n\alpha'_n \nu'_n(A) / \mu(L_n).$$

Consequently,  $\mu_n \leq [n\alpha'_n / \mu(L_n)] \nu'_n$ , whence Proposition 3 shows that  $\mu_n$  lies in the face generated by  $X$ .

We have  $\sum_{n \in I} \mu(L_n) = 1$  and  $\mu(A) = \sum_{n \in I} \mu(A \cap L_n) = \sum_{n \in I} \mu(L_n) \mu_n(A)$  for all Borel sets  $A \subseteq K$ . Therefore  $\mu = \sum_{n \in I} \mu(L_n) \mu_n$  is a  $\sigma$ -convex combination of the  $\mu_n$ , hence  $\mu$  lies in the  $\sigma$ -convex hull of the face generated by  $X$ .

(b)  $\Rightarrow$  (a) is clear.

In particular, Theorem 4 shows that a measure  $\mu \in M_1^+(K)$  lies in the  $\sigma$ -convex face generated by a measure  $\nu \in M_1^+(K)$  if and only if  $\mu \ll \nu$ . The corresponding statement for norm-closed faces is given in [4, Proposition 1.3]:  $\mu$  lies in the norm-closure of the face generated by  $\nu$  if and only if  $\mu \ll \nu$ . Thus the  $\sigma$ -convex face generated by  $\nu$  coincides with the norm-closure of the face generated by  $\nu$ . In general, the  $\sigma$ -convex faces in  $M_1^+(K)$  coincide with the norm-closed faces, as the next theorem shows.

**THEOREM 5.** *Let  $K$  be a compact Hausdorff space. For any face  $F$  of  $M_1^+(K)$ , the following conditions are equivalent:*

- (a)  $F$  is a split-face.
- (b)  $F$  is norm-closed.
- (c)  $F$  is  $\sigma$ -convex.

*Proof.* (a)  $\Leftrightarrow$  (b) follows from [3, Corollary to Theorem 1], and also appears in [4, Theorem 2.4].

(b)  $\Rightarrow$  (c) follows from the observation that infinite convex combinations in  $M_1^+(K)$  must also converge in the norm topology.

(c)  $\Rightarrow$  (b): Suppose that  $\mu_1, \mu_2, \dots \in F$  and  $\mu \in M_1^+(K)$  such that  $\|\mu_n - \mu\| \rightarrow 0$ . It follows easily from Urysohn's Lemma and the regularity of the measures that  $\mu_n(A) \rightarrow \mu(A)$  for all Borel sets  $A \subseteq K$ . Setting  $\nu = \sum_{n=1}^{\infty} \mu_n / 2^n$ , we thus see that  $\nu \in F$  and  $\mu \ll \nu$ . According to Theorem 4,  $\mu \in F$ .

**DEFINITION.** As in [2], any compact convex set  $K$  (in a locally convex, Hausdorff, linear topological space) is affinely homeomorphic to a weak\*-compact convex subset of the dual space  $A(K)^*$  (where  $A(K)$  denotes the Banach space of all real-valued affine continuous functions on  $K$ ). Because of this,  $K$  inherits a norm topology from  $A(K)^*$ .

**PROPOSITION 6.** *Let  $K$  be a Choquet simplex, and let  $K^*$  denote the set of maximal measures in  $M_1^+(K)$ . Then  $K^*$  is a  $\sigma$ -convex face of  $M_1^+(K)$ , and the rule  $\phi(\mu) = x_\mu$  defines a continuous affine isomorphism  $\phi$  of  $K^*$  onto  $K$ . The maps  $\phi$  and  $\phi^{-1}$  both preserve  $\sigma$ -convex combinations and norms.*

*Proof.* It is well-known that  $K^*$  is a face of  $M_1^+(K)$ , and that  $\phi$  is a continuous affine isomorphism. The  $\sigma$ -convexity of  $K^*$  follows easily from Mokobodzki's characterization of maximal measures [2, Proposition I.4.5].

Since  $\phi$  is continuous and affine, it must preserve  $\sigma$ -convex combinations, hence so does  $\phi^{-1}$ .

According to [4, Lemma 2.6],  $\phi^{-1}$  preserves norms, hence  $\phi$  does also.

With the help of Proposition 6, Theorems 4 and 5 imply the corresponding results for arbitrary Choquet simplexes.

**THEOREM 7.** *Let  $K$  be a Choquet simplex,  $x \in K$ ,  $Y \subseteq K$ . Then the following conditions are equivalent:*

- (a)  $x$  lies in the  $\sigma$ -convex face generated by  $Y$ .
- (b)  $x$  lies in the  $\sigma$ -convex hull of the face generated by  $Y$ .
- (c) There is some  $y$  in the  $\sigma$ -convex hull of  $Y$  such that  $\mu_x \ll \mu_y$ .

COROLLARY 8. *If  $F$  is a face of a Choquet simplex  $K$ , then the  $\sigma$ -convex hull of  $F$  is also a face of  $K$ .*

THEOREM 9. *If  $F$  is a face of a Choquet simplex  $K$ , then the following conditions are equivalent:*

- (a)  *$F$  is a split-face.*
- (b)  *$F$  is norm-closed.*
- (c)  *$F$  is  $\sigma$ -convex.*

The equivalence (a)  $\Leftrightarrow$  (b) of Theorem 9 has appeared in [3, Corollary to Theorem 1] and [4, Theorem 2.8]. We note that the characterization (a)  $\Leftrightarrow$  (c) has an apparent advantage, in that it depends only on the topology intrinsic to  $K$ , rather than on the (external) norm topology. While we have utilized the norm results as the fastest means of proving Theorems 5 and 9, it is also possible to prove the equivalence (a)  $\Leftrightarrow$  (c) in these theorems without any use of norms.

THEOREM 10. *The lattice  $\mathcal{F}$  of  $\sigma$ -convex faces of a Choquet simplex  $K$  forms a complete Boolean algebra. For  $\{F_i\} \subseteq \mathcal{F}$ ,  $\wedge F_i = \cap F_i$ . For  $F, G \in \mathcal{F}$ ,  $F \vee G$  is the convex hull of  $F \cup G$ .*

*Proof.* Obviously  $\mathcal{F}$  is a complete lattice in which arbitrary infima are given by intersections. For  $F, G \in \mathcal{F}$ , we see from Theorem 9 and [2, Corollary II.6.8] that the convex hull of  $F \cup G$  is a  $\sigma$ -convex face of  $K$ . (This is also easy to prove directly, using [1, Proposition 3].) Thus the convex hull of  $F \cup G$  equals  $F \vee G$ .

Given  $F, G, H \in \mathcal{F}$ , we automatically have  $(F \wedge G) \vee (F \wedge H) \subseteq F \wedge (G \vee H)$ . Now consider any  $x \in F \wedge (G \vee H)$ . Inasmuch as  $G \vee H$  is the convex hull of  $G \cup H$ , there must be a convex combination  $x = \alpha y + (1 - \alpha)z$  with  $y \in G, z \in H$ . If  $\alpha = 0$  or 1, then either  $x \in F \wedge H$  or  $x \in F \wedge G$ . If  $0 < \alpha < 1$ , then since  $F$  is a face we obtain  $y \in F \wedge G, z \in F \wedge H$ . Thus  $x \in (F \wedge G) \vee (F \wedge H)$  in any case, whence  $F \wedge (G \vee H) = (F \wedge G) \vee (F \wedge H)$ . Therefore  $\mathcal{F}$  is a distributive lattice.

Given  $F \in \mathcal{F}$ , we see from Theorem 9 that  $F' \in \mathcal{F}$  as well (which is also easy to prove directly). Obviously  $F'$  is a complement for  $F$  in  $\mathcal{F}$ , whence  $\mathcal{F}$  is a complemented lattice.

Therefore  $\mathcal{F}$  is a complete, complemented, distributive lattice, i.e., a complete Boolean algebra.

DEFINITION. Let  $K$  be a Choquet simplex, let  $\{x_i\} \subseteq K$ , and for each  $i$  let  $F_i$  be the face generated by  $x_i$  in  $K$ . If  $F_i$  and  $F_j$  are disjoint for all  $i \neq j$ , we shall say that the points  $x_i$  are *facially independent* (in  $K$ ).

COROLLARY 11. *Any  $\sigma$ -convex face  $F$  in a Choquet simplex  $K$  can be generated by facially independent points of  $K$ .*

*Proof.* Let  $\mathcal{F}$  denote the lattice of  $\sigma$ -convex faces of  $K$ , and let  $\mathcal{F}_0$  be the set of those faces in  $\mathcal{F}$  which can be obtained as the  $\sigma$ -convex face generated by a single point of  $K$ . Note that every nonempty face in  $\mathcal{F}$  contains a (nonempty) face from  $\mathcal{F}_0$ . Thus, since  $\mathcal{F}$  is a complete Boolean algebra, there exists a family  $\{F_i\}$  of pairwise disjoint faces in  $\mathcal{F}_0$  such that  $F = \vee F_i$  in  $\mathcal{F}$ . For each  $i$ ,  $F_i$  is the  $\sigma$ -convex face generated by some  $x_i \in K$ . Then the  $x_i$  are facially independent points of  $K$ , and  $F$  is the  $\sigma$ -convex face generated by  $\{x_i\}$ .

#### REFERENCES

1. E. M. Alfsen, *On the decomposition of a Choquet simplex into a direct convex sum of complementary faces*, Math. Scand., **17** (1965), 169–176.
2. ———, *Compact Convex Sets and Boundary Integrals*, Berlin (1971), Springer-Verlag (Ergebnisse der Math., Vol. 57).
3. L. Asimow and A. J. Ellis, *Facial decomposition of linearly compact simplexes and separation of functions on cones*, Pacific J. Math., **34** (1970), 301–309.
4. Á. Lima, *On simplicial and central measures, and split faces*, Proc. London Math. Soc., **26** (1973), 707–728.

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