

# Pacific Journal of Mathematics

**SOME NONOSCILLATION CRITERIA FOR HIGHER ORDER  
NONLINEAR DIFFERENTIAL EQUATIONS**

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**Sufficient conditions for an  $n$ th order nonlinear differential equation to be nonoscillatory are given. An essential part of the hypotheses is that a related linear equation be disconjugate.**

The linear differential equation

$$(1) \quad x^{(n)} + p(t)x = 0,$$

where  $p: [t_0, \infty) \rightarrow R$  is continuous, is said to be eventually disconjugate if there exists  $T \geq t_0$  such that no solution of (1) has more than  $n - 1$  zeros (counting multiplicities) on  $[T, \infty)$ . A solution  $x(t)$  of (1) (or equation (2) below) will be called nonoscillatory if there exists  $t_1 \geq t_0$  such that  $x(t) \neq 0$  for  $t \geq t_1$ . Equation (1) (or (2)) will be called nonoscillatory if all its solutions are nonoscillatory. Clearly, disconjugacy implies nonoscillation. On the other hand, for  $n = 2, 3$  or  $4$  and either  $p(t) > 0$  or  $p(t) < 0$ , if equation (1) is nonoscillatory, then (1) is eventually disconjugate. Whether this is true for  $n > 4$  remains an open question (see Nehari [11]).

In this paper we consider the nonlinear differential equation

$$(2) \quad x^{(n)} + q(t)f(t, x, x', \dots, x^{(n-1)}) = 0$$

where  $q: [t_0, \infty) \rightarrow R$  and  $f: [t_0, \infty) \times R^n \rightarrow R$  are continuous, and obtain some nonoscillation results by making assumptions on the disconjugacy of certain related linear equations. A discussion of disconjugacy criteria for linear differential equations can be found in Coppel [2], Levin [10], Nehari [11], Trench [12], or Willett [13]. For a discussion of nonoscillation criteria for second order nonlinear equations we refer the reader to the recent papers of Coffman and Wong [1], Graef and Spikes [3–5], Wong [14], and the references contained therein. There appears to be no known sufficient conditions for nonoscillation of higher order nonlinear equations.

We will assume that there is a continuous function  $W: [t_0, \infty) \times R^n \rightarrow R$  such that

$$(3) \quad |f(t, u_1, \dots, u_n)| \leq W(t, u_1, \dots, u_n) |u_1|$$

for all  $(t, u_1, \dots, u_n) \in [t_0, \infty) \times R^n$ , and

$$(4) \quad f(t, u_1, \dots, u_n)/u_1 \rightarrow A \text{ as } u_1 \rightarrow 0.$$

THEOREM 1. Suppose that conditions (3) and (4) hold,  $W(t, u_1, \dots, u_n) \leq B$  and  $M = \max\{|A|, B\}$ . If the equations

$$(5) \quad x^{(n)} \pm M|q(t)|x = 0$$

are eventually disconjugate, then equation (2) is nonoscillatory.

*Proof.* Suppose that equations (5) are disconjugate on  $[T, \infty)$  where  $T \geq t_0$  and let  $x(t)$  be a solution of (1). Define  $Q: [T, \infty) \rightarrow \mathbb{R}$  by

$$Q(t) = \begin{cases} q(t)f(t, x(t), \dots, x^{(n-1)}(t))/x(t), & \text{if } x(t) \neq 0 \\ Aq(t), & \text{if } x(t) = 0. \end{cases}$$

It then follows that  $Q(t)$  is continuous and  $x(t)$  is a solution of

$$(6) \quad x^{(n)} + Q(t)x = 0.$$

Kondrat'ev [9] showed that if  $p_1(t) \leq p_2(t)$  and the equations

$$x^{(n)} + p_i(t)x = 0, \quad i = 1, 2$$

are disconjugate on  $[T, \infty)$ , then for any  $p(t)$  with  $p_1(t) \leq p(t) \leq p_2(t)$  the equation

$$x^{(n)} + p(t)x = 0$$

is disconjugate on  $[T, \infty)$ . Here we have  $|Q(t)| \leq M|q(t)|$  so  $-M|q(t)| \leq Q(t) \leq M|q(t)|$ . Hence equation (6) is disconjugate and so  $x(t)$  is nonoscillatory.

REMARK 1. If  $q(t) \geq 0$  and  $u_1 f(t, u_1, \dots, u_n) \geq 0$ , then  $Q(t) \geq 0$ . Since the equation  $x^{(n)} = 0$  is disconjugate on  $[T, \infty)$  for any  $T \geq t_0$ , we would only need to assume that equation (5) with “+” is eventually disconjugate. Note also that condition (4) is only needed to insure that  $Q$  is continuous.

REMARK 2. Equations (5) are eventually disconjugate if, for example,

$$\int_{t_0}^{\infty} t^{n-1}|q(t)| dt < \infty$$

(see Kiguradze [8], Kondrat'ev [9], or Willett [13]). In this regard we would then have a generalization of a result of Kartsatos [7; Theorem 2].

Willett [13; Theorem 1.4] has shown that if for each  $i = 1, 2, \dots, n$ ,  $p_i : [t_0, \infty) \rightarrow \mathcal{R}$  is continuous and

$$(7) \quad \int_{t_0}^{\infty} t^{i-1} |p_i(t)| dt < \infty,$$

then the equation

$$(8) \quad x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = 0$$

is eventually disconjugate. (Recently Gustafson [6] showed that even though nonoscillation implies disconjugacy for equation (8) with  $n = 2$ , this is not the case for  $n > 2$ .) Employing the method of proof used above we can obtain that all solutions of

$$(9) \quad x^{(n)} + p_1(t)f_1(x^{(n-1)}) + \dots + p_n(t)f_n(x) = 0$$

are nonoscillatory.

**THEOREM 2.** *Suppose that condition (7) holds and there are bounded continuous functions  $W_i : [t_0, \infty) \rightarrow \mathcal{R}$ ,  $i = 1, 2, \dots, n$  such that*

$$|f_i(u)| \leq W_i(u) |u|$$

and

$$f_i(u)/u \rightarrow A_i \quad \text{as } u \rightarrow 0.$$

Then all solutions of (9) are nonoscillatory.

*Proof.* If  $x(t)$  is a solution of (9), then  $x(t)$  is also a solution of

$$(10) \quad x^{(n)} + Q_1(t)x^{(n-1)} + \dots + Q_n(t)x = 0$$

where

$$Q_i(t) = \begin{cases} p_i(t)f_i(x^{(n-i)}(t))/x^{(n-i)}(t), & \text{if } x^{(n-i)}(t) \neq 0 \\ A_i p_i(t), & \text{if } x^{(n-i)}(t) = 0 \end{cases}$$

In addition, for each  $i = 1, 2, \dots, n$

$$\begin{aligned}
\int_{t_0}^{\infty} t^{i-1} |Q_i(t)| dt &\leq \int_{t_0}^{\infty} t^{i-1} [ |p_i(t)| |f_i(x^{(n-i)}(t))| / |x^{(n-i)}(t)| ] dt \\
&\leq \int_{t_0}^{\infty} t^{i-1} |p_i(t)| W_i(x^{(n-i)}(t)) dt \\
&\leq K_i \int_{t_0}^{\infty} t^{i-1} |p_i(t)| dt \\
&< \infty
\end{aligned}$$

where  $K_i$  is a constant. It follows from Willett's theorem that equation (10) is disconjugate and hence  $x(t)$  is nonoscillatory.

Clearly various other forms of equation (9) can be handled in a similar fashion.

As an example of the above results, consider the equation

$$(11) \quad x^{(n)} + x^3(\sin t)/t^{n+1}(x^2 + 1) = 0.$$

The corresponding linear equation

$$x^{(n)} + x(\sin t)/t^{n+1} = 0$$

is disconjugate, so equation (11) is nonoscillatory.

#### REFERENCES

1. C. V. Coffman and J. S. W. Wong, *Oscillation and nonoscillation theorems for second order ordinary differential equations*, Funkcial. Ekvac., **15** (1972), 119–130.
2. W. A. Coppel, *Disconjugacy*, Lecture Notes in Math., **220**, Springer-Verlag, New York, 1971.
3. J. R. Graef and P. W. Spikes, *A nonoscillation result for second order ordinary differential equations*, Rend. Accad. Sci. Fis. Mat. Napoli, (4) **41** (1974), 3–12.
4. ———, *Nonoscillation theorems for forced second order nonlinear differential equations*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., to appear.
5. ———, *Sufficient conditions for nonoscillation of a second order nonlinear differential equation*, Proc. Amer. Math. Soc., **50** (1975), 289–292.
6. G. B. Gustafson, *The nonequivalence of oscillation and nondisconjugacy*, Proc. Amer. Math. Soc., **25** (1970), 254–260.
7. A. G. Kartsatos, *Maintenance of oscillations under the effect of a periodic forcing term*, Proc. Amer. Math. Soc., **33** (1972), 377–383.
8. I. T. Kiguradze, *Oscillation properties of solutions of certain ordinary differential equations*, Soviet Math. Dokl., **3** (1962), 649–652.
9. V. A. Kondrat'ev, *Oscillatory properties of solutions of the equation  $y^{(n)} + p(x)y = 0$* , Trudy Moskov. Mat. Obšč., **10** (1961), 419–436.
10. A. Ju. Levin, *Non-oscillation of solutions of the equation  $x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = 0$* , Russian Math. Surveys, **24** (1969), 43–99.
11. Z. Nehari, *Nonlinear techniques for linear oscillation problems*, Trans. Amer. Math. Soc., **210** (1975), 387–406.

12. W. F. Trench, *A sufficient condition for eventual disconjugacy*, Proc. Amer. Math. Soc., **52** (1975), 139–146.
13. D. Willett, *Disconjugacy tests for singular linear differential equations*, SIAM J. Math. Anal., **2** (1971), 536–545.
14. J. S. W. Wong, *On the generalized Emden–Fowler equation*, SIAM Review, **17** (1975), 339–360.

Received March 15, 1976. Research supported by the Mississippi State University Biological and Physical Sciences Research Institute.

MISSISSIPPI STATE UNIVERSITY







# Pacific Journal of Mathematics

Vol. 66, No. 1

November, 1976

Helen Elizabeth. Adams, <i>Factorization-prime ideals in integral domains</i> . . . . .	1
Patrick Robert Ahern and Robert Bruce Schneider, <i>The boundary behavior of Henkin's kernel</i> . . . . .	9
Daniel D. Anderson, Jacob R. Matijevic and Warren Douglas Nichols, <i>The Krull intersection theorem. II</i> . . . . .	15
Efraim Pacillas Armendariz, <i>On semiprime P.I.-algebras over commutative regular rings</i> . . . . .	23
Robert H. Bird and Charles John Parry, <i>Integral bases for bicyclic biquadratic fields over quadratic subfields</i> . . . . .	29
Tae Ho Choe and Young Hee Hong, <i>Extensions of completely regular ordered spaces</i> . . . . .	37
John Dauns, <i>Generalized monoform and quasi injective modules</i> . . . . .	49
F. S. De Blasi, <i>On the differentiability of multifunctions</i> . . . . .	67
Paul M. Eakin, Jr. and Avinash Madhav Sathaye, <i>R-endomorphisms of <math>R[[X]]</math> are essentially continuous</i> . . . . .	83
Larry Quin Eifler, <i>Open mapping theorems for probability measures on metric spaces</i> . . . . .	89
Garret J. Etgen and James Pawlowski, <i>Oscillation criteria for second order self adjoint differential systems</i> . . . . .	99
Ronald Fintushel, <i>Local <math>S^1</math> actions on 3-manifolds</i> . . . . .	111
Kenneth R. Goodearl, <i>Choquet simplexes and <math>\sigma</math>-convex faces</i> . . . . .	119
John R. Graef, <i>Some nonoscillation criteria for higher order nonlinear differential equations</i> . . . . .	125
Charles Henry Heiberg, <i>Norms of powers of absolutely convergent Fourier series: an example</i> . . . . .	131
Les Andrew Karlovitz, <i>Existence of fixed points of nonexpansive mappings in a space without normal structure</i> . . . . .	153
Gangaram S. Ladde, <i>Systems of functional differential inequalities and functional differential systems</i> . . . . .	161
Joseph Michael Lambert, <i>Conditions for simultaneous approximation and interpolation with norm preservation in <math>C[a, b]</math></i> . . . . .	173
Ernest Paul Lane, <i>Insertion of a continuous function</i> . . . . .	181
Robert F. Lax, <i>Weierstrass points of products of Riemann surfaces</i> . . . . .	191
Dan McCord, <i>An estimate of the Nielsen number and an example concerning the Lefschetz fixed point theorem</i> . . . . .	195
Paul Milnes and John Sydney Pym, <i>Counterexample in the theory of continuous functions on topological groups</i> . . . . .	205
Peter Johanna I. M. De Paepe, <i>Homomorphism spaces of algebras of holomorphic functions</i> . . . . .	211
Judith Ann Palagallo, <i>A representation of additive functionals on <math>L^p</math>-spaces, <math>0 &lt; p &lt; 1</math></i> . . . . .	221
S. M. Patel, <i>On generalized numerical ranges</i> . . . . .	235
Thomas Thornton Read, <i>A limit-point criterion for expressions with oscillatory coefficients</i> . . . . .	243
Elemer E. Rosinger, <i>Division of distributions</i> . . . . .	257
Peter S. Shoenfeld, <i>Highly proximal and generalized almost finite extensions of minimal sets</i> . . . . .	265
R. Sirois-Dumais and Stephen Willard, <i>Quotient-universal sequential spaces</i> . . . . .	281
Robert Charles Thompson, <i>Convex and concave functions of singular values of matrix sums</i> . . . . .	285
Edward D. Tymchatyn, <i>Some <math>n</math>-arc theorems</i> . . . . .	291
Jang-Mei Gloria Wu, <i>Variation of Green's potential</i> . . . . .	295