SOME NONOSCILLATION CRITERIA FOR HIGHER ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Sufficient conditions for an \( n \)th order nonlinear differential equation to be nonoscillatory are given. An essential part of the hypotheses is that a related linear equation be disconjugate.

The linear differential equation

\[
x^{(n)} + p(t)x = 0,
\]

where \( p: [t_0, \infty) \to \mathbb{R} \) is continuous, is said to be eventually disconjugate if there exists \( T \geq t_0 \) such that no solution of (1) has more than \( n - 1 \) zeros (counting multiplicities) on \([T, \infty)\). A solution \( x(t) \) of (1) (or equation (2) below) will be called nonoscillatory if there exists \( t_1 \geq t_0 \) such that \( x(t) \neq 0 \) for \( t \geq t_1 \). Equation (1) (or (2)) will be called nonoscillatory if all its solutions are nonoscillatory. Clearly, disconjugacy implies nonoscillation. On the other hand, for \( n = 2, 3 \) or 4 and either \( p(t) > 0 \) or \( p(t) < 0 \), if equation (1) is nonoscillatory, then (1) is eventually disconjugate. Whether this is true for \( n > 4 \) remains an open question (see Nehari [11]).

In this paper we consider the nonlinear differential equation

\[
x^{(n)} + q(t)f(t, x, x', \ldots, x^{(n-1)}) = 0
\]

where \( q: [t_0, \infty) \to \mathbb{R} \) and \( f: [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R} \) are continuous, and obtain some nonoscillation results by making assumptions on the disconjugacy of certain related linear equations. A discussion of disconjugacy criteria for linear differential equations can be found in Coppel [2], Levin [10], Nehari [11], Trench [12], or Willett [13]. For a discussion of nonoscillation criteria for second order nonlinear equations we refer the reader to the recent papers of Coffman and Wong [1], Graef and Spikes [3–5], Wong [14], and the references contained therein. There appears to be no known sufficient conditions for nonoscillation of higher order nonlinear equations.

We will assume that there is a continuous function \( W: [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R} \) such that

\[
|f(t, u_1, \ldots, u_n) - W(t, u_1, \ldots, u_n)| \leq W(t, u_1, \ldots, u_n)|u_1|
\]

for all \((t, u_1, \ldots, u_n) \in [t_0, \infty) \times \mathbb{R}^n\), and
(4) \[ f(t, u_1, \ldots, u_n)/u_1 \to A \text{ as } u_1 \to 0. \]

**Theorem 1.** Suppose that conditions (3) and (4) hold, \( W(t, u_1, \ldots, u_n) \leq B \) and \( M = \max \{ |A|, B \} \). If the equations

(5) \[ x^{(n)} + M |q(t)| x = 0 \]

are eventually disconjugate, then equation (2) is nonoscillatory.

**Proof.** Suppose that equations (5) are disconjugate on \([T, \infty)\) where \( T \geq t_0 \) and let \( x(t) \) be a solution of (1). Define \( Q: [T, \infty) \to \mathbb{R} \) by

\[
Q(t) = \begin{cases} 
q(t)f(t, x(t), \ldots, x^{(n-1)}(t))/x(t), & \text{if } x(t) \neq 0 \\
Aq(t), & \text{if } x(t) = 0.
\end{cases}
\]

It then follows that \( Q(t) \) is continuous and \( x(t) \) is a solution of

(6) \[ x^{(n)} + Q(t)x = 0. \]

Kondrat’ev [9] showed that if \( p_1(t) \leq p_2(t) \) and the equations

\[ x^{(n)} + p_i(t)x = 0, \quad i = 1, 2 \]

are disconjugate on \([T, \infty)\), then for any \( p(t) \) with \( p_1(t) \leq p(t) \leq p_2(t) \) the equation

\[ x^{(n)} + p(t)x = 0 \]

is disconjugate on \([T, \infty)\). Here we have \(|Q(t)| \leq M|q(t)| \) so \(-M|q(t)| \leq Q(t) \leq M|q(t)|\). Hence equation (6) is disconjugate and so \( x(t) \) is nonoscillatory.

**Remark 1.** If \( q(t) \geq 0 \) and \( u_1 f(t, u_1, \ldots, u_n) \geq 0 \), then \( Q(t) \geq 0 \). Since the equation \( x^{(n)} = 0 \) is disconjugate on \([T, \infty)\) for any \( T \geq t_0 \), we would only need to assume that equation (5) with "+" is eventually disconjugate. Note also that condition (4) is only needed to insure that \( Q \) is continuous.

**Remark 2.** Equations (5) are eventually disconjugate if, for example,

\[ \int_{t_0}^{\infty} t^{n-1} |q(t)| \, dt < \infty \]
(see Kiguradze [8], Kondrat'ev [9], or Willett [13]). In this regard we would then have a generalization of a result of Kartsatos [7; Theorem 2].

Willett [13; Theorem 1.4] has shown that if for each $i = 1, 2, \ldots, n$, $p_i : [t_0, \infty) \rightarrow \mathbb{R}$ is continuous and

$$(7) \quad \int_{t_0}^{\infty} t^{-1} |p_i(t)| \, dt < \infty,$$

then the equation

$$(8) \quad x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0$$

is eventually disconjugate. (Recently Gustafson [6] showed that even though nonoscillation implies disconjugacy for equation (8) with $n = 2$, this is not the case for $n > 2$.) Employing the method of proof used above we can obtain that all solutions of

$$(9) \quad x^{(n)} + p_1(t) f_1(x^{(n-1)}) + \cdots + p_n(t) f_n(x) = 0$$

are nonoscillatory.

**Theorem 2.** Suppose that condition (7) holds and there are bounded continuous functions $W_i : [t_0, \infty) \rightarrow \mathbb{R}$, $i = 1, 2, \ldots, n$ such that

$$|f_i(u)| \leq W(u) |u|$$

and

$$f_i(u)/u \rightarrow A_i \quad \text{as} \quad u \rightarrow 0.$$

Then all solutions of (9) are nonoscillatory.

**Proof.** If $x(t)$ is a solution of (9), then $x(t)$ is also a solution of

$$(10) \quad x^{(n)} + Q_1(t)x^{(n-1)} + \cdots + Q_n(t)x = 0$$

where

$$Q_i(t) = \begin{cases} 
 p_i(t)f_i(x^{(n-i)}(t))/x^{(n-1)}(t), & \text{if } x^{(n-i)}(t) \neq 0 \\
 A_i p_i(t), & \text{if } x^{(n-i)}(t) = 0
\end{cases}$$

In addition, for each $i = 1, 2, \ldots, n$
\[
\int_{t_0}^{\infty} t^{-1} |Q_i(t)| \, dt \leq \int_{t_0}^{\infty} t^{-1} \left[ |p_i(t)| / |f_i(x^{(n-i)}(t))| / |x^{(n-i)}(t)| \right] \, dt
\leq \int_{t_0}^{\infty} t^{-1} |p_i(t)| / {\mathcal{W}}(x^{(n-i)}(t)) \, dt
\leq K_i \int_{t_0}^{\infty} t^{-1} |p_i(t)| \, dt
< \infty
\]

where \(K_i\) is a constant. It follows from Willett's theorem that equation (10) is disconjugate and hence \(x(t)\) is nonoscillatory.

Clearly various other forms of equation (9) can be handled in a similar fashion.

As an example of the above results, consider the equation

\[
(11) \quad x^{(n)} + x^3(\sin t)/t^{n+1}(x^2 + 1) = 0.
\]

The corresponding linear equation

\[
(\text{11}) \quad x^{(n)} + x(\sin t)/t^{n+1} = 0
\]

is disconjugate, so equation (11) is nonoscillatory.

References

10. A. Ju. Levin, Non-oscillation of solutions of the equation \(x^{(n)} + p_i(t)x^{(n-i)} + \cdots + p_n(t)x = 0\), Russian Math. Surveys, 24 (1969), 43–90.

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