

Pacific Journal of Mathematics

**CONVEX AND CONCAVE FUNCTIONS OF SINGULAR VALUES
OF MATRIX SUMS**

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A new set of inequalities for functions of singular values of matrix sums is established. These inequalities are complementary to a number of classically known inequalities in that the direction of the inequality sign is reversed. A matrix valued triangle inequality is also given. Special cases of these results are due to S. Yu. Rotfel'd.

1. Introduction. Let $A, B, C = A + B$ be n -square matrices with singular values $\alpha_1 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \cdots \geq \beta_n$, $\gamma_1 \geq \cdots \geq \gamma_n$ respectively. In [5] S. Yu. Rotfel'd stated and in [6] proved that

$$(1) \quad f(\gamma_1) + \cdots + f(\gamma_n) \leq f(\alpha_1) + \cdots + f(\alpha_n) + f(\beta_1) + \cdots + f(\beta_n)$$

when f is an increasing concave function of a nonnegative real variable, with $f(0) = 0$. This inequality is of some interest as in previously published work a convexity (rather than concavity) hypothesis has usually been necessary to establish results of the general type of (1). See, for example, Gohberg and Krein [3], page 49, or Marcus and Minc [4], pages 103 and 116. In this paper we shall uncover the algebraic foundation of (1) by giving a short proof of a generalization. Our proof, which is simpler and more direct than the proof of (1) given by Rotfel'd, will be based on an interesting matrix valued triangle inequality, a special case of which was given by Rotfel'd. We note that the methods used by Rotfel'd are very much adapted to the inequality (1) that he wished to prove, and do not appear capable of proving the extensions of his results to be presented below.

2. Positive semidefinite matrices. In this section we discuss an important special case for which sharper results are possible. Throughout §2 we let $A, B, C = A + B$ be $n \times n$ Hermitian positive semidefinite matrices with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \cdots \geq \beta_n$, $\gamma_1 \geq \cdots \geq \gamma_n$, respectively. Let $F(x_1, \cdots, x_{2n})$ and $G(x_1, \cdots, x_{2n})$ be functions of $2n$ nonnegative variables, symmetric in these variables (i.e., remaining unchanged if the variables are permuted), with F concave and G convex: for all nonnegative vectors x, y and real numbers θ with $0 \leq \theta \leq 1$,

$$F(\theta x + (1 - \theta)y) \geq \theta F(x) + (1 - \theta)F(y),$$

$$G(\theta x + (1 - \theta)y) \leq \theta G(x) + (1 - \theta)G(y).$$

Then we have:

THEOREM 1. *For Hermitian positive semidefinite $A, B, C = A + B$ and functions F, G as above, the following inequalities hold:*

$$(2.1) \quad F(\gamma_1, \dots, \gamma_n, 0, \dots, 0) \leq F(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n),$$

$$(2.2) \quad G(\gamma_1, \dots, \gamma_n, 0, \dots, 0) \geq G(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n),$$

$$(2.3) \quad F\left(\frac{\gamma_1 + \dots + \gamma_n}{2n}, \dots, \frac{\gamma_1 + \dots + \gamma_n}{2n}\right) \geq F(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n),$$

$$(2.4) \quad G\left(\frac{\gamma_1 + \dots + \gamma_n}{2n}, \dots, \frac{\gamma_1 + \dots + \gamma_n}{2n}\right) \leq G(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n).$$

Proof. Since A and B are semidefinite, we may write $A = A_1 A_1^*$, $B = B_1 B_1^*$, where $*$ denotes adjoint. Then we have

$$C = A_1 A_1^* + B_1 B_1^* = [A_1, B_1] \begin{bmatrix} A_1^* \\ B_1^* \end{bmatrix}$$

Reversing the order of the factors in the last product, and using the fact that this reversal leaves unchanged the nonzero eigenvalues, we see that

$$(3) \quad \begin{bmatrix} A_1^* \\ B_1^* \end{bmatrix} [A_1, B_1] = \begin{bmatrix} A_1^* A_1 & A_1^* B_1 \\ B_1^* A_1 & B_1^* B_1 \end{bmatrix}$$

has $\gamma_1, \dots, \gamma_n, 0, \dots, 0$ as its eigenvalues. Making a block diagonal unitary similarity to diagonalize the blocks $A_1^* A_1$ and $B_1 B_1^*$ in (3), we next see that $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ are diagonal elements of a Hermitian matrix having $\gamma_1, \dots, \gamma_n, 0, \dots, 0$ as its eigenvalues. Consequently

$$(4) \quad \begin{bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \\ \beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \beta_n \end{bmatrix} = S \begin{bmatrix} \gamma_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \gamma_n \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

where S is a $2n$ -square doubly stochastic matrix. By Birkhoff's theorem this implies that the left hand side of (4) belongs to the convex hull of all

rearrangements of the column vector $v = [\gamma_1, \dots, \gamma_n, 0, \dots, 0]^T$. Now it is well known that a symmetric concave function F on the convex polytope spanned by all rearrangements of a fixed vector v assumes its least value at the vector v and its greatest value at the vector \bar{v} in which each component is the average of the components of v . See, for example, [4, p. 104]. (The argument there only shows that F achieves its maximum at a vector in the convex polytope having all components equal; however, it is easy to show that \bar{v} is the unique such vector.) This proves the first and third inequalities of the theorem; the second and fourth follow from the corresponding fact for convex functions G in which the words “least” and “greatest” above are interchanged.

Previously known inequalities for functions on the eigenvalues of A , B , $C = A + B$ take the form

$$(5.1) \quad F_1(\gamma_1, \dots, \gamma_n) \geq F_1(\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n),$$

$$(5.2) \quad G_1(\gamma_1, \dots, \gamma_n) \leq G_1(\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n),$$

$$(5.3) \quad F_1(\gamma_1, \dots, \gamma_n) \leq F_1\left(\frac{\alpha_1 + \beta_1 + \dots + \alpha_n + \beta_n}{n}, \dots, \frac{\alpha_1 + \beta_1 + \dots + \alpha_n + \beta_n}{n}\right)$$

$$(5.4) \quad G_1(\gamma_1, \dots, \gamma_n) \geq G_1\left(\frac{\alpha_1 + \beta_1 + \dots + \alpha_n + \beta_n}{n}, \dots, \frac{\alpha_1 + \beta_1 + \dots + \alpha_n + \beta_n}{n}\right)$$

where $F_1(x_1, \dots, x_n)$ and $G_1(x_1, \dots, x_n)$ are symmetric functions of n nonnegative variables, with F_1 concave and G_1 convex. These follow from the well known results that

$$\gamma_1 + \dots + \gamma_k \leq \alpha_1 + \dots + \alpha_k + \beta_1 + \dots + \beta_k \quad \text{for } k = 1, \dots, n,$$

(with equality for $k = n$), and therefore

$$\begin{bmatrix} \gamma_1 \\ \cdot \\ \cdot \\ \cdot \\ \gamma_n \end{bmatrix} = S_1 \begin{bmatrix} \alpha_1 + \beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n + \beta_n \end{bmatrix}$$

for some n -square doubly stochastic matrix S_1 . Evidently (2.1) and (2.2)

may be regarded as inequalities complementary to the previously known inequalities (5.1) and (5.2).

In the special case $F(x_1, \dots, x_{2n}) = f(x_1) + \dots + f(x_{2n})$, with $f(x)$ concave and $f(0) = 0$, inequality (2.1) reduces to the inequality (1) found by Rotfel'd.

The device of proving matrix inequalities by reversing the order of the factors, as in (3), was first introduced by Wielandt, in another context, and appears in his lecture notes [9].

3. A triangle inequality for matrices. If A is a square matrix, not necessarily Hermitian, define

$$|A| = (AA^*)^{1/2},$$

where the positive semidefinite determination of the square root is used. If H and K are Hermitian matrices, the notation $H \leq K$ will signify that $K - H$ is positive semidefinite.

THEOREM 2. *Let A and B be square matrices, not necessarily Hermitian. Then unitary matrices U and V exist such that*

$$(6) \quad |A + B| \leq U|A|U^* + V|B|V^*.$$

REMARK 1. Without the presence of the unitary matrices U and V , this result would be false.

REMARK 2. For the special case of Hermitian A and B , with rank $B = 1$, this result appears in Rotfel'd's paper [6]. The method used by Rotfel'd can, after some simplification, be used to obtain (6) for arbitrary Hermitian A and B , but does not immediately extend to the non-Hermitian case.

Proof. We first present a lemma giving a result of Fan and Hoffman [2], reproved in [1], and reproved again in [7]. For completeness we include the short proof from [7].

LEMMA. *Let M be a matrix, not necessarily Hermitian, and denote the eigenvalues of $\frac{1}{2}(M + M^*)$ and $(MM^*)^{1/2}$ respectively by $\lambda_1 \geq \dots \geq \lambda_n$, $s_1 \geq \dots \geq s_n$. Then*

$$\lambda_i \leq s_i \quad \text{for } i = 1, \dots, n.$$

Proof of lemma. Take v_1, \dots, v_n and w_1, \dots, w_n to be orthonormal sets of eigenvectors for $\frac{1}{2}(M + M^*)$ and MM^* respectively:

$$\frac{1}{2}(M + M^*)v_i = \lambda_i v_i, \quad MM^* w_i = s_i^2 w_i, \quad i = 1, \dots, n.$$

Let k be fixed, $1 \leq k \leq n$. The spans of v_1, \dots, v_k and w_k, \dots, w_n intersect in at least one dimension; let x be a unit vector in this intersection. Then

$$\left(\frac{1}{2}(M + M^*)x, x\right) \geq \lambda_k, \quad (MM^*x, x) \leq s_k^2$$

where $(\ , \)$ denotes inner product. We now have:

$$\begin{aligned} \lambda_k &\leq \left(\frac{1}{2}(M + M^*)x, x\right) = \operatorname{Re}(M^*x, x) \leq |(M^*x, x)| \\ &\leq \|M^*x\| = (MM^*x, x)^{1/2} \leq s_k. \end{aligned}$$

This completes the proof of the lemma.

Turning to the proof of Theorem 2, let $C = A + B$. Passing to $CW = AW + BW$ with W unitary, we see that no generality is lost if we take C to be positive semidefinite. From $C = A + B$ we then obtain, after forming the adjoint and adding, the equation

$$(7) \quad C = \frac{1}{2}(A + A^*) + \frac{1}{2}(B + B^*).$$

We claim: $\frac{1}{2}(A + A^*) \leq U |A| U^*$ for some unitary U . Indeed, take U to be such that a unitary similarity simultaneously brings $\frac{1}{2}(A + A^*)$ and $U |A| U^* = U(AA^*)^{1/2}U^*$ into diagonal form with diagonal elements in nonincreasing order. By the lemma, each eigenvalue of $\frac{1}{2}(A + A^*)$ is dominated by the corresponding eigenvalue of $U |A| U^*$. This means:

$$\frac{1}{2}(A + A^*) \leq U |A| U^*.$$

In the same way unitary V exists such that $\frac{1}{2}(B + B^*) \leq V |B| V^*$. Substituting these inequalities into (7) and using $C = |C|$, we obtain (6).

This lemma may be used to give a variant of certain proofs in [8].

4. Singular value inequalities. We continue to let $F(x_1, \dots, x_{2n})$ and $G(x_1, \dots, x_{2n})$ be symmetric functions of $2n$ nonnegative variables, with F concave and G convex. We now let $A, B, C = A + B$ be not necessarily Hermitian matrices with singular values $\alpha_1 \geq \dots \geq \alpha_n, \beta_1 \geq \dots \geq \beta_n$ and $\gamma_1 \geq \dots \geq \gamma_n$ respectively.

THEOREM 3. *Let arbitrary matrices $A, B, C = A + B$ and symmetric functions F, G be as described above. Then:*

- (i) *inequality (2.1) holds if F is nondecreasing as well as concave,*
- (ii) *inequality (2.2) holds if G is nonincreasing as well as convex,*

- (iii) inequality (2.3) holds if F is nonincreasing as well as concave,
 (iv) inequality (2.4) holds if G is nondecreasing as well as convex.

Here, for example, F nondecreasing means that $F(x_1, \dots, x_{2n})$ is a nondecreasing function of x_1 for fixed x_2, x_3, \dots, x_{2n} .

Proof. It is well known that if H, K are Hermitian matrices with $H \leq K$ then the eigenvalues $h_1 \geq \dots \geq h_n$ and $k_1 \geq \dots \geq k_n$ of H and K satisfy $h_i \leq k_i$ for each i . Indeed, set $K = H + P$, with P positive semidefinite, and use the easily proved inequality $k_i \geq h_i + p_n$ where p_n is the smallest eigenvalue of P . From $C = A + B$ we get

$$|C| \leq U |A| U^* + V |B| V^*;$$

thus the singular values $\gamma_1 \geq \dots \geq \gamma_n$ of C are dominated, term by term, by the singular values $\gamma'_1 \geq \dots \geq \gamma'_n$ of $U |A| U^* + V |B| V^*$. Using first the nondecreasing nature of F , then Theorem 1 applied to $U |A| U^*$ and $V |B| V^*$, we get

$$F(\gamma_1, \dots, \gamma_n, 0, \dots, 0) \leq F(\gamma'_1, \dots, \gamma'_n, 0, \dots, 0) \leq F(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$$

The remaining parts of the theorem are proved similarly.

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Received February 6, 1976. The preparation of this paper was supported in part by Grant 72-2164 of the U.S. Airforce Scientific Research Office.

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