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CHOQUET SIMPLEXES AND σ -CONVEX FACES

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The purpose of this paper is to present a simple characterization of the split-faces in a Choquet simplex K , i.e., those faces F such that K is a direct convex sum of F and its complementary face. It is shown that a face F is a split-face if and only if it is σ -convex, i.e., closed under infinite convex combinations. This is proved by means of a measure-theoretic characterization of the σ -convex faces of K . As a consequence, it is shown that the lattice of σ -convex faces of a Choquet simplex forms a complete Boolean algebra.

It is well-known that infinite convex combinations are always available in a compact convex subset K of a locally convex, Hausdorff, linear topological space: given points x_1, x_2, \dots in K and nonnegative real numbers $\alpha_1, \alpha_2, \dots$ such that $\sum \alpha_k = 1$, the series $\sum \alpha_k x_k$ must converge to some point in K . We refer to $\sum \alpha_k x_k$ as a σ -convex combination of the x_k . We define σ -convex subsets of K and σ -convex hulls in K in the obvious manner. A face of K which happens to be σ -convex is called a σ -convex face, and we note that given any subset $X \subseteq K$, there is a smallest σ -convex face containing X , called the σ -convex face generated by X . We also note that if F is any face of K (σ -convex or not), and if we have a σ -convex combination $\sum \alpha_k x_k \in F$ with all $\alpha_k > 0$, then all $x_k \in F$. (This follows from the defining property of a face, since

$$\alpha_j x_j + (1 - \alpha_j) \left[\sum_{k \neq j} \alpha_k x_k / (1 - \alpha_j) \right] = \sum \alpha_k x_k \in F$$

for all j .)

DEFINITION. Let F be a face of a compact convex set K . Then F' is defined to be the union of those faces of K which are disjoint from F . We say that F is a split-face of K [2] provided F' is a face and for each $x \in K - (F \cup F')$ there is a unique convex combination $x = \alpha y + (1 - \alpha)z$ with $y \in F$, $z \in F'$. In this event, F' is a complement for F in the lattice of faces of K .

THEOREM 1. Let F be a face of a Choquet simplex K . Then F' is a face of K . For any $x \in K - (F \cup F')$, there is at most one convex combination $x = \alpha y + (1 - \alpha)z$ with $y \in F$ and $z \in F'$.

Proof. [1, Theorem 1 and Proposition 2].

COROLLARY 2. *Let F be a face of a Choquet simplex K . Then F is a split-face of K if and only if the convex hull of $F \cup F'$ is K .*

DEFINITION. For any compact Hausdorff space K , we use $M_1^+(K)$ to denote the set of all probability measures on K , equipped with the vague (weak*) topology. There is also a natural norm topology on $M_1^+(K)$, obtained from the identification of the space $M(K)$ [of all signed regular Borel measures on K] with the dual Banach space $C(K)^*$. If K is a compact convex set, then for any $\mu \in M_1^+(K)$ we use x_μ to denote the resultant (barycenter) of μ in K . If K is a Choquet simplex, then for any $x \in K$ we use μ_x to denote the unique maximal measure in $M_1^+(K)$ whose resultant is x .

The existence of σ -convex combinations is more straightforward in $M_1^+(K)$ than in general compact convex sets. Given measures μ_1, μ_2, \dots in $M_1^+(K)$ and nonnegative real numbers $\alpha_1, \alpha_2, \dots$ such that $\sum \alpha_k = 1$, we may define a Borel measure μ on K by setting $\mu(A) = \sum \alpha_k \mu_k(A)$ for all Borel sets $A \subseteq K$. It is clear that $\mu \in M_1^+(K)$. Observing that $\mu(f) = \sum \alpha_k \mu_k(f)$ for all $f \in C(K)$, we see that $\sum \alpha_k \mu_k$ converges to μ in the vague topology. Therefore μ coincides with the σ -convex combination $\sum \alpha_k \mu_k$ in $M_1^+(K)$.

PROPOSITION 3. *Let K be a compact Hausdorff space, $\mu \in M_1^+(K)$, $X \subseteq M_1^+(K)$. Then μ lies in the face generated by X if and only if there exists ν in the convex hull of X such that $\mu \leq \alpha \nu$ for some $\alpha > 0$.*

Proof. Analogous to [4, Proposition 1.2].

THEOREM 4. *Let K be a compact Hausdorff space, $\mu \in M_1^+(K)$, $X \subseteq M_1^+(K)$. Then the following conditions are equivalent:*

- (a) μ lies in the σ -convex face generated by X .
- (b) μ lies in the σ -convex hull of the face generated by X .
- (c) There is some ν in the σ -convex hull of X for which $\mu \ll \nu$.

Proof. (a) \Rightarrow (c): Let F denote the set of those measures $\mu' \in M_1^+(K)$ which are absolutely continuous with respect to some ν' (depending on μ') in the σ -convex hull of X . We claim that F is a σ -convex face of $M_1^+(K)$.

First consider a σ -convex combination $\mu' = \sum \alpha_k \mu_k$ in $M_1^+(K)$ such that each $\mu_k \in F$. For each k , there is some ν_k in the σ -convex hull of X such that $\mu_k \ll \nu_k$. Then $\nu' = \sum \alpha_k \nu_k$ lies in the σ -convex hull of X , and we infer that $\mu' \ll \nu'$, whence $\mu' \in F$. Thus F is σ -convex.

Next consider a proper convex combination $\mu' = \alpha \mu_1 + (1 - \alpha) \mu_2$ in $M_1^+(K)$ such that $\mu' \in F$. There is some ν' in the σ -convex hull of X such

that $\mu' \ll \nu'$. Since $\mu_1 \leq \alpha^{-1}\mu'$ and $\mu_2 \leq (1-\alpha)^{-1}\mu'$, we find that $\mu_1, \mu_2 \ll \nu'$, and consequently $\mu_1, \mu_2 \in F$. Thus F is a face of $M_1^+(K)$.

Clearly $X \subseteq F$, hence F must contain the σ -convex face generated by X . Therefore $\mu \in F$.

(c) \Rightarrow (b): There is a σ -convex combination $\nu = \sum \alpha_k \nu_k$ with each $\nu_k \in X$. Renumbering if necessary, we may assume that $\alpha_1 > 0$. For each k , set $\alpha'_k = \alpha_1 + \dots + \alpha_k$ and $\nu'_k = (\alpha_1 \nu_1 + \dots + \alpha_k \nu_k) / \alpha'_k$, and note that ν'_k is a measure in $M_1^+(K)$ which lies in the convex hull of X .

For each positive integer n , take a Hahn Decomposition of the signed measure $n\alpha'_n \nu'_n - \mu$. This gives us a Borel set $K_n \subseteq K$ such that $\mu(A) \leq n\alpha'_n \nu'_n(A)$ for all Borel sets $A \subseteq K_n$ and $n\alpha'_n \nu'_n(A) \leq \mu(A)$ for all Borel sets $A \subseteq K - K_n$.

Since $2\alpha'_2 \nu'_2(K_1 - K_2) \leq \mu(K_1 - K_2) \leq \alpha'_1 \nu'_1(K_1 - K_2) \leq \alpha'_2 \nu'_2(K_1 - K_2)$, we see that $\mu(K_1 - K_2) = \nu'_2(K_1 - K_2) = 0$. Thus we may replace K_2 by $K_1 \cup K_2$, so that now $K_1 \subseteq K_2$. Continuing in this manner, we see that we may assume that $K_n \subseteq K_{n+1}$ for all n .

Set $J = K - (\cup K_n)$, and note that $\alpha'_n \nu'_n(J) \leq \mu(J)/n$ for all n . For all $k \geq n$, $\alpha'_k \nu'_k(J) \leq \mu(J)/k \leq \mu(J)/n$, hence $\nu(J) = \lim_{k \rightarrow \infty} \alpha'_k \nu'_k(J) \leq \mu(J)/n$. Since this holds for all n , we obtain $\nu(J) = 0$. Since $\nu'_n \leq \nu/\alpha'_n$ and $\mu \ll \nu$, it follows that $\nu'_n(J) = 0$ for all n and $\mu(J) = 0$. Thus we may replace each K_n by $K_n \cup J$, without affecting the properties obtained above. As a result, we now have $\cup K_n = K$.

Now set $L_1 = K_1$ and $L_n = K_n - K_{n-1}$ for all $n > 1$, so that L_1, L_2, \dots are pairwise disjoint Borel sets whose union is K . Set $I = \{n \mid \mu(L_n) > 0\}$. For $n \in I$, define $\mu_n \in M_1^+(K)$ by setting $\mu_n(A) = \mu(A \cap L_n) / \mu(L_n)$ for all Borel sets $A \subseteq K$. For such A , we have $A \cap L_n \subseteq L_n \subseteq K_n$ and so

$$\mu_n(A) = \mu(A \cap L_n) / \mu(L_n) \leq n\alpha'_n \nu'_n(A \cap L_n) / \mu(L_n) \leq n\alpha'_n \nu'_n(A) / \mu(L_n).$$

Consequently, $\mu_n \leq [n\alpha'_n / \mu(L_n)] \nu'_n$, whence Proposition 3 shows that μ_n lies in the face generated by X .

We have $\sum_{n \in I} \mu(L_n) = 1$ and $\mu(A) = \sum_{n \in I} \mu(A \cap L_n) = \sum_{n \in I} \mu(L_n) \mu_n(A)$ for all Borel sets $A \subseteq K$. Therefore $\mu = \sum_{n \in I} \mu(L_n) \mu_n$ is a σ -convex combination of the μ_n , hence μ lies in the σ -convex hull of the face generated by X .

(b) \Rightarrow (a) is clear.

In particular, Theorem 4 shows that a measure $\mu \in M_1^+(K)$ lies in the σ -convex face generated by a measure $\nu \in M_1^+(K)$ if and only if $\mu \ll \nu$. The corresponding statement for norm-closed faces is given in [4, Proposition 1.3]: μ lies in the norm-closure of the face generated by ν if and only if $\mu \ll \nu$. Thus the σ -convex face generated by ν coincides with the norm-closure of the face generated by ν . In general, the σ -convex faces in $M_1^+(K)$ coincide with the norm-closed faces, as the next theorem shows.

THEOREM 5. *Let K be a compact Hausdorff space. For any face F of $M_1^+(K)$, the following conditions are equivalent:*

- (a) *F is a split-face.*
- (b) *F is norm-closed.*
- (c) *F is σ -convex.*

Proof. (a) \Leftrightarrow (b) follows from [3, Corollary to Theorem 1], and also appears in [4, Theorem 2.4].

(b) \Rightarrow (c) follows from the observation that infinite convex combinations in $M_1^+(K)$ must also converge in the norm topology.

(c) \Rightarrow (b): Suppose that $\mu_1, \mu_2, \dots \in F$ and $\mu \in M_1^+(K)$ such that $\|\mu_n - \mu\| \rightarrow 0$. It follows easily from Urysohn's Lemma and the regularity of the measures that $\mu_n(A) \rightarrow \mu(A)$ for all Borel sets $A \subseteq K$. Setting $\nu = \sum_{n=1}^{\infty} \mu_n / 2^n$, we thus see that $\nu \in F$ and $\mu \ll \nu$. According to Theorem 4, $\mu \in F$.

DEFINITION. As in [2], any compact convex set K (in a locally convex, Hausdorff, linear topological space) is affinely homeomorphic to a weak*-compact convex subset of the dual space $A(K)^*$ (where $A(K)$ denotes the Banach space of all real-valued affine continuous functions on K). Because of this, K inherits a norm topology from $A(K)^*$.

PROPOSITION 6. *Let K be a Choquet simplex, and let K^* denote the set of maximal measures in $M_1^+(K)$. Then K^* is a σ -convex face of $M_1^+(K)$, and the rule $\phi(\mu) = x_\mu$ defines a continuous affine isomorphism ϕ of K^* onto K . The maps ϕ and ϕ^{-1} both preserve σ -convex combinations and norms.*

Proof. It is well-known that K^* is a face of $M_1^+(K)$, and that ϕ is a continuous affine isomorphism. The σ -convexity of K^* follows easily from Mokobodzki's characterization of maximal measures [2, Proposition I.4.5].

Since ϕ is continuous and affine, it must preserve σ -convex combinations, hence so does ϕ^{-1} .

According to [4, Lemma 2.6], ϕ^{-1} preserves norms, hence ϕ does also.

With the help of Proposition 6, Theorems 4 and 5 imply the corresponding results for arbitrary Choquet simplexes.

THEOREM 7. *Let K be a Choquet simplex, $x \in K$, $Y \subseteq K$. Then the following conditions are equivalent:*

- (a) *x lies in the σ -convex face generated by Y .*
- (b) *x lies in the σ -convex hull of the face generated by Y .*
- (c) *There is some y in the σ -convex hull of Y such that $\mu_x \ll \mu_y$.*

COROLLARY 8. *If F is a face of a Choquet simplex K , then the σ -convex hull of F is also a face of K .*

THEOREM 9. *If F is a face of a Choquet simplex K , then the following conditions are equivalent:*

- (a) *F is a split-face.*
- (b) *F is norm-closed.*
- (c) *F is σ -convex.*

The equivalence (a) \Leftrightarrow (b) of Theorem 9 has appeared in [3, Corollary to Theorem 1] and [4, Theorem 2.8]. We note that the characterization (a) \Leftrightarrow (c) has an apparent advantage, in that it depends only on the topology intrinsic to K , rather than on the (external) norm topology. While we have utilized the norm results as the fastest means of proving Theorems 5 and 9, it is also possible to prove the equivalence (a) \Leftrightarrow (c) in these theorems without any use of norms.

THEOREM 10. *The lattice \mathcal{F} of σ -convex faces of a Choquet simplex K forms a complete Boolean algebra. For $\{F_i\} \subseteq \mathcal{F}$, $\bigwedge F_i = \bigcap F_i$. For $F, G \in \mathcal{F}$, $F \vee G$ is the convex hull of $F \cup G$.*

Proof. Obviously \mathcal{F} is a complete lattice in which arbitrary infima are given by intersections. For $F, G \in \mathcal{F}$, we see from Theorem 9 and [2, Corollary II.6.8] that the convex hull of $F \cup G$ is a σ -convex face of K . (This is also easy to prove directly, using [1, Proposition 3].) Thus the convex hull of $F \cup G$ equals $F \vee G$.

Given $F, G, H \in \mathcal{F}$, we automatically have $(F \wedge G) \vee (F \wedge H) \subseteq F \wedge (G \vee H)$. Now consider any $x \in F \wedge (G \vee H)$. Inasmuch as $G \vee H$ is the convex hull of $G \cup H$, there must be a convex combination $x = \alpha y + (1 - \alpha)z$ with $y \in G, z \in H$. If $\alpha = 0$ or 1, then either $x \in F \wedge H$ or $x \in F \wedge G$. If $0 < \alpha < 1$, then since F is a face we obtain $y \in F \wedge G, z \in F \wedge H$. Thus $x \in (F \wedge G) \vee (F \wedge H)$ in any case, whence $F \wedge (G \vee H) = (F \wedge G) \vee (F \wedge H)$. Therefore \mathcal{F} is a distributive lattice.

Given $F \in \mathcal{F}$, we see from Theorem 9 that $F' \in \mathcal{F}$ as well (which is also easy to prove directly). Obviously F' is a complement for F in \mathcal{F} , whence \mathcal{F} is a complemented lattice.

Therefore \mathcal{F} is a complete, complemented, distributive lattice, i.e., a complete Boolean algebra.

DEFINITION. Let K be a Choquet simplex, let $\{x_i\} \subseteq K$, and for each i let F_i be the face generated by x_i in K . If F_i and F_j are disjoint for all $i \neq j$, we shall say that the points x_i are *facially independent* (in K).

COROLLARY 11. *Any σ -convex face F in a Choquet simplex K can be generated by facially independent points of K .*

Proof. Let \mathcal{F} denote the lattice of σ -convex faces of K , and let \mathcal{F}_0 be the set of those faces in \mathcal{F} which can be obtained as the σ -convex face generated by a single point of K . Note that every nonempty face in \mathcal{F} contains a (nonempty) face from \mathcal{F}_0 . Thus, since \mathcal{F} is a complete Boolean algebra, there exists a family $\{F_i\}$ of pairwise disjoint faces in \mathcal{F}_0 such that $F = \vee F_i$ in \mathcal{F} . For each i , F_i is the σ -convex face generated by some $x_i \in K$. Then the x_i are facially independent points of K , and F is the σ -convex face generated by $\{x_i\}$.

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