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A REPRESENTATION OF ADDITIVE FUNCTIONALS ON L^p-SPACES, $0<p<1$

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Mizel and Sundaresan have given an integral representation for a class of nonlinear functionals, called additive functionals, on the Banach spaces L^p, $p \geqq 1$. In this paper, analogous results for these additive functionals on the spaces L^p, $0<p<1$, are presented. The convergence of additive functionals is also investigated whenever three types of convergence are imposed on the members of L^p: almost everywhere convergence, convergence in measure, and convergence in the metric d, where $d(x,y)=\int |x-y|^p d\mu$. In all three cases an integral representation for the functional is obtained, and necessary and sufficient conditions are given for the continuity of the functional.

Introduction. A topological vector space is an F-space if its topology is induced by a complete invariant metric. This definition corresponds to that of Rudin **[16]** and does not assume local convexity. In fact, if (T,β,μ) is a finite measure space, $L^p = \left\{x: \int_T |x|^p d\mu < \infty\right\}$, with the metric $d(x,y)=\int_T |x-y|^p d\mu$ for $0<p<1$, is an example of an F-space which is not locally convex. Hence, on L^p there need be no continuous linear functionals. It is known (cf. **[3]**) that $(L^p)^*$ separates the elements of L^p if, and only if, μ is atomic. However, even in this situation, L^p is not locally convex.

Recall that if (T,β,μ) is a σ-finite measure space, then $T = A \cup N$, where A is atomic and N is nonatomic. Since there are no continuous linear functionals supported on N, it is natural to ask what types of functionals do exist on N. This has led to an investigation of nonlinear functionals. Representations of nonlinear functionals on various Banach spaces have been extensively studied (cf. 1, 2, 5, 6, 8, 9, 10, 11, 12, 17, 18, 19). In particular, Martin and Mizel **[8]** have given an integral representation for a class of nonlinear functionals, called additive functionals. A functional F is *additive* if it has the following properties:

(1) F is additive on functions of disjoint support: $F(x+y)=F(x)+F(y)$, provided the intersection of the supports of x and y is of measure zero;

(2) F is statistical: If x and y are two real-valued functions on T which are equimeasurable, then $F(x)=F(y)$. Two real-valued func-

tions x and y on T are said to be *equimeasurable* if for every Borel set S on $(-\infty,\infty)$, $x^{-1}(S)$ and $y^{-1}(S)$ are measurable and have equal measure.

The major result in [8] is the following theorem which is also basic to this development:

THEOREM A. *Let (T,μ) be a nonatomic measure space of finite measure, $\mu T\neq 0$. Let L^∞ denote the set of all essentially bounded, real-valued measurable functions on T, and F a finite, real-valued, additive functional on L^∞. Then there exists a unique continuous real-valued function f on $(-\infty,\infty)$ for which $f(0)=0$ and*

$$F(x)=\int_T f(x(t))d\mu(t),$$

for all x in L^∞.

If T is discrete and $x\in l^\infty(T)$, then $F(x)$ has the representation $F(x)=\Sigma_{t\in T}f(x(t))\mu(t)$.

A similar integral representation has been given by Mizel and Sundaresan [11] for additive functionals on the Banach spaces L^p, $p\geqq 1$. We are concerned with obtaining an analogous result for the spaces L^p, $0<p<1$. To this end we study the convergence of the additive functional F whenever three types of convergence are imposed on the members of L^p: almost everywhere convergence, convergence in measure, and convergence in the metric d, where $d(x,y)=\int_T |x-y|^p d\mu$. In all three cases we give necessary and sufficient conditions for the continuity of F and obtain the integral representation for F.

The discussion is divided into two main parts: the finite case where $\mu T<\infty$ and the σ-finite case where $\mu T=\infty$.

1. The finite non-atomic case. On the spaces $L^p(T)$, $0<p<1$, we define a function d by

$$d(x,y)=\int_T |x(t)-y(t)|^p d\mu(t).$$

The function d is a complete, invariant metric on $L^p(T)$, when $\mu T<\infty$. [4]

The principal result of this section is Theorem 1.1 which gives necessary and sufficient conditions for an additive functional F on L^p to be continuous with respect to the metric d; that is, $d(x_n,x)\to 0$ implies $F(x_n)\to F(x)$.

Before stating Theorem 1.1, we recall three lemmas that are used in the proof of the theorem. The first is the Vitali convergence theorem [4, p. 150] for $p = 1$.

LEMMA 1.1 *Let (T, μ) be a measure space and $\{f_n\}$ be a sequence of functions in $L^1(T)$ converging almost everywhere to a function f. Then $f \in L^1(T)$ and $|f_n - f|_1 = \int_T |f_n - f|\, d\mu$ converges to zero if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that if E is a measurable subset of T and $\mu E < \delta$, then $\int_E |f_n|\, d\mu < \epsilon$ for all $n \geqq 1$.*

LEMMA 1.2. *If T is a measure space, then the simple functions in $L^p(T)$ are dense in $L^p(T)$ with respect to the metric d: given $\epsilon > 0$ and $x \in L^p(T)$, there exists a simple function ϕ such that $d(x, \phi) < \epsilon$.*

LEMMA 1.3. *If there is a sequence $\{x_n\}$ in $L^p(T)$ such that $d(x_n, x)$ converges to zero, then for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $\mu E < \delta$, then $\int_E |x_n|^p d\mu < \epsilon$ for all $n \geqq 1$.*

THEOREM 1.1. *Let (T, μ) be a nonatomic finite measure space, $\mu T > 0$. Let F be an additive functional on L^p. Then F satisfies the continuity condition*

$$d(x_n, x) \to 0 \quad \Rightarrow \quad F(x_n) \to F(x)$$

if and only if there exists a continuous function f: $R \to R$ such that $f(0) = 0$ and $|f(r)| \leqq k(1 + |r|)^p$ for all $r \in R$ and some $k \geqq 0$, and $F(x) = \int_T (f \circ x) d\mu$ for all x in L^p.

Proof. To establish necessity, suppose that F is continuous on L^p. Then the restriction F_1 of F to L^∞ is an additive functional on L^∞. Thus by Theorem A there exists a unique continuous function f: $R \to R$ such that $f(0) = 0$ and $F_1(x) = \int_T f(x(t))d\mu$, for all x in L^∞. We claim that f satisfies the condition $|f(r)| \leqq k(1 + |r|)^p$ for all $r \in R$. Suppose not. Then there exists a sequence $\{r_n\}$ in R such that $|f(r_n)| \geqq n(1 + |r_n|)^p$. Let $\{E_n\}$ be a sequence of measurable sets in T such that $\mu E_n \doteq (1\backslash|f(r_n)|)\mu T$. Then

$$\begin{aligned} d(r_n\chi_{E_n}, 0) &= \int_T |r_n\chi_{E_n}|^p d\mu = \int_{E_n} |r_n|^p d\mu = |r_n|^p \mu E_n \\ &= |r_n|^p \left[\frac{\mu T}{|f(r_n)|}\right] \leqq |r_n|^p \left[\frac{\mu T}{n(1+|r_n|)^p}\right] \\ &= \frac{1}{n}\left[\frac{|r_n|}{1+|r_n|}\right]^p \mu T < \frac{1}{n}\mu T \to 0 \end{aligned}$$

as $n \to \infty$. However,

$$F(r_n\chi_{E_n}) = \int_T (f \circ r_n\chi_{E_n}) d\mu = f(r_n)\mu E_n = \pm \mu T \neq 0.$$

This contradicts the continuity of F. Hence f must satisfy the desired condition.

Next we need to verify that the functional F has the given representation. First we show that $f \circ x$ is in L^1 if $x \in L^p$. We know that $|f(r)| \leqq k(1+|r|)^p$ for all $r \in R$ and some $k \geqq 0$. Choose $K > k$ so that $(K^{1/p} - k^{1/p}) \geqq 1$ and set $c = k^{1/p}(K^{1/p} - k^{1/p})$. Whenever $|r| > c$,

$$\begin{aligned} (K^{1/p} - k^{1/p})|r| &> k^{1/p}, \\ K^{1/p}|r| &> k^{1/p}(1+|r|), \end{aligned}$$

and thus $K|r|^p \geqq k(1+|r|)^p \geqq |f(r)|$. Let $A_1 = \{t \in T\colon |x(t)| < c\}$ and $A_2 = T - A_1$. For $t \in A_1$, $|f(x(t))| \leqq k(1+|x(t)|)^p \leqq k(1+c)^p$, so $f \circ x$ is bounded on A_1 and

$$\int_{A_1} |f \circ x| d\mu \leqq k(1+c)^p \mu_{A_1} < \infty.$$

Also, $|f(x\chi_{A_2})| \leqq K|x\chi_{A_2}|^p$ so

$$\int_T |f(x\chi_{A_2})| d\mu \leqq K \int_T |x\chi_{A_2}|^p d\mu < \infty$$

since $x\chi_{A_2} \in L^p$. Thus

$$\begin{aligned} \int_T |f \circ x| d\mu &= \int_{A_1} |f \circ x| d\mu + \int_{A_2} |f \circ x| d\mu \\ &= \int_{A_1} |f \circ x| d\mu + \int_T |f \circ x\chi_{A_2}| d\mu < \infty \end{aligned}$$

and $f \circ x$ is in L^1.

Now let x be in L^p. By Lemma 1.2 the simple functions are dense in L^p. Thus there exists a sequence $\{x_n\}$ of simple functions such that $d(x_n, x)$ converges to zero. Then there is a subsequence $\{x_{n_j}\}$ such that x_{n_j} converges to x almost everywhere. Thus we could have chosen the sequence $\{x_n\}$ of simple functions so that it converged to x in the metric d and almost everywhere. Since the functions x_n are simple, they are in L^∞. Also F is continuous, so

$$F(x) = \lim F(x_n) = \lim \int_T (f \circ x_n) d\mu.$$

We must show that $\lim \int_T (f \circ x_n) d\mu = \int_T (f \circ x) d\mu$ or that $f(x_n)$ converges to $f(x)$ in L^1. By Lemma 1.1 it suffices to show that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $\mu E < \delta$, then $\int_E |f(x_n)| d\mu < \epsilon$ for all n.

Since f is continuous and x_n converges to x almost everywhere, $f(x_n)$ converges to $f(x)$ almost everywhere. Also there exist constants c and K such that $|f(t)| \leqq K|t|^p$ whenever $|t| > c$. Let $K_1 = \sup\{|f(t)|: |t| \leqq c\}$. The sequence $\{x_n\}$ of simple functions is in L^p. By Lemma 1.3, for $\epsilon > 0$ there exists a $\delta_1 > 0$ such that if $\mu E < \delta_1$, then $\int_E |x_n|^p d\mu < \epsilon/2K$ for all $n \geqq 1$. Given $\epsilon > 0$, choose $\delta < \min(\delta_1, \epsilon/2K_1)$. Then with $\mu E < \delta$,

$$\int_E |f(x_n)| d\mu = \int_{E \cap \{t: |x_n(t)| \leqq c\}} |f(x_n)| d\mu + \int_{E \cap \{t \cdot |x_n(t)| > c\}} |f(x_n)| d\mu$$
$$< K_1 \delta + \frac{K\epsilon}{2K} < \epsilon$$

for all $n \geqq 1$. Thus, by Lemma 1.1, $f(x_n)$ converges to $f(x)$ in L^1; so $F(x) = \lim F(x_n) = \lim \int_T (f \circ x_n) d\mu = \int_T f(x) d\mu$.

Conversely, if $f: R \to R$ is a continuous function satisfying $f(0) = 0$ and $|f(r)| \leqq k(1 + |r|)^p$ for all $r \in R$ and some $k \geqq 0$, then the functional $F(x) = \int_T f(x) d\mu$ is well-defined and additive on $L^p(T)$. Let $\{x_n\}$ be any sequence in $L^p(T)$ such that $d(x_n, x)$ converges to zero as $n \to \infty$. Then x_n converges in measure to x; that is, given $\delta > 0$ there exists an integer N such that for $n \geqq N$, $\mu\{t: |x_n(t) - x(t)| \geqq \delta\} < \delta$. Since f is a continuous function, it must follow that $f(x_n)$ converges to $f(x)$ in measure. By hypothesis, $|f(x_n)| \leqq k(1 + |x_n|)^p$, so

$$\int_T |f(x_n)|\, d\mu \leqq k \int_T (1+|x_n|)^p d\mu \leqq k\mu T + k\int_T |x_n|^p d\mu < \infty$$

since $\mu T < \infty$ and $x_n \in L^p(T)$. Thus $f(x_n)$ is in $L^1(T)$. If the functions $\{f(x_n)\}$ have equiabsolutely continuous integrals, then $\lim_{n\to\infty}\int_T f(x_n)d\mu = \int_T f(x)d\mu$ [**14**, p. 152]. Now assume for any measurable set $E \subset T$ that $\mu E < \delta$. Then

$$\left|\int_E f(x_n)d\mu\right| \leqq \int_E |f(x_n)|\, d\mu \leqq k\int_E (1+|x_n|^p)d\mu.$$

By Lemma 1.3, for each $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $\mu E < \delta$, $\int_E |x_n|^p d\mu < \epsilon$. Thus

$$\left|\int_E f(x_n)d\mu\right| < k\cdot\delta + k\cdot\epsilon$$

and the functions $\{f(x_n)\}$ have equiabsolutely continuous integrals. Therefore,

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty}\int_T f(x_n)d\mu = \int_T f(x)d\mu = F(x).$$

In Theorem 1.2 we examine the convergence of $F(x_n)$ when the sequence $\{x_n\}$ in L^p converges to x almost everywhere.

THEOREM 1.2. *Let (T, μ) be a nonatomic finite measure space, $\mu T \neq 0$, and let F be an additive functional on $L^p(T)$. Then F satisfies the continuity condition*

x_n converges to x almost everywhere $\Rightarrow$ $F(x_n)$ converges to $F(x)$

if and only if there exists a continuous function f: $R \to R$ such that $f(0) = 0$, f is bounded, and $F(x) = \int_T (f\circ x)d\mu$, for all x in L^p.

Proof. Let f be as described and consider F: $L^p(T) \to R$ defined by

$$F(x) = \int_T (f\circ x)d\mu.$$

If $\{x_n\}$ is a sequence in L^p and $x \in L^p$ such that x_n converges to x almost everywhere, then $f(x_n)$ converges to $f(x)$ almost everywhere since f is continuous. Then, since f is bounded, we can apply the bounded convergence theorem and get that

$$F(x) = \int_T (f \circ x) d\mu = \int_T \lim (f \circ x_n) d\mu$$
$$= \lim \int_T (f \circ x_n) d\mu = \lim F(x_n).$$

Conversely, suppose F is an additive functional satisfying the continuity requirement. If, as in Theorem 1.1, we let $F_1 = F | L^\infty(T)$, then F_1 is an additive functional on L^∞. Thus by Theorem A there exists a unique continuous function $f\colon R \to R$ such that $f(0) = 0$ and for all x in L^∞, $F(x) = \int_T (f \circ x) d\mu$. Next we need that f is bounded. Suppose not. Then there exists a sequence $\{r_n\}$ of real numbers such that $|r_n| \to \infty$ and $1 \leqq |f(r_n)| \to \infty$. Since T is nonatomic, there exists a sequence of measurable sets $\{E_n\}$ such that $\mu E_n = \mu T / |f(r_n)|$. Then $\mu E_n \to 0$ as $n \to \infty$. Let $x_n = r_n \chi_{E_n}$. For each n, $x_n \in L^p$ and there is a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to 0$ almost everywhere. But x_{n_k} is a bounded measurable function, and, therefore we can write

$$F(x_{n_k}) = \int_T (f \circ x_{n_k}) d\mu = \int_{E_{n_k}} f(r_{n_k}) d\mu$$
$$= f(r_{n_k}) \left[\frac{\mu T}{|f(r_{n_k})|} \right] = \pm \mu T < \infty.$$

Thus we have that $F(x_{n_k})$ does not approach zero even though x_{n_k} converges to zero almost everywhere. We have reached a contradiction, so the f must be bounded.

Now let x be any function in L^p. Then by Lemma 1.2 there exists a sequence $\{\phi_n\}$ of simple functions such that $d(x, \phi_n)$ converges to zero. Hence there is a subsequence $\{\phi_{n_k}\}$ such that ϕ_{n_k} converges to x almost everywhere. By the continuity condition on F, $F(\phi_{n_k})$ converges to $F(x)$. Since f is continuous $f(\phi_{n_k})$ converges to $f(x)$ almost everywhere. Finally, since f is bounded, $f \circ x$ is in L^1 and we can apply the bounded convergence theorem to get

$$\int_T (f \circ x) d\mu = \lim \int_T (f \circ \phi_{n_k}) d\mu = \lim F(\phi_{n_k}) = F(x).$$

In Theorem 1.3 the sequence $\{x_n\}$ in L^p converges in measure to x. The results are analogous to those in the previous theorem, and, in fact, the necessity condition follows directly from Theorem 1.2 [cf. **19**].

THEOREM 1.3. *Let (T, μ) be a nonatomic finite measure space and let F be an additive functional on L^p. Then F satisfies the continuity condition*

$$x_n \to x \textit{ in measure } \Rightarrow F(x_n) \to F(x)$$

if and only if there exists a continuous function $f\colon R \to R$ *such that* $f(0) = 0$, f *is bounded, and* $F(x) = \int_T (f \circ x)d\mu$, *for all* $x \in L^p$.

Proof. On a finite measure space, almost everywhere convergence implies convergence in measure; that is, if $x_n \to x$ almost everywhere, then for $\epsilon > 0$ there exists an integer N such that for $n \geqq N$, $\mu\{t\colon |x_n(t) - x(t)| \geqq \epsilon\} < \epsilon$. Hence if the given continuity condition holds for F, then $x_n \to x$ almost everywhere implies that $F(x_n)$ converges to $F(x)$. Hence we can apply Theorem 1.1 and conclude that there exists a continuous function $f\colon R \to R$ satisfying the stated conditions.

Conversely, suppose f exists as described and there is a sequence $\{x_n\}$ in L^p such that $x_n \to x$ in measure. We must show that $\int_T (f \circ x_n)d\mu = F(x_n)$ converges to $F(x) = \int_T (f \circ x)d\mu$. Since f is bounded and $\mu T < \infty$, $f(x_n)$ and $f(x)$ are in L^1. Since $x \in L^p$,

$$\mu\{t\colon |x(t)| > N\} \to 0 \text{ as } N \to \infty.$$

We will choose $\delta < 1$ and partition T into three sets as follows:

$$\begin{aligned}
A_1 &= \{t\colon |x(t)| \leqq N \text{ and } |x_n(t) - x(t)| < \delta\},\\
A_2 &= \{t\colon |x(t)| > N \text{ and } |x_n(t) - x(t)| < \delta\}, \quad \text{and}\\
A_3 &= \{t\colon |x_n(t) - x(t)| \geqq \delta\}.
\end{aligned}$$

For $\epsilon > 0$ choose N such that $2\|f\|\mu A_2 < \epsilon/3$, where $\|f\| = \sup_{t\in T} |f(t)|$.

The function f is continuous and thus is uniformly continuous on $[-(N+1), N+1]$; so there exists δ' such that if s, t are in the interval $[-(N+1), N+1]$ and $|s - t| < \delta'$, then $|f(s) - f(t)| < \epsilon/3\mu T$. Since x_n converges to x in measure, there exists n_0 such that for $n > n_0$, $\mu A_3 < \epsilon/6\|f\|$. So for $n > n_0$ and $\delta < \delta'$, we have

$$\begin{aligned}
|F(x) - F(x_n)| &\leqq \int_T |f(x) - f(x_n)|\, d\mu = \int_{A_1} |f(x) - f(x_n)|\, d\mu\\
&\quad + \int_{A_2} |f(x) - f(x_n)|\, d\mu + \int_{A_3} |f(x) - f(x_n)|\, d\mu\\
&\leqq \frac{\epsilon}{3\mu T}\mu A_1 + 2\|f\|\mu A_2 + 2\|f\|\mu A_3 < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

So $F(x_n)$ converges to $F(x)$.

2. The σ-finite nonatomic case. In this section we consider additive functionals on $L^p(T)$ where (T, μ) is a σ-finite measure space: $T = \bigcup_{i=1}^{\infty} T_i$, $T_i \cap T_j = \emptyset$ for $i \neq j$, and $\mu T = \infty$ but $\mu T_i < \infty$ for every i. The results in Theorems 2.2 and 2.3 are similar to those for a space of finite measure. The next theorem, however, is quite different from its analogue given in Theorem 1.2.

THEOREM 2.1. *Let (T, μ) be a nonatomic σ-finite measure space, $\mu T = \infty$. Let F be an additive functional on L^p. Then F satisfies the continuity condition*

$$x_n \to x \text{ almost everywhere } \Rightarrow F(x_n) \to F(x)$$

if and only if $F \equiv 0$.

Proof. Let x_1 be an element of L^p which is nonzero and has finite support: $x_1 = x\chi_{A_1}$, where $\mu A_1 < \infty$. Since $\mu T = \infty$ and T is nonatomic, we can construct a sequence $\{x_n\}$ of functions in L^p in the following way: Let $\{A_n\}_{n \geqq 1}$ be a sequence of pairwise disjoint measurable sets with $\mu A_1 = \mu A_n$, for $n = 2, 3, \cdots$; then (cf. [**11**, p. 108]) let $x_n = x_n \chi_{A_n}$ in such a way that x_n and x_1 are equimeasurable functions. Since x_1 has finite support, the sequence $\{x_n\}$ must converge to zero almost everywhere. The functions are equimeasurable, so $F(x_1) = F(x_n) = 0$. But x_1 is an arbitrary function with finite support. So for any function x in L^p with finite support, $F(x) = 0$. Let f be any function in L^p. Then there exists a sequence of functions $\{f_n\}$ with finite support such that f_n converges to f almost everywhere. By hypothesis, $F(f_n)$ converges to $F(f)$. Hence $F(f) = 0$.

THEOREM 2.2. *Let (T, μ) be a nonatomic σ-finite measure space, $\mu T = \infty$. Let F be an additive functional on L^p. Then F satisfies the continuity condition*

$$x_n \to x \text{ in measure } \Rightarrow F(x_n) \to F(x)$$

if and only if there exists a continuous function $f \colon R \to R$ such that $f([-h, h]) = 0$ for some $h > 0$, f is bounded, and

$$F(x) = \int_T (f \circ x) d\mu,$$

for all x in L^p.

Proof. Suppose that F is an additive functional satisfying the given continuity condition. Let B be any measurable set in T with $0 < \mu B <$

∞, let μ_B be the restriction of μ to the set B, and let $y' = y\chi_B$ where $y \in L^p(T, \mu)$. Define a functional F_B on $L^p(T, \mu_B)$ by

$$F_B(y) = F(y\chi_B) = F(y').$$

Note that F_B is a well-defined, additive functional on (T, μ_B) and F_B satisfies the same continuity condition as F. Thus we can apply Theorem 1.3 to F_B. There exists a continuous function $f: R \to R$ such that $f(0) = 0$, f is bounded, and $F_B(y) = \int_B (f \circ y) d\mu$, for all $y \in L^p(T, \mu_B)$.

We claim that f is independent of the choice of B. Suppose C is another measurable set in T such that $0 < \mu C < \mu B$ and g is the representing function for F_C. Then, since T is nonatomic, there is a set $B_1 \subset B$ such that $\mu B_1 = \mu C$ and, therefore, $r\chi_{B_1}$ and $r\chi_C$ are equimeasurable for all $r \in R$. Then

$$F_{B_1}(r) = F(r\chi_{B_1}) = F(r\chi_C) = F_C(r).$$

$$F_{B_1}(r) = \int_{B_1} (f \circ r) d\mu = \int_C (g \circ r) d\mu = F_C(r).$$

Or, $f(r)\mu B_1 = g(r)\mu C$. Then $f(r) = g(r)$, for all $r \in R$, since $\mu B_1 = \mu C$.

Now let x be a member of $L^p(T, \mu)$ such that if A = support of x, then $\mu A < \infty$. By the result of the preceding paragraphs, we can write

$$F(x) = F_A(x) = \int_A (f \circ x) d\mu = \int_T (f \circ x) d\mu.$$

We need to verify that $f([-h, h]) = 0$ for some $h > 0$. Suppose not. Then there is a sequence $\{\epsilon_n\}$ converging to zero such that $f(\epsilon_n) \neq 0$ for all n. Since T is nonatomic, we can find a sequence of pairwise disjoint measurable sets $\{A_n\}_{n=1}^{\infty}$ with $0 < \mu A_1 = \mu A_2 = \cdots < \infty$.

For any integer m,

$$F\left(\epsilon_n \chi \bigcup_{i=1}^{m} A_i\right) = \sum_{i=1}^{m} F(\epsilon_n \chi_{A_i}) = mF(\epsilon_n \chi_{A_1}).$$

The function $h_n = \epsilon_n \chi \bigcup_{i=1}^{m} A_i$ has finite support, so

$$F(h_n) = mF(\epsilon_n \chi_{A_1}) = m \int_{A_1} (f \circ \epsilon_n \chi_{A_1}) d\mu = mf(\epsilon_n)\mu A_1.$$

Select m so that $|F(h_n)| \geqq 1$. We note that a sequence $\{h_n\}$ so constructed converges in measure to zero as $\epsilon_n \to 0$, but $|F(x_n)| \geqq 1$. This contradicts the continuity condition on F. Thus there exists $h > 0$ such that $f([-h, h]) = 0$.

Now let x be an arbitrary function in L^p. Let $E_C = \{t \in T: |x(t)| \geqq C\}$ and let $x_n = x\chi_{E_{1/n}}$. Then $x_n \to x$ in measure and $\mu(\text{support } x_n) = \mu E_{1/n} < \infty$. So

$$F(x) = \lim F(x_n) = \lim \int_T (f \circ x_n) d\mu = \lim \int_T (f \circ x\chi_{E_{1/n}}) d\mu.$$

If $1/n < h$, then $f \circ x\chi_{E_{1/n}}(t) = f \circ x(t)$ if $t \in E_{1/n}$ and 0 elsewhere. So $F(x) = \displaystyle\int_T (f \circ x) d\mu$.

Conversely, assume $f: R \to R$ is continuous, f is bounded, and $f([-h, h]) = 0$ for some $h > 0$. If x is in L^p, then for each integer m, $\mu\{t: |x(t)| \geqq 1/m\} < \infty$. If $1/m < h$, then $f \circ x = f \circ x\chi_{E_{1/m}}$ and

$$\int_T |f \circ x| \, d\mu = \int_T |f \circ x\chi_{E_{1/m}}| \, d\mu = \int_{E_{1/m}} |f \circ x| \, d\mu < \infty,$$

since $\mu E_{1/m} < \infty$ and f is bounded. So $f \circ x$ is in L^1. Thus $F(x)$ is well-defined and additive. Suppose $\{x_n\}$ is a sequence in L^p such that $x_n \to x$ in measure. Then given $\epsilon > 0$ there exists an integer N such that, for $n \geqq N$, $\mu\{t: |x_n(t) - x(t)| \geqq \epsilon\} < \epsilon$. Furthermore, $\mu\{t: |x_n\chi_{E_{1/m}}(t) - x\chi_{E_{1/m}}(t)| \geqq \epsilon\} < \epsilon$, so $x_n\chi_{E_{1/m}} \to x\chi_{E_{1/m}}$ in measure. Since $f([-h, h]) = 0$, $f(0) = 0$. If we choose $1/m < h/2$, then we can apply Theorem 1.3 and

$$\begin{aligned} \int (f \circ x_n\chi_{E_{1/m}}) d\mu &= \int_{E_{1/m}} (f \circ x_n) d\mu \to \int_{E_{1/m}} (f \circ x) d\mu \\ &= \int_T (f \circ x\chi_{E_{1/m}}) d\mu = \int_T (f \circ x) d\mu. \end{aligned}$$

Let $A_m = \{t: |x(t)| < 1/m\} = T - E_{1/m}$. We must verify that $\displaystyle\int_T (f \circ x_n\chi_{A_m}) d\mu \to 0$ as $n \to \infty$.

Since $x_n \to x$ in measure, $x_n\chi_{A_m} \to x\chi_{A_m}$ in measure. The support of $f \circ x_n\chi_{A_m}$ is contained in $C_{n,h} = \{t \in A_m: |x_n(t)| \geqq h\}$, and $\mu C_{n,h} \to 0$ as $n \to \infty$ because $C_{n,h} \subset \{t: |x_n(t) - x(t)| \geqq h/2\}$. Thus

$$\int_T (f \circ x_n\chi_{A_m}) d\mu = \int_{C_{n,h}} (f \circ x_n\chi_{A_m}) d\mu \to 0,$$

since f is bounded and $\mu C_{n,h} \to 0$. Therefore, we have

$$F(x_n)=\int_T (f\circ x_n)d\mu = \int_T (f\circ x_n\chi_{E_{1/m}})+\int_T (f\circ x_n\chi_{A_m})d\mu$$
$$\to \int_T (f\circ x)d\mu = F(x).$$

THEOREM 2.3. *Let (T,μ) be a nonatomic σ-finite measure space, $\mu T=\infty$. Let F be an additive functional on L^p. Then F satisfies the continuity condition*

$$d(x_n, x)\to 0 \;\Rightarrow\; F(x_n)\to F(x)$$

if and only if there exists a continuous function $f\colon R\to R$ such that $f(0)=0$, $|f(r)|\leqq k|r|^p$ for all $r\in R$ and some $k\geqq 0$, and

$$F(x)=\int_T (f\circ x)d\mu,$$

for all x in L^p.

Proof. Suppose F is an additive functional satisfying the given continuity condition. As in Theorem 2.2, we let B be any measurable set in T with $0<\mu B<\infty$ and define a corresponding additive functional F_B satisfying the continuity condition. Then we can apply Theorem 1.1 to the functional F_B and obtain a continuous function $f\colon R\to R$ such that $f(0)=0$, $|f(r)|\leqq k(1+|r|)^p$ for all $r\in R$ and some $k\geqq 0$, and

$$F_B(y)=\int_B (f\circ y)d\mu,$$

for all $y\in L^p(T,\mu_B)$. We want to show that f satisfies the condition $|f(r)|\leqq k|r|^p$, or that

$$\sup_{r\to 0}\frac{|f(r)|}{|r|^p}<\infty.$$

Suppose not. Then there exists a sequence $\{\epsilon_n\}$ such that $\epsilon_n\to 0$ and $|f(\epsilon_n)|>n|\epsilon_n|^p$. Let $\{A_n\}$ be a sequence of measurable sets in T with $\mu A_n = 1/|f(\epsilon_n)|$. The functions $h_n=\epsilon_n\chi_{A_n}$ are in L^p and converge to zero in d, since

$$d(h_n,0)=\int_T |h_n|^p d\mu = \int_T |\epsilon_n\chi_{A_n}|^p d\mu = \int_{A_n} |\epsilon_n|^p d\mu = |\epsilon_n|^p\mu A_n$$
$$=\frac{|\epsilon_n|^p}{|f(\epsilon_n)|}<\frac{1}{n}.$$

$$F(h_n)=F_{A_n}(\epsilon_n\chi_{A_n})=\int_{A_n}(f\circ\epsilon_n\chi_{A_n})d\mu = f(\epsilon_n)\int_T \chi_{A_n}d\mu$$
$$=f(\epsilon_n)\mu A_n = \pm 1.$$

This contradicts the continuity of F_{A_n}, so we must have that

$$\sup_{r\to 0} \frac{|f(r)|}{|r|^p} < \infty.$$

Next let x be any function L^p. As in Theorem 2.2, let $E_C = \{t \in X: |x(t)| \geqq C\}$. Define a sequence $\{x_n\}$ by $x_n = x\chi_{E_{1/n}}$; then

$$d(x_n, x) = \int_T |x - x_n|^p d\mu = \int_T |x - x\chi_{E_{1/n}}|^p d\mu = \int_{\{t:|x(t)|<1/n\}} |x|^p d\mu \to 0$$

as $n \to \infty$, $x \in L^p$. Then there is a subsequence of $\{x_n\}$ which converges to x almost everywhere. Let us consider only the subsequence which we call $\{x_n\}$. Since f is continuous and x_n converges to x almost everywhere, $f(x_n)$ converges to $f(x)$ almost everywhere. For all t, $|x_n(t)| \leqq |x(t)|$. Hence $|f(x_n)| \leqq k|x_n|^p \leqq k|x|^p$. By the Lebesgue Convergence Theorem [**15**, p. 88],

$$\begin{aligned} F(x) &= \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} F_{E_{1/n}}(x_n) = \lim_{n\to\infty} \int_{E_{1/n}} (f \circ x\chi_{E_{1/n}}) d\mu \\ &= \lim_{n\to\infty} \int_T (f \circ x_n) d\mu = \int_T \lim_{n\to\infty} (f \circ x_n) d\mu = \int_T f(x) d\mu, \end{aligned}$$

for all x in L^p.

To prove the converse we need to show that $F(x) = \int_T (f \circ x) d\mu$, with f as described, satisfies the desired continuity condition. The proof that F is an additive functional is standard. Let $\{x_n\}$ be a sequence in L^p such that $d(x_n, x)$ converges to zero. Then x_n converges to x in measure; that is, given $\delta > 0$ there exists an integer N such that for $n \geqq N$, $\mu\{t: |x_n(t) - x(t)| \geqq \delta\} < \delta$. Since f is a continuous function, it must follow that $f(x_n)$ converges to $f(x)$ in measure. By hypothesis, $|f(x_n)| \leqq k|x_n|^p$, so $f(x_n)$ is in L^1 since x_n is in L^p. If the functions of the sequence $\{f(x_n)\}$ have equiabsolutely continuous integrals, then

$$\lim_{n\to\infty} \int_T f(x_n) d\mu = \int_T f(x) d\mu$$

[**14**, p. 152]. Now assume for any measurable set $E \subset T$ that $\mu E < \delta$. Then

$$\left| \int_E f(x_n) d\mu \right| \leqq \int_E |f(x_n)| d\mu \leqq k \int_E |x_n|^p d\mu.$$

By Lemma 1.3, for each $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $\mu E < \delta$, $\int_E |x_n|^p d\mu < \epsilon$. Thus $\left| \int_E f(x_n) d\mu \right| < k \cdot \epsilon$ and the functions $\{f(x_n)\}$ have equiabsolutely continuous integrals. Therefore,

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \int_T f(x_n) d\mu = \int_T f(x) d\mu = F(x).$$

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