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**EXISTENCE OF FIXED POINTS OF NONEXPANSIVE  
MAPPINGS IN A SPACE WITHOUT NORMAL STRUCTURE**

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## EXISTENCE OF FIXED POINTS OF NONEXPANSIVE MAPPINGS IN A SPACE WITHOUT NORMAL STRUCTURE

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A mapping  $T: C \rightarrow X$  defined on a subset  $C$  of a Banach space  $X$ , with norm  $\|\cdot\|$ , is said to be *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ . If  $C$  is assumed to be convex and weakly compact and if  $T: C \rightarrow C$  then one of the main open questions is whether  $T$  has a fixed point in  $C$ , i.e., whether there exists  $x \in C$  so that  $Tx = x$ . If  $X$  is reflexive and uniformly convex or, more generally, if  $X$  is reflexive and has normal structure then the answer is affirmative. Our purpose is to give an example of a classical reflexive space which does not have normal structure and for which the answer is nevertheless affirmative.

Nonexpansive mappings have played an important role in recent developments of nonlinear functional analysis (e.g., Browder [1], de-Figueiredo [3]). The analysis of the uniformly convex and normal structure cases is due to Browder [2], Göhde [4] and Kirk [6].

Let  $X_J$  be the space  $l_2$  renormed according to

$$\|x\| = \max\{\|x\|_\infty, \|x\|_2/\sqrt{2}\},$$

where  $\|\cdot\|_\infty$  denotes the  $l_\infty$  norm and  $\|\cdot\|_2$  the  $l_2$  norm. This space originates with R. C. James. A space is said to have *normal structure* if for each bounded convex subset  $C$  consisting of more than one point there is a point  $w \in C$  so that  $\sup\{\|w - y\|: y \in C\} < \sup\{\|x - y\|: x, y \in C\}$ . The set  $\{x: x = x(i), x(i) \geq 0, \|x\|_2 \leq 1\}$  in  $X_J$  contains no such point  $w$ .

**THEOREM.** *Let  $C$  be a bounded closed convex subset of  $X_J$ . Let  $T: C \rightarrow C$  be nonexpansive. Then  $T$  has a fixed point in  $C$ .*

One of the main tools in the proof is the following general lemma which has been used in Karlovitz [5].

**LEMMA.** *Let  $X$  be a Banach space. Let  $C_0$  be a weakly compact convex subset of  $X$ . Let  $T: C_0 \rightarrow C_0$  be nonexpansive. Suppose that  $C_0$  is minimal in the sense that it contains no proper closed convex subset which is*

invariant under  $T$ . Let  $\{x_n\}$  be a sequence of approximate fixed points, i.e.,  $x_n \in C_0$  and  $\|Tx_n - x_n\| \rightarrow 0$ . Then for each  $x \in C_0$

$$(1) \quad \lim_n \|x - x_n\| = \text{diameter } C_0.$$

If  $X$  is reflexive then a standard argument establishes the existence of a subset  $C_0 \subset C$  which is minimal in the sense of the lemma. If  $C_0$  consists of more than one point, another standard argument yields a sequence of approximate fixed points in  $C_0$ . It follows from (1) that  $X$  does not have normal structure. Hence if  $X$  does have normal structure then  $C_0$  is necessarily a singleton and thus a fixed point of  $T$ . This is an alternate proof of the known result. In order to use the lemma to show that in  $X_J$   $C_0$  is necessarily a singleton, we need to make more explicit use of the nonexpansiveness of  $T$  and of the geometry of  $X_J$ . While some modest generalizations will become apparent, the implications for the general reflexive case are not yet clear.

*Proof of the lemma.* Choose  $y \in C_0$  and let  $s = \limsup \|y - x_n\|$ . Let  $D = \{x : x \in C_0, \limsup \|x - x_n\| \leq s\}$ , which is nonempty closed and convex. It is also invariant under  $T$ . For

$$\|Tx - x_n\| \leq \|Tx - Tx_n\| + \|Tx_n - x_n\| \leq \|x - x_n\| + \|Tx_n - x_n\|$$

and  $\|Tx_n - x_n\| \rightarrow 0$ . By the minimality of  $C_0$ ,  $D = C_0$ . Extract a subsequence  $\{x_{n'}\}$  so that  $\lim \|y - x_{n'}\| = s'$  exists. Suppose that there exists  $z \in C_0$  and a subsequence  $\{x_{n''}\}$  of  $\{x_{n'}\}$  so that  $\lim \|z - x_{n''}\| = t$ . Let  $E = \{x : x \in C_0, \limsup \|x - x_{n''}\| \leq \min\{t, s'\}\}$ . Repeating the argument, we find  $E = C_0$ . Hence  $y, z \in E$ , and so  $t = s'$ . Thus for each  $x \in C_0$ ,  $\lim \|x - x_{n'}\|$  exists and equals  $s'$ .

We complete the proof by showing that  $s' = r = \text{diam } C_0$ . From this it follows that  $\|y - x_n\| \rightarrow r$  whenever  $\{\|y - x_{n'}\|\}$  converges. Whence, by boundedness,  $\|y - x_n\| \rightarrow r$  for the entire sequence. By repeating the argument above with  $\{x_{n'}\}$  replaced by the entire sequence  $\{x_n\}$ , we shall have proved (1).

To this end, consider  $F = \{u : u \in C_0, \|u - x\| \leq s' \text{ for each } x \in C_0\}$ .  $F$  is nonempty because we can extract a weakly convergent subsequence, again denoted by  $\{x_{n'}\}$ , with limit  $z$ . Since  $\|x - x_n\| \rightarrow s'$  for each  $x \in C_0$  it follows that  $\|x - z\| \leq s'$  for each  $x \in C_0$ ; hence  $z \in F$ . Now if  $s' < r$  then  $F \subsetneq C_0$ . However, this contradicts the minimality of  $C_0$  because  $F$  is invariant under  $T$ . To see the latter, we first note that as a consequence of minimality closed convex hull  $(TC_0) = C_0$ . Hence if  $u$  is an arbitrary element of  $C_0$  then for given  $\epsilon > 0$  we can choose  $v = \sum_{i=1}^k \lambda_i Tx_i$  with  $x_i \in C_0$ ,  $\lambda_i > 0$ ,  $\sum \lambda_i = 1$  and  $\|u - v\| \leq \epsilon$ . Choose arbitrary  $w$  in  $F$ . Then

$$\begin{aligned} \|Tw - u\| &\leq \|Tw - v\| + \|v - u\| \\ &\leq \sum \lambda_i \|Tw - Tx_i\| + \|v - u\| \\ &\leq \sum \lambda_i \|w - x_i\| + \|v - u\| \leq s' + \epsilon. \end{aligned}$$

Since  $\epsilon > 0$  and  $u \in C_0$  are arbitrary this shows  $Tw \in F$ . Thus  $F$  is invariant under  $T$  and hence  $s' = r$ . This finishes the proof.

*Proof of the theorem.* Given  $x \in X_j$  we represent its components by  $x(j), j = 1, 2, \dots$ . Since  $X_j$  is a renorming of  $l_2$  there exists a component  $x(j)$  so that  $\|x\|_\infty = |x(j)|$ .

By a standard Zorn's lemma argument there exists a closed convex subset  $C_0$  of  $C$  which is invariant under  $T$  and minimal in the sense of the lemma. We propose to show that  $C_0$  consists of a single point. By invariance this is then a fixed point of  $T$ .

We proceed by contradiction. Suppose that  $C_0$  consists of more than one point. We may assume without loss of generality that  $0 \in C_0$ , and we let diameter  $C_0 = r > 0$ . For each  $s, 0 < s < 1$ , we define  $T_s = (1 - s)T$ . Clearly  $T_s: C_0 \rightarrow C_0$  and it is a strict contraction. Hence by the Banach contraction principle there exists a unique  $x_s \in C_0$  so that  $T_s x_s = x_s$ . Thus

$$Tx_s = (1 - s)^{-1}x_s, \text{ for } 0 < s < 1.$$

By the minimality of  $C_0, x_s \neq 0$ . The desired contradiction results from a study of the points  $x_s$ . Several propositions are needed.

PROPOSITION 1. For each  $x \in C_0, \lim_{s \rightarrow 0} \|x - x_s\|_\infty = r$ .

*Proof.* By contradiction. Suppose that for some  $x \neq 0 \in C_0$  and sequence  $\{x_{s_n}\}$  with  $s_n \rightarrow 0$ , denoted by  $\{x_n\}, \|x - x_n\|_\infty \leq r - \delta, \delta > 0$ . Since  $\|0 - x_n\|_\infty \leq \text{diam } C_0 = r$  it follows that  $\|x/2 - x_n\|_\infty \leq r - \delta/2$  for all  $n$ . By the uniform convexity of  $\|\cdot\|_2$ , it follows from  $\|x - x_n\|_2/\sqrt{2}, \|0 - x_n\|_2/\sqrt{2} \leq \text{diam } C_0 = r$  that  $\|x/2 - x_n\|_2/\sqrt{2} \leq r - \tau$  for some  $\tau > 0$ . Hence,  $\|x/2 - x_n\| \leq r - \min\{\tau, \delta/2\}$  for all  $n$ , which contradicts (1) because  $\|Tx_n - x_n\| = s_n(1 - s_n)^{-1}\|x_n\| \rightarrow 0$ .

PROPOSITION 2. For each  $s, 0 < s < 1, \lim_{t \rightarrow s} \|x_t - x_s\| = 0$ .

*Proof.* We denote  $x_s$  by  $x$  and  $x_t$  by  $y$ . Suppose that  $\|x - y\| = \|x - y\|_\infty = |x(k) - y(k)|$ . By nonexpansiveness

$$(2) \quad |(1 - s)^{-1}x(k) - (1 - t)^{-1}y(k)| \leq |x(k) - y(k)|.$$

For  $s \neq t$  it readily follows that  $\operatorname{sgn} x(k) = \operatorname{sgn} y(k) = \sigma = \pm 1$ . Suppose that  $0 < t < s$ . If  $\sigma x(k) > \sigma y(k)$  then

$$(1-s)^{-1}\sigma x(k) - (1-t)^{-1}\sigma y(k) > (1-t)^{-1}(\sigma x(k) - \sigma y(k)),$$

contradicting (2). Hence  $\sigma y(k) \geq \sigma x(k)$ . If  $(1-t)^{-1}\sigma y(k) - (1-s)^{-1}\sigma x(k) \geq 0$  then, by (2),  $(1-s)t(1-t)^{-1}s^{-1}\sigma y(k) \leq \sigma x(k) \leq \sigma y(k)$ . If  $(1-s)^{-1}\sigma x(k) - (1-t)^{-1}\sigma y(k) \geq 0$  then, directly,  $(1-s)(1-t)^{-1}\sigma y(k) \leq \sigma x(k) \leq \sigma y(k)$ . If  $s < t < 1$ , analogous inequalities are derived. It follows that

$$(3) \quad |x(k) - y(k)| \leq \begin{cases} (s-t)s^{-1}(1-t)^{-1}|y(k)|, & 0 < t < s, \\ (t-s)t^{-1}(1-s)^{-1}|x(k)|, & s < t < 1. \end{cases}$$

Hence if  $\|x_t - x_s\| = \|x_s - x_t\|_\infty$  and  $s/2 < t < (1+s)/2$ ,

$$(4) \quad \|x_s - x_t\| \leq A |s - t|, \quad \text{for some } A = A(s) > 0.$$

Now suppose that  $\|x - y\| = \|x - y\|_2/\sqrt{2}$ . By nonexpansiveness

$$\|(1-s)^{-1}x - (1-t)^{-1}y\|_2 \leq \|x - y\|_2.$$

We divide the positive integers according to:

$$I_1 = \{i: |(1-s)^{-1}x(i) - (1-t)^{-1}y(i)| \leq |x(i) - y(i)|\}$$

and

$$I_2 = \{i: |(1-s)^{-1}x(i) - (1-t)^{-1}y(i)| > |x(i) - y(i)|\}.$$

Then

$$(5) \quad \sum_{I_2} [((1-s)^{-1}x(i) - (1-t)^{-1}y(i))^2 - (x(i) - y(i))^2] \leq \sum_{I_1} (x(i) - y(i))^2.$$

By definition, (2) holds for  $k \in I_1$ . We can deduce, as above, that (3) holds. Whence, for  $s/2 < t < (1+s)/2$

$$(6) \quad \sum_{I_1} (x(i) - y(i))^2 \leq B(s-t)^2 \quad \text{for some } B = B(s) > 0.$$

We note the identity

$$(1-t)^{-1}y(i) - (1-s)^{-1}x(i) = (1-s)^{-1}(y(i) - x(i) - \gamma(s,t)y(i)),$$

where  $\gamma(s, t) = (s - t)(1 - t)^{-1}$ . Substitution into (5) yields

$$\sum_{I_2} [(1 - s)^{-2}((y(i) - x(i)) - \gamma(s, t)y(i))^2 - (x(i) - y(i))^2] \cong \sum_{I_1} (x(i) - y(i))^2.$$

By the Schwarz inequality and some simple manipulation

$$\sum_{I_2} (x(i) - y(i))^2 \cong s^{-1}(1 - s)^2 \sum_{I_1} (x(i) - y(i))^2 + 2s^{-1}\gamma(s, t)\|y\|_2\|x - y\|_2.$$

Combining this with (6) we find that if  $\|x_s - x_t\| = \|x_s - x_t\|_2/\sqrt{2}$  and  $s/2 < t < (1 + s)/2$  then

$$(7) \quad \|x_s - x_t\| \cong K(s - t)^{1/2} \quad \text{for some } K = K(s) > 0.$$

The proposition now follows from (4) and (7).

For each positive integer  $i$  and  $\epsilon > 0$  we introduce the notation:

$$A^\epsilon(i) = \{s : 0 < s < 1, |x_s(i)| \cong r - \epsilon\} \quad \text{and} \quad \alpha^\epsilon(i) = \inf A^\epsilon(i).$$

PROPOSITION 3. *For each positive integer  $i$  and  $\epsilon, 0 < \epsilon \cong r/4$ , there exists  $s_1, 0 < s_1 < 1$ , with the property that for each  $s, 0 < s \cong s_1$ , there exists a positive integer  $k(s)$  such that  $k(s) \neq i$  and  $|x_s(k(s))| \cong r - \epsilon$ .*

*Proof.* If  $A^\epsilon(i) = \emptyset$  this follows from Proposition 1 with  $x = 0$ . Otherwise choose  $s_0 \in A^\epsilon(i)$ . Let  $\epsilon_1 = \min\{s_0(1 - s_0)^{-1}(r - \epsilon), \epsilon/2, rs_0/2, s_0(1 - s_0)^{-1}\epsilon/2\}$ . By Proposition 1 choose  $s_1$  so that  $\|x_{s_0} - x_s\|_\infty \cong r - \epsilon_1$  for all  $0 < s \cong s_1$ . By virtue of  $Tx_{s_0} = (1 - s_0)^{-1}x_{s_0} \in C_0$  we have  $\|x_{s_0}\|_\infty \cong r(1 - s_0)$ . Choose  $s, 0 < s \cong s_1$ . Suppose that  $\text{sgn}x_{s_0}(i) = \text{sgn}x_s(i)$  or  $x_s(i) = 0$ . Then from  $3r/4 \cong |x_{s_0}(i)| \cong r(1 - s_0)$  we deduce  $|x_{s_0}(i) - x_s(i)| \cong r - rs_0, r/4 < r - \epsilon_1$ . If  $\text{sgn}x_{s_0}(i) = -\text{sgn}x_s(i)$  then

$$\begin{aligned} r &\cong \|Tx_{s_0} - Tx_s\| \cong |(1 - s_0)^{-1}x_{s_0}(i) - (1 - s)^{-1}x_s(i)| > |x_{s_0}(i) - x_s(i)| \\ &\quad + s_0(1 - s_0)^{-1}|x_{s_0}(i)| \cong |x_{s_0}(i) - x_s(i)| \\ &\quad + s_0(1 - s_0)^{-1}(r - \epsilon) \cong |x_{s_0}(i) - x_s(i)| + \epsilon_1, \end{aligned}$$

and hence  $|x_{s_0}(i) - x_s(i)| < r - \epsilon_1$ . Thus there exists a positive integer  $j \neq i$  so that  $\|x_{s_0} - x_s\|_\infty = |x_{s_0}(j) - x_s(j)| \cong r - \epsilon_1$ . We assert that  $k(s) = j$  satisfies the proposition. If  $\text{sgn}x_{s_0}(j) = \text{sgn}x_s(j)$  then  $r - \epsilon_1 \cong |x_{s_0}(j) - x_s(j)| < \max\{|x_{s_0}(j)|, |x_s(j)|\}$ . Since  $|x_{s_0}(j)| \cong r(1 - s_0) < r - \epsilon_1$  it follows that  $|x_s(j)| > r - \epsilon_1 > r - \epsilon$ , as desired. If  $\text{sgn}x_{s_0}(j) \neq \text{sgn}x_s(j)$ , then

$$\begin{aligned}
r &\geq \|Tx_{s_0} - Tx_s\| \geq |(1-s_0)^{-1}x_{s_0}(j) - (1-s)^{-1}x_s(j)| = (1-s_0)^{-1}|x_{s_0}(j)| \\
&\quad + (1-s)^{-1}|x_s(j)| \geq |x_{s_0}(j)| + |x_s(j)| + s_0(1-s_0)^{-1}|x_{s_0}(j)| \\
&= |x_{s_0}(j) - x_s(j)| + s_0(1-s_0)^{-1}|x_{s_0}(j)| \geq r - \epsilon_1 + s_0(1-s_0)^{-1}|x_{s_0}(j)|.
\end{aligned}$$

Hence  $|x_{s_0}(j)| \leq s_0^{-1}(1-s_0)\epsilon_1$ . So  $|x_{s_0}(j) - x_s(j)| \geq r - \epsilon_1$  implies  $|x_s(j)| \geq r - \epsilon_1 - s_0^{-1}(1-s_0)\epsilon_1 \geq r - \epsilon_1 - \epsilon/2 \geq r - \epsilon$ , as desired. Since  $s \leq s_1$  was arbitrarily chosen this finishes the proof.

**PROPOSITION 4.** *Suppose  $\alpha^\epsilon(i) = \alpha^\delta(j) = 0$  for some  $\epsilon, \delta$  with  $0 < \epsilon, \delta \leq r/64$ . Then  $i = j$ .*

*Proof.* For  $i$  and  $\epsilon$  we choose  $s_1$  according to Proposition 3. Thus if  $s \in A^\epsilon(i)$  and  $s \leq s_1$  then  $|x_s(m)| \geq r - \epsilon$  for  $m = i, k(s)$ . Since  $i \neq k(s)$  and  $\|x_s\|_2^2 \leq 2r^2$  we readily find that  $|x_s(m)| \leq r/4$  for  $m \neq i, k(s)$ . By Proposition 1 we choose  $s_2, s_3 \in A^\epsilon(i)$ ,  $s_2, s_3 < s_1$ , so that  $\|x_{s_p} - x_{s_q}\|_\infty \geq r - \epsilon$  for  $p \neq q$  and  $p, q = 1, 2, 3$ . Now suppose that  $k(s_1) = k(s_2)$ . Then  $|x_{s_1}(m) - x_{s_2}(m)| \leq |x_{s_1}(m)| + |x_{s_2}(m)| \leq r/2 < r - \epsilon$  for  $m \neq i, k(s_1)$ . Moreover,  $\text{sgn}x_{s_1}(i) = \text{sgn}x_{s_2}(i)$ ; otherwise  $\|x_{s_1} - x_{s_2}\|_\infty \geq |x_{s_1}(i)| + |x_{s_2}(i)| \geq 2r - 2\epsilon > r$ . Thus  $|x_{s_1}(i) - x_{s_2}(i)| \leq r - (r - \epsilon) = \epsilon$ . By the same argument  $|x_{s_1}(k(s_1)) - x_{s_2}(k(s_1))| \leq \epsilon$ . Thus  $|x_{s_1}(i) - x_{s_2}(i)| < r - \epsilon$  for all positive integers  $i$ , which is a contradiction. Hence  $k(s_1) \neq k(s_2)$ . Similarly  $k(s_3) \neq k(s_1), k(s_2)$ . Now if  $i \neq j$  we repeat the argument and find  $t_1, t_2, t_3 \in A^\delta(j)$  so that  $|x_{t_p}(j)|, |x_{t_p}(k(t_p))| \geq r - \delta$ ,  $p = 1, 2, 3$  and so that  $j, k(t_1), k(t_2)$  and  $k(t_3)$  are disjoint. Thus we can find  $s_q$  and  $t_p$  so that  $\{i, k(s_q)\} \cap \{j, k(t_p)\} = \emptyset$ . Then from  $|x_{s_q}(i)|, |x_{s_q}(k(s_q))| \geq r - \epsilon, |x_{t_p}(j)|, |x_{t_p}(k(t_p))| \geq r - \delta$  and  $\|x_{t_p}\|_2, \|x_{s_q}\|_2 \leq \sqrt{2}r$  it follows that  $\|x_{s_q} - x_{t_p}\|_{2/\sqrt{2}} > r$  which contradicts  $x_{s_q}, x_{t_p} \in C_0$ . Hence  $i = j$ .

*Completion of the proof of the theorem.* Let  $\epsilon = r/128$ . If there exists a positive integer  $i$  so that  $\alpha^\epsilon(i) = 0$ , let  $i_0 = i$ . Otherwise  $\alpha^\epsilon(i) > 0$  for each  $i$  and we let  $i_0 = 1$ . Apply Proposition 3 to find  $s_1 = s_1(i_0, \epsilon)$ . In the sequel the positive integers  $k(\cdot)$  will be those given by the proposition for this  $s_1$ . Denote  $k(s_1)$  by  $k_1$ . Let  $s_2 = \alpha^\epsilon(k_1)$ . If  $\alpha^\epsilon(i_0) = 0$  it follows from  $i_0 \neq k_1$  and Proposition 4 that  $s_2 > 0$ ; otherwise  $s_2 > 0$  by hypothesis. By Proposition 2  $\|x_{s_2-\mu} - x_{s_2}\| \rightarrow 0$  as  $\mu \rightarrow 0$ . Hence we can choose  $\mu > 0$  so that  $r - 2\epsilon \leq |x_{s_2-\mu}(k_1)|$ . Since  $s_2 - \mu < s_2 < s_1$ ,  $k(s_2 - \mu)$  is well defined. Since  $s_2 - \mu < \alpha^\epsilon(k_1)$ ,  $|x_{s_2-\mu}(k_1)| < r - \epsilon$ ; and hence  $k(s_2 - \mu) \neq k_1$ . Denote  $x_{s_2-\mu}$  by  $y$  and  $k(s_2 - \mu)$  by  $k_2$ . Thus  $|y(k_1)|, |y(k_2)| \geq r - 2\epsilon$ . Since  $k_1, k_2 \neq i_0$ , reasoning as above,  $\alpha^\epsilon(k_1), \alpha^\epsilon(k_2) > 0$ . Hence we can choose  $s_3, 0 < s_3 < \alpha^\epsilon(k_1), \alpha^\epsilon(k_2)$ . Then  $|x_{s_3}(k_1)|, |x_{s_3}(k_2)| < r - \epsilon$ , and hence  $k_3 = k(s_3) \neq k_1, k_2$ . Repeating the argument we find  $z = x_{s_3-\eta}, \eta > 0$ , and  $k_4 = k(s_3 - \eta) \neq k_3$  so that  $|z(k_3)| \geq r - 2\epsilon$  and  $|z(k_4)| \geq$

$r - \epsilon$ . Moreover,  $s_3 - \eta < \alpha^\epsilon(k_1), \alpha^\epsilon(k_2)$ , hence  $k_4 \neq k_1, k_2$ . Thus  $k_1, k_2, k_3$  and  $k_4$  are disjoint. Hence from  $\|y\|, \|z\| \leq r$  and  $|y(k_1)|, |y(k_2)|, |z(k_3)|, |z(k_4)| \geq r - 2\epsilon$  we readily calculate  $\|y - z\|_2/\sqrt{2} > r$  which contradicts  $y, z \in C_0$ . This contradiction proves that  $C_0$  cannot consist of more than one point and finishes the proof of the theorem.

*Added in proof.* Additional applications of the lemma as well as some of its points of contact with work of M. Edelstein will be discussed elsewhere. P. M. Fitzpatrick has informed us that he independently developed the lemma.

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# Pacific Journal of Mathematics

Vol. 66, No. 1

November, 1976

Helen Elizabeth. Adams, <i>Factorization-prime ideals in integral domains</i> .....	1
Patrick Robert Ahern and Robert Bruce Schneider, <i>The boundary behavior of Henkin's kernel</i> .....	9
Daniel D. Anderson, Jacob R. Matijevic and Warren Douglas Nichols, <i>The Krull intersection theorem. II</i> .....	15
Efraim Pacillas Armendariz, <i>On semiprime P.I.-algebras over commutative regular rings</i> .....	23
Robert H. Bird and Charles John Parry, <i>Integral bases for bicyclic biquadratic fields over quadratic subfields</i> .....	29
Tae Ho Choe and Young Hee Hong, <i>Extensions of completely regular ordered spaces</i> .....	37
John Dauns, <i>Generalized monoform and quasi injective modules</i> .....	49
F. S. De Blasi, <i>On the differentiability of multifunctions</i> .....	67
Paul M. Eakin, Jr. and Avinash Madhav Sathaye, <i>R-endomorphisms of <math>R[[X]]</math> are essentially continuous</i> .....	83
Larry Quin Eifler, <i>Open mapping theorems for probability measures on metric spaces</i> .....	89
Garret J. Etgen and James Pawlowski, <i>Oscillation criteria for second order self adjoint differential systems</i> .....	99
Ronald Fintushel, <i>Local <math>S^1</math> actions on 3-manifolds</i> .....	111
Kenneth R. Goodearl, <i>Choquet simplexes and <math>\sigma</math>-convex faces</i> .....	119
John R. Graef, <i>Some nonoscillation criteria for higher order nonlinear differential equations</i> .....	125
Charles Henry Heiberg, <i>Norms of powers of absolutely convergent Fourier series: an example</i> .....	131
Les Andrew Karlovitz, <i>Existence of fixed points of nonexpansive mappings in a space without normal structure</i> .....	153
Gangaram S. Ladde, <i>Systems of functional differential inequalities and functional differential systems</i> .....	161
Joseph Michael Lambert, <i>Conditions for simultaneous approximation and interpolation with norm preservation in <math>C[a, b]</math></i> .....	173
Ernest Paul Lane, <i>Insertion of a continuous function</i> .....	181
Robert F. Lax, <i>Weierstrass points of products of Riemann surfaces</i> .....	191
Dan McCord, <i>An estimate of the Nielsen number and an example concerning the Lefschetz fixed point theorem</i> .....	195
Paul Milnes and John Sydney Pym, <i>Counterexample in the theory of continuous functions on topological groups</i> .....	205
Peter Johanna I. M. De Paepe, <i>Homomorphism spaces of algebras of holomorphic functions</i> .....	211
Judith Ann Palagallo, <i>A representation of additive functionals on <math>L^p</math>-spaces, <math>0 &lt; p &lt; 1</math></i> .....	221
S. M. Patel, <i>On generalized numerical ranges</i> .....	235
Thomas Thornton Read, <i>A limit-point criterion for expressions with oscillatory coefficients</i> .....	243
Elemer E. Rosinger, <i>Division of distributions</i> .....	257
Peter S. Shoenfeld, <i>Highly proximal and generalized almost finite extensions of minimal sets</i> .....	265
R. Sirois-Dumais and Stephen Willard, <i>Quotient-universal sequential spaces</i> .....	281
Robert Charles Thompson, <i>Convex and concave functions of singular values of matrix sums</i> .....	285
Edward D. Tymchatyn, <i>Some <math>n</math>-arc theorems</i> .....	291
Jang-Mei Gloria Wu, <i>Variation of Green's potential</i> .....	295