DIVISION OF DISTRIBUTIONS

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This paper deals with division in an associative commutative algebra containing the distributions in $\mathbb{R}^n$.

1. Introduction. In [5] and [6], a family $(A_{p,\lambda} | p \in \tilde{N}^n, \lambda \in \Lambda)$ of associative, commutative algebras with unit element were constructed, with the following main properties:

1. $\mathcal{D}'(\mathbb{R}^n) \subset A_{p,\lambda}, \forall p \in \tilde{N}^n, \lambda \in \Lambda$, (here, $N = \{0, 1, 2, \cdots \}$, $\tilde{N} = N \cup \{\infty\}$ and $n \in N$, $n \geq 1$);

2. The multiplication in each of the algebras $A_{p,\lambda}, p \in \tilde{N}^n, \lambda \in \Lambda$, induces on $\mathcal{C}^\infty(\mathbb{R}^n)$ the usual multiplication of functions and the function $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$, with $\psi(x) = 1$, $\forall x \in \mathbb{R}^n$, is the unit element in the algebras;

3. for each $\lambda \in \Lambda$, there exist linear mappings $D^p: A_{q+p,\lambda} \rightarrow A_{q,\lambda}$, with $p \in N^n$, $q \in \tilde{N}^n$, such that

3.1. $D^p$ satisfies on $A_{q+p,\lambda}$ the Leibnitz rule of product derivative.

3.2. $D^p$ is the usual distribution derivative on $\mathcal{C}^\infty(\mathbb{R}^n) \oplus \mathcal{D}'(\mathbb{R}^n)$, where $\mathcal{D}'(\mathbb{R}^n) = \{S \in \mathcal{D}'(\mathbb{R}^n) | \text{supp} S \text{ is finite}\}$;

4. The following relations hold for the Dirac $\delta_{x_0}$ distribution, concentrated in $x_0 \in \mathbb{R}^n$:

$$(x - x_0)^r \cdot D^s \delta_{x_0} = 0 \in A_{p,\lambda}, \forall p \in N^n, \lambda \in \Lambda,$$

if $q, r \in N^n$, $r \geq p + e$, $r \geq q + e$, where $e = (1, \cdots, 1) \in N^n$.

In the present paper, within the one dimensional case $n = 1$, necessary or sufficient conditions are given for $T \in A_{p,\lambda}$, in order to be a solution of one of the equations $x^m \cdot T = 0 \in A_{p,\lambda}$ and $x^m \cdot T = S \in A_{p,\lambda}$, with $m \in N$, $m \geq 1$.

2. Notations. Several classes of sequences of complex valued smooth functions (see [5] and [6]) will be needed.

1. $\mathcal{W} = N \rightarrow \mathcal{C}^\infty(\mathbb{R}^1)$; if $s \in \mathcal{W}$, $\nu \in N$, $x \in \mathbb{R}^1$, then $s(\nu) \in \mathcal{C}^\infty(\mathbb{R}^1)$, $s(\nu)(x) \in C^1$; for $\psi \in \mathcal{C}^\infty(\mathbb{R}^1)$ denote $u(\psi) \in \mathcal{W}$, where $u(\psi)(\nu) = \psi, \forall \nu \in N$; $\mathcal{W}$ is in a natural way an associative, commutative algebra (the vector spaces and algebras are considered over the field $C^1$ of
complex numbers), with the unit element $u(1)$ and zero element $u(0)$; thus, $\mathcal{O} = \{u(0)\}$ is the null space in $\mathcal{W}$;

(2) $D : \mathcal{W} \to \mathcal{W}$ is defined by $(Ds)(\nu)(x) = (Ds(\nu))(x), \forall s \in \mathcal{W}, \nu \in N, x \in R^1$; for given $x_0 \in R^1$, define $\tau_{x_0} : \mathcal{W} \to \mathcal{W}$ by $(\tau_{x_0}s)(\nu)(x) = s(\nu)(x - x_0), \forall s \in \mathcal{W}, \nu \in N, x \in R^1$;

(3) $\mathcal{U} = \{u(\psi) | \psi \in \mathcal{C}^\infty(R^1)\}$;

(4) $\mathcal{S}_0$ is the set of $s \in \mathcal{W}$, weakly convergent in $\mathcal{D}'(R^1)$; $\mathcal{V}_0$ is the kernel of the linear surjection:

$$\mathcal{S}_0 \ni s \to (s, \cdot) \in \mathcal{D}'(R^1),$$

where

$$\langle s, \psi \rangle = \lim_{\nu \to 0} \int_{R^1} s(\nu)(x)\psi(x)dx, \quad \forall \psi \in \mathcal{D}(R^1);$$

One of the basic ideas in the construction of the associative and commutative distribution multiplication in [5] and [6], is the way the weakly convergent sequences of smooth functions representing the Dirac $\delta$ distribution are chosen:

(5) $\mathcal{S}_0^0$ is the set of $s \in \mathcal{S}_0$, satisfying the conditions:

(5.1) $\langle s, \cdot \rangle = \delta,$

(5.2) $\forall \epsilon > 0 : \exists \nu_\epsilon \in N : \forall \nu \in N,$

$\nu \geq \nu_\epsilon, x \in R^1, |x| \geq \epsilon : s(\nu)(x) = 0$

(5.3) $\forall p \in N : \exists \nu_p \in N : \forall \nu \in N,$

$\nu \geq \nu_p : W(s(\nu), \cdots, s(\nu + p))(0) \neq 0.$

where $W(\psi_1, \cdots, \psi_m)(x), x \in R^1$, denotes the Wronskian function of $\psi_1, \cdots, \psi_m \in \mathcal{C}^\infty(R^1)$.

The condition (5.3), called "strong local presence of $s$ in $x = 0$" and replaced in [6] by a weaker form, plays a central role in the associative, commutative distribution multiplication presented in [5] and [6].

(6) for $p \in \bar{N}$, denote by $\mathcal{V}_0^p$ the set of $\nu \in \mathcal{V}_0$, satisfying the above condition (5.2), as well as

$$\forall q \in N, q \leq p : \exists \nu_q \in N : \forall \nu \in N : \nu \geq \nu_q \Rightarrow D^\nu\nu(\nu)(0) = 0;$$

(7) $\mathcal{S}_0^p = \{s \in \mathcal{S}_0 | \text{supp}(s, \cdot) \subseteq \{0\}\};$

(8) $\mathcal{V}_0^p$, with $p \in \bar{N}$, and $\mathcal{S}_0$ are the vector subspaces generated in $\mathcal{W}$ by $\bigcup_{x \in R^1} \tau_x \mathcal{S}_0^p$, respectively $\bigcup_{x \in R^1} \tau_x \mathcal{S}_0$;

(9) $\mathcal{S}_0 = \bigcup_{x \in R^1} \tau_x \mathcal{S}_0^0$;

(10) for $\Sigma = (s, x \in R^1) \in \mathcal{S}_0$, denote by $\mathcal{S}(\Sigma)$ the vector subspace generated in $\mathcal{S}_0$ by the sequences $D^\nu s$, with $x \in R^1, p \in N$.

And now, the definition of the associative, commutative algebras
(A_{p, \lambda} | p \in \mathbb{N}, \lambda \in \Lambda), where \Lambda is the set of all \lambda = (\Sigma, \mathcal{F}) with \Sigma \in \mathcal{F}_0 and \mathcal{F}_1 vector subspace in \mathcal{F}_0, such that (\mathcal{U} + \mathcal{F}_0) \cap \mathcal{F}_1 = \mathcal{O} and \mathcal{F}_0 = \mathcal{U} + \mathcal{F}_\delta + \mathcal{F}_1.

Suppose p \in \mathbb{N}, \lambda = (\Sigma, \mathcal{F}) \in \Lambda and denote

11. \mathcal{S}_{p, \lambda} = \mathcal{V}_{p, \lambda} \mathcal{S}_{p, \lambda} \mathcal{S}_0 \mathcal{F}_1;

12. \mathcal{A}_{p, \lambda} the smallest subalgebra in \mathcal{W}, containing \mathcal{S}_{p, \lambda} and invariant of the mapping \mathcal{D}: \mathcal{W} \rightarrow \mathcal{W};

13. \mathcal{I}_{p, \lambda} the vector subspace generated in \mathcal{W} by \mathcal{V}_{p, \lambda} \cdot \mathcal{A}_{p, \lambda}.

Then (see [5] and [6])

1. A_{p, \lambda} = \mathcal{A}_{p, \lambda} / \mathcal{S}_{p, \lambda},
2. D: A_{p+1, \lambda} \rightarrow A_{p, \lambda} is given by

\[ D(t + \mathcal{I}_{p+1, \lambda}) = Dt + \mathcal{I}_{p, \lambda}, \quad \forall t \in A_{p+1, \lambda} \]

3. Multiplication by $1/x^m$, $m = 1, 2, \cdots$. It is shown (see Corollary 2) that in the algebras $A_{p, \lambda}$, the multiplication by $1/x^m$ does not represent the division by $x^m$.

**Theorem 1.** Suppose $T \in A_{p, \lambda}$, with given $p \in \mathbb{N}$, $\lambda \in \Lambda$.

Suppose $\psi \in \mathcal{C}^\infty(R^1)$ such that for a certain $m \in \mathbb{N}$

\[ D^q\psi(0) = 0, \quad \forall q \in \mathbb{N}, \quad q \leq m. \]

If there exists $\chi \in \mathcal{C}^\infty(R^1)$ such that $\psi \cdot T = \chi$ in $A_{p, \lambda}$, then:

\[ D^q\chi(0) = 0, \quad \forall q \in \mathbb{N}, \quad q \leq \min\{p, m\}. \]

**Proof.** Assume $T = t + \mathcal{I}_{p, \lambda}$, with $t \in \mathcal{A}_{p, \lambda}$. Then $\psi \cdot T = \chi$ in $A_{p, \lambda}$ implies $u(\chi) = u(\psi) \cdot t + w$, with $w \in \mathcal{I}_{p, \lambda}$. Therefore,

\[ \forall q \in \mathbb{N}, \quad q \leq p : \exists \nu_q \in \mathbb{N} : \forall \nu \in \mathbb{N}, \nu \geq \nu_q : D^q w(\nu)(0) = 0. \]

Since $\chi = \psi \cdot t(\nu) + w(\nu)$, $\forall \nu \in \mathbb{N}$, the proof is completed.

**Corollary 1.** Suppose $T \in A_{p, \lambda}$, with given $p \in \mathbb{N}$, $\lambda \in \Lambda$.

If $\psi \in \mathcal{C}^\infty(R^1)$ such that $\psi(0) \neq 0$, then, $x^m \cdot T \neq \psi$ in $A_{p, \lambda}$, $\forall m \in \mathbb{N}, m \geq 1$.

**Corollary 2.** If $m \in \mathbb{N}$, $m \geq 1$, then, $x^m \cdot (1/x^m) \neq 1$, in each of the algebras $A_{p, \lambda}$, $p \in \mathbb{N}$, $\lambda \in \Lambda$.

4. Division by $x^m$, $m = 1, 2, \cdots$. First, in Theorem 2, a
sufficient condition is given for \( T \in A_{p,\lambda} \), in order to be a solution of the equation \( x^m \cdot T = 0 \in A_{p,\lambda} \), where \( m \in \mathbb{N}, \ m \geq 1 \).

For \( p \in \mathbb{N} \) and \( \lambda \in \Lambda \), denote by \( B^0_{p,\lambda} \) all the elements \( T \in A_{p,\lambda} \) of the form \( T = t + \mathcal{J}_{p,\lambda} \), where \( t \in \mathcal{A}_{p,\lambda} \cap \mathcal{Y}_0 \) and satisfies also (5.2) in §2.

**Proposition 1.** Suppose given \( p \in \mathbb{N}, \lambda \in \Lambda \) and \( \psi \in \mathcal{C}(\mathbb{R}^r) \), such that, for a certain \( q \in \mathbb{N}, \ q \geq p \):

\[
D^\ast \psi(0) = 0, \quad \forall r \in \mathbb{N}, \quad r \leq q.
\]

Then, \( \psi \cdot B^0_{p,\lambda} = \{0\} \subset A_{p,\lambda} \).

**Proof.** Assume \( T \in B^0_{p,\lambda} \) and \( T = t + \mathcal{J}_{p,\lambda} \), with \( t \in \mathcal{A}_{p,\lambda} \cap \mathcal{Y}_0 \) and satisfying (5.2) in §2. Then, \( \psi \cdot T = u(\psi) \cdot t + \mathcal{J}_{p,\lambda} \). But, obviously, \( u(\psi) \cdot t \in \mathcal{Y}_{\delta,q} \subset \mathcal{Y}_{\delta,q} \subset \mathcal{J}_{p,\lambda} \), hence, \( T = 0 \in A_{p,\lambda} \).

**Theorem 2.** Suppose given \( p \in \mathbb{N}, \lambda \in \Lambda \) and \( m \in \mathbb{N}, \ m \geq 1 \).

Then, any

\[
T_0 = \sum_{0 \leq i \leq k} x^r_{\ast} \cdot T_{i_1} + \sum_{0 \leq j \leq h} x^q_{\ast} \cdot D^{^p} \delta \cdot T_{3_j}
\]

with \( k, h, r, q, p, r_{\ast} \in \mathbb{N}, \ r_{\ast} > p - m, \)

\( q_{\ast} > \max\{p, p_{\ast}\} - m, \)

and \( T_{i_1} \in B^0_{p,\lambda}, \ T_{2_j}, \ T_{3_j} \in A_{p,\lambda}, \)

will be a solution in \( A_{p,\lambda} \) of the equation \( x^m \cdot T = 0 \).

**Proof.** According to Proposition 1, \( x^m \cdot x^r_{\ast} \cdot T_{i_1} = x^{m+r}_{\ast} \cdot T_{i_1} = 0 \in A_{p,\lambda} \), since \( m + r > p \). According to (4) in §1 (see also 3) in Theorem 6, §8 [5], \( x^m \cdot x^q_{\ast} \cdot D^{^p} \delta = x^{m+q}_{\ast} \cdot D^{^p} \delta = 0 \in A_{p,\lambda} \), since \( m + q_{\ast} > \max\{p, p_{\ast}\} \).

It results the following sufficient condition on \( T \in A_{p,\lambda} \), solution of the equation \( x^m \cdot T = S \in A_{p,\lambda} \).

**Corollary 3.** Suppose \( S \in A_{p,\lambda} \), with \( p \in \mathbb{N}, \lambda \in \Lambda \) given and \( m \in \mathbb{N}, \ m \geq 1 \).

If \( T_{i_1} \) is any solution in \( A_{p,\lambda} \) of the equation \( x^m \cdot T = S \) and \( T_0 \) is given as in Theorem 2, then \( T = T_{i_1} + T_0 \) will be again a solution of that equation.

Before a necessary condition is given on \( T \in A_{p,\lambda} \), solution of the equation \( x^m \cdot T = 0 \in A_{p,\lambda} \), the notion of *support* of the elements in \( A_{p,\lambda} \) will be defined.
Suppose $T \in A_{p, \lambda}$, with $p \in \tilde{N}$, $\lambda \in \Lambda$ given and $E \subset R^1$. Then,

1. $T$ vanishes on $E$, only if $T = t + \mathcal{I}_{p, \lambda}$, with $t \in \mathcal{A}_{p, \lambda}$, such that $t(v)(x) = 0$, $\forall v \in N$, $v \geq \nu_0$, $x \in E$.

2. $T$ strictly vanishes on $E$, only if $T$ vanishes on a certain open set $G \subset R^1$, containing $E$.

3. $T$ is supported by $E$, only if for every open set $G \subset R^1$, containing $E$, one can write $T = t + \mathcal{I}_{p, \lambda}$, with $t \in \mathcal{A}_{p, \lambda}$, such that $\text{supp } t(v) \subset G$, $\forall v \in N$, $v \geq \nu_0$.

The support of $T$ is defined as the closed set

$$\text{supp } T = R^1 \setminus \{x \in R^1 \mid T \text{ strictly vanishes on } \{x\}\}.$$ 

Obviously, for the distributions in $\mathcal{E}^*(R^1) \oplus \mathcal{B}'(R^1)$, the above notion of support is identical with the usual one for distributions.

**Proposition 2.** Suppose $x_0 \in R^1$ and $q \in N$, then, $D^q \delta_{x_0} \in A_{p, \lambda}$, for $p \in \tilde{N}$, $\lambda \in \Lambda$, and

1. $D^q \delta_{x_0}$ is supported by $\{x_0\}$ and $\text{supp } D^q \delta_{x_0} = \{x_0\}$,
2. if $E \subset R^1$ and $x_0 \notin \text{closure } E$, then $D^q \delta_{x_0}$ strictly vanishes on $E$,
3. $D^q \delta_{x_0}$ does not vanish on $R^1 \setminus \{x_0\}$,
4. $D^q \delta_{x_0}$ does not vanish on $\{x_0\}$.

**Proof.** (1), (2) and (3) follow easily.

(4) Assume $\lambda = (\Sigma, \mathcal{F})$ and $\Sigma = (s_x \mid x \in R^1)$, then, $D^q \delta_{x_0} = D^q s_{x_0} + \mathcal{F}_{p, \lambda}$ and $s_{x_0} \in \tau_{x_0} \mathcal{X}_0$. Suppose, $D^q \delta_{x_0}$ vanishes on $\{x_0\}$, then, there exists $t \in \mathcal{A}_{p, \lambda}$, such that $t - D^q s_{x_0} \in \mathcal{F}_{p, \lambda}$ and $t(v)(x_0) = 0$, $\forall v \in N$, $v \geq \nu_0$. Denoting $v = t - D^q s_{x_0}$, the relation $v \in \mathcal{F}_{p, \lambda}$ implies $v(v)(x_0) = 0$, $\forall v \in N$, $v \geq \nu_1$. Therefore, it results

$$D^q s_{x_0}(v)(x_0) = t(v)(x_0) - v(v)(x_0) = 0, \quad \forall v \in N, \quad v \geq \nu_2.$$ 

But, that relation implies $W(s_{x_0}(v), \cdots, s_{x_0}(v + q))(x_0) = 0, \forall v \in N$, $v \geq \nu_2$, which contradicts the assumption $s_{x_0} \in \tau_{x_0} \mathcal{X}_0$.

**Remark.** The property of the Dirac distributions that $D^q \delta_{x_0}$ does not vanish on $\{x_0\}, \forall x_0 \in R^1$, $q \in N$, is a direct consequence of the "condition of strong local presence" (see (5.3) in §2) and it is proper for the distribution multiplication presented in [5] and [6]. The "delta sequences" generally used (see [2]) do not necessarily prevent the vanishing of $D^q \delta_{x_0}$ on $\{x_0\}$.

**Theorem 3.** Suppose $T \in A_{p, \lambda}$ with $p \in \tilde{N}$, $\lambda \in \Lambda$ given.
If \( x^m \cdot T = 0 \in A_{p, \lambda} \), for a certain \( m \in N \), \( m \geq 1 \), then \( T \) is supported by \( \{0\} \), hence \( \text{supp } T \subset \{0\} \).

**Proof.** Assume \( T = t + \mathcal{J}_{p, \lambda} \), with \( t \in A_{p, \lambda} \). Then \( x^m \cdot T = 0 \in A_{p, \lambda} \) implies \( u(x^m) \cdot t \in \mathcal{J}_{p, \lambda} \), therefore, according to the definition of \( \mathcal{J}_{p, \lambda} \) (see (13), §2), it results

\[
u(x^m) \cdot t = \sum_{0 \leq i \leq k} v_i \cdot a_i
\]

with \( k \in N \), \( v_i \in \mathcal{V}_{\delta, p} \), \( a_i \in A_{p, \lambda} \).

Now, due to the definition \( \mathcal{V}_{\delta, p} \) (see (8) and (6), §2), it follows that:

\[
\forall i \in \{0, \ldots, k\}: \exists X \subset R^1 \times X \text{ finite: } v_i = \sum_{x \in X} v_{x_i} \text{, where } v_{\alpha} \in \tau \mathcal{V}_{\delta, p}^0
\]

Concluding, there exists \( X \subset R^1 \), \( X \) finite, such that

\[
u(x^m) \cdot t = \sum_{x \in X} \sum_{0 \leq i \leq h} v_{x_i} \cdot b_{x_i} \text{ with } h \in N \text{, } v_{x_i} \in \tau \mathcal{V}_{\delta, p}^0, \text{ } b_{x_i} \in A_{p, \lambda}.
\]

It will be shown now, that in the above relation, one can consider \( X = \{0\} \). Indeed, suppose \( x_0 \in X \setminus \{0\} \), then \( v_{x_0} \in \tau \mathcal{V}_{\delta, p}^0 \) with \( 0 \leq j \leq h \). The condition (5.2) in §2, results in the existence of \( w_{x_0} \in \mathcal{W} \), with \( 0 \leq j \leq h \), such that \( v_{x_0}(v)(x) = x^m \cdot w_{x_0}(v)(x), \forall 0 \leq j \leq h, \text{ } x \in R^1, \text{ } v \in N, \text{ } v \equiv v_{x_0} \). Moreover, \( w_{x_0} \in \tau \mathcal{V}_{\delta, p}^0 \), \( \forall 0 \leq j \leq h \), since \( v_{x_0} \in \tau \mathcal{V}_{\delta, p}^0 \) with \( 0 \leq j \leq h \), and \( x_0 \neq 0 \).

Denoting

\[
v = \sum_{x \in X} \sum_{0 \leq j \leq h} w_{x_0} \cdot b_{x_0}
\]

it results \( v \in \mathcal{J}_{p, \lambda} \), hence, \( T = t_1 + \mathcal{J}_{p, \lambda} \), where \( t_1 = t - v \in A_{p, \lambda} \). But \( u(x^m) \cdot t_1 = u(x^m) \cdot t - u(x^m) \cdot v = \sum_{0 \leq j \leq h} v_{x_0} \cdot b_{x_0} \).

Since \( v_{x_0} \), with \( 0 \leq j \leq h \), satisfy (5.2) in §2, it follows that \( u(x^m) \cdot t_1 \) and, therefore \( t_1 \) satisfy the same condition. Thus, \( T = t_1 + \mathcal{J}_{p, \lambda} \) is supported by \( \{0\} \), which obviously results in \( \text{supp } T \subset \{0\} \).

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TECHNION — ISRAEL INSTITUTE OF TECHNOLOGY
Helen Elizabeth Adams, *Factorization-prime ideals in integral domains* .............. 1
Patrick Robert Ahern and Robert Bruce Schneider, *The boundary behavior of Henkin’s kernel* ................................................................. 9
Efrain Pacillas Armendariz, *On semiprime PI-algebras over commutative regular rings* ................................................................. 23
Robert H. Bird and Charles John Parry, *Integral bases for bicyclic biquadratic fields over quadratic subfields* ................................................................. 29
Tae Ho Choe and Young Hee Hong, *Extensions of completely regular ordered spaces* .................................................................................................................. 37
John Dauns, *Generalized monoform and quasi injective modules* .................. 49
F. S. De Blasi, *On the differentiability of multifunctions* ................................................. 67
Paul M. Eakin, Jr. and Avinash Madhav Sathaye, *R-endomorphisms of R[[X]] are essentially continuous* ................................................................. 83
Larry Quin Eifler, *Open mapping theorems for probability measures on metric spaces* ................................................................. 89
Garret J. Etgen and James Pawlowski, *Oscillation criteria for second order self adjoint differential systems* ................................................................. 99
Ronald Fintushel, *Local $S^1$ actions on 3-manifolds* ................................................. 111
Kenneth R. Goodearl, *Choquet simplexes and $\sigma$-convex faces* .................. 119
John R. Graef, *Some nonoscillation criteria for higher order nonlinear differential equations* .................................................................................................................. 125
Charles Henry Heiberg, *Norms of powers of absolutely convergent Fourier series: an example* .................................................................................................................. 131
Les Andrew Karlovitz, *Existence of fixed points of nonexpansive mappings in a space without normal structure* ................................................................. 153
Gangaram S. Ladde, *Systems of functional differential inequalities and functional differential systems* ................................................................. 161
Joseph Michael Lambert, *Conditions for simultaneous approximation and interpolation with norm preservation in $C[a, b]$* ................................................. 173
Ernest Paul Lane, *Insertion of a continuous function* ................................................. 181
Robert F. Lax, *Weierstrass points of products of Riemann surfaces* .................. 191
Dan McCord, *An estimate of the Nielsen number and an example concerning the Lefschetz fixed point theorem* ................................................................. 195
Paul Milnes and John Sydney Pym, *Counterexample in the theory of continuous functions on topological groups* ................................................................. 205
Peter Johanna I. M. De Paepe, *Homomorphism spaces of algebras of holomorphic functions* ................................................................. 211
Judith Ann Palagallo, *A representation of additive functionals on $L^p$-spaces, $0 < p < 1$* ................................................................. 221
S. M. Patel, *On generalized numerical ranges* .......................................................... 235
Thomas Thornton Read, *A limit-point criterion for expressions with oscillatory coefficients* ................................................................. 243
Elemer E. Rosinger, *Division of distributions* .......................................................... 257
Peter S. Shoenfeld, *Highly proximal and generalized almost finite extensions of minimal sets* ................................................................. 265
R. Siros-Dumais and Stephen Willard, *Quotient-universal sequential spaces* .......... 281
Robert Charles Thompson, *Convex and concave functions of singular values of matrix sums* ................................................................. 285
Edward D. Tymchatyn, *Some n-arc theorems* .......................................................... 291
Jang-Mei Gloria Wu, *Variation of Green’s potential* ................................................ 295