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G. T. Whyburn gave an inductive proof of the n-arc theorem for complete, locally connected, metric spaces. In this note Whyburn's proof is modified to generalize this theorem to the class of regular, T_1, locally connected spaces. This result is then used to obtain an affirmative solution to a conjecture of J. H. V. Hunt.

Our notation will follow that of Whyburn [3] and Hunt [1].

Let X be a topological space and let P and Q be disjoint closed sets in X. A set C is said to *separate P and Q in the broad sense in X* if $X\backslash C = A \cup B$ where A is separated from B, $P\backslash C \subset A$ and $Q\backslash C \subset B$. The space X is said to be *n-point strongly connected between P and Q* if no subset of X with fewer than n points separates P and Q in the broad sense in X. A subset of X is said to *join* P and Q if some component of the set meets both P and Q.

1. The second n-arc theorem.

THEOREM 1. *The locally connected, regular, T_1 space X is n-point strongly connected between two disjoint closed sets P and Q if and only if there exist n disjoint open sets in X which join P and Q.*

Proof. The sufficiency is obvious. We shall prove necessity by induction on n. The case $n = 1$ follows from the fact that the components of X are open (as X is locally connected) and hence some component of X meets both P and Q. Suppose the theorem holds for all positive integers less than n.

Suppose X is n-point strongly connected between the disjoint closed sets P and Q. Let S denote the set of all $x \in X$ such that there exists a set S_x which is the union of n disjoint open connected sets $n-1$ of which join P and Q and the nth one joins P and x. Then S is clearly open in X. If $y \in X$ then $X\backslash\{y\}$ is $(n-1)$-point strongly connected between $P\backslash\{y\}$ and $Q\backslash\{y\}$. By induction $X\backslash\{y\}$ contains a set $U_1, \cdots, U_{n-1}$ of disjoint open connected sets joining P and Q. Since X is regular and locally connected there exist by the chaining lemma open connected sets $V_1, \cdots, V_{n-1}$ such that for each i $\bar{V}_i \subset U_i$ and V_i joins P and Q. The sets $V_1, \cdots, V_{n-1}$ have closures which are disjoint from y and from each other. If $y \in P$ then y is clearly in S so $P \subset S$.

The set S is also closed in X. For let $y \in \bar{S}$. We may suppose $y \notin P$. Let A be the union of $(n-1)$ connected open sets $V_1, \cdots, V_{n-1}$ with disjoint closures joining P and Q such that $y \notin \bar{A}$. Let R be a connected region containing y such that $\bar{R} \cap (P \cup \bar{A}) = \varnothing$. Let $x \in R \cap S$. Let S_x be the union of n open connected sets $U_1, \cdots, U_n$ with pairwise disjoint closures such that $U_1, \cdots, U_{n-1}$ join P and Q and U_n joins P and x. For each $i = 1, \cdots, n-1$ let $\alpha_i = \alpha_{i,1} \cup \cdots \cup \alpha_{i,n_i}$ be an irreducible chain of open connected sets in V_i such that $\alpha_{i,1} \cap Q \neq \varnothing$, $\alpha_{i,n_i} \cap P \neq \varnothing$ and each $\alpha_{i,j}$ meets at most one of $\bar{U}_1, \cdots, \bar{U}_n$. For each $i = 1, \cdots, n$ let $\beta_i = \beta_{i,1} \cup \cdots \cup \beta_{i,m_i}$ be an irreducible chain of open connected sets in U_i with $\beta_{i,1} \cap P \neq \varnothing$, $\beta_{i,m_i} \cap (Q \cup R) \neq \varnothing$ and each $\beta_{i,j}$ meets at most one of the disjoint closed sets $\bar{V}_1, \cdots, \bar{V}_n$, $\bar{R}$. Let $B = \beta_1 \cup \cdots \cup \beta_n$.

In α_i let n_{1_i} be the smallest integer such that $\alpha_{i,n_{1_i}}$ meets $B \cup P$. Let ${}_1\alpha_i = \alpha_{i,1} \cup \cdots \cup \alpha_{i,n_{1_i}}$. Let $A_1 = {}_1\alpha_1 \cup \cdots \cup {}_1\alpha_{n-1}$. In β_i let m_{1_i} be the smallest integer such that $\beta_{i,m_{1_i}}$ meets $Q \cup R \cup A_1$. For each i let ${}_1\beta_i = \beta_{i,1} \cup \cdots \cup \beta_{i,m_{1_i}}$ and let $B_1 = {}_1\beta_1 \cup \cdots \cup {}_1\beta_n$. In α_i let n_{2_i} be the smallest integer such that $\alpha_{i,n_{2_i}}$ meets $B_1 \cup P$. Let ${}_2\alpha_i = \alpha_{i,1} \cup \cdots \cup \alpha_{i,n_{2_i}}$ and let $A_2 = {}_2\alpha_1 \cup \cdots \cup {}_2\alpha_{n-1}$. In β_i let m_{2_i} be the smallest integer such that $\beta_{i,m_{2_i}}$ meets $Q \cup R \cup A_2$. Let ${}_2\beta_i = \beta_{i,1} \cup \cdots \cup \beta_{i,m_{2_i}}$ and let $B_2 = {}_2\beta_i \cup \cdots \cup {}_2\beta_n$. We can continue this process indefinitely. For each i $1 \leqq m_{r+1_i} \leqq m_{r_i}$ and $n_{r_i} \leqq n_{r+1_i} \leqq n_i$. It follows that there exists a positive integer s such that $A_j = A_s$ and $B_j = B_s$ for all $j \geqq s$.

Now A_s and B_s are unions of $n-1$ and n respectively disjoint chains of open connected sets. For each $j = 1, \cdots, n$ ${}_s\beta_j$ meets at most one ${}_s\alpha_i$ and ${}_s\beta_j \cap {}_s\alpha_i \subset \alpha_{i,n_{s_i}}$. Also, for each $i = 1, \cdots, n-1$ ${}_s\alpha_i$ meets at most one ${}_s\beta_j$. For each $i = 1, \cdots, n-1$ let $e_i = {}_s\alpha_i$ if ${}_s\alpha_i$ meets P and let $e_i = {}_s\alpha_i \cup {}_s\beta_j$ where j is the unique integer such that ${}_s\beta_j$ meets ${}_s\alpha_i$ if ${}_s\alpha_i$ does not meet P. The sets $e_1, \cdots, e_{n-1}$ are disjoint chains of connected open sets such that e_i joins P to Q. Note that each e_i is disjoint from R. Since each ${}_s\alpha_i$ for $i = 1, \cdots, n-1$ meets at most one ${}_s\beta_j$ for $j = 1, \cdots, n$ there exists an ${}_s\beta_j$ which is disjoint from each of $e_1, \cdots, e_{n-1}$. If ${}_s\beta_j$ meets Q then $e_1, \cdots, e_{n-1}$, ${}_s\beta_j$ are n disjoint open sets which join P and Q and the theorem is true for X. If ${}_s\beta_j$ is disjoint from Q then ${}_s\beta_j$ meets R and so $e_1, \cdots, e_{n-1}$, ${}_s\beta_j \cup R$ are n disjoint open connected sets such that $e_1, \cdots, e_{n-1}$ join P and Q and ${}_s\beta_j \cup R$ joins P and y. Hence $y \in S$ and S is closed. It follows that S is a union of components of X. Since $P \subset S$ and X is n-point strongly connected between P and Q some component of X meets both P and Q. Hence $Q \cap S \neq \varnothing$. If $x \in Q \cap S$ then S_x satisfies the theorem.

The following result is called the second n-arc theorem by Menger **[2]**. It was first proved in the form given below by Whyburn **[3]**.

COROLLARY 2. *If X is a complete, locally connected, metric space* *that is n-point strongly connected between the two disjoint closed sets P and Q, then X contains n disjoint arcs joining P and Q.*

Proof. The corollary follows immediately from Theorem 1 since an open connected set in a complete, locally connected, metric space is arcwise connected.

2. n-large point connectedness. Let $\mathscr{C}$ be a family of disjoint closed subsets of a topological space X. Following Hunt **[1]**, we call a subset S of X a *large point* of X with respect to $\mathscr{C}$ if S is a point or S is a member of $\mathscr{C}$. We shall say that X is *n-large point strongly connected between* two disjoint closed sets *A and B with respect to $\mathscr{C}$* provided no set of fewer than n large points with respect to $\mathscr{C}$ separates A and B in the broad sense in X.

If $A_1, \cdots, A_n$ and B are disjoint closed subsets of a topological space X we say that a set of n disjoint sets $\alpha_1, \cdots, \alpha_n$ in X *joins $A_1, \cdots, A_n$ and B* if each α_i joins $A_1 \cup \cdots \cup A_n$ and B, each α_i meets exactly one A_j and each A_j meets exactly one α_i.

The following theorem was proved by Hunt **[1]** for the case X a locally compact, locally connected, metric space. It is obtained here as an easy corollary of our Theorem 1.

COROLLARY (Hunt) 3. *Let X be a normal, T_1, locally connected space and let $A_1, \cdots, A_n$ and B be disjoint closed sets in X. Let $\mathscr{C} = \{A_1, \cdots, A_n\}$. A necessary and sufficient condition that there be n disjoint open sets in X joining $A_1, \cdots, A_n$ and B is that X be n-large point strongly connected between $A_1 \cup \cdots \cup A_n$ and B with respect to $\mathscr{C}$.*

Proof. Define an equivalence relation $\sim$ on X by setting $x \sim y$ if and only if $x = y$ or $x, y \in A_i$ for some $i \in \{1, \cdots, n\}$. Then $\sim$ is a closed equivalence relation on X. Let $\pi: X \to X/\sim$ be the natural projection of X onto the quotient space $X/\sim$. Then $X/\sim$ is T_1. Since X is normal and $\sim$ has only a finite number of nondegenerate equivalence classes it follows that $X/\sim$ is regular. It is well-known (and easy to prove) that the quotient space of a locally connected space is locally connected. It is easy to check that $X/\sim$ is n-point strongly connected between $A = \pi(A_1 \cup \cdots \cup A_n)$ and B. By Theorem 1 there exist n-disjoint open connected sets $U_1, \cdots, U_n$ joining A and B. If $U_i \cap A = \pi(A_{j_i})$ then it is easy to see that $\pi^{-1}(U_i)$ joins A_{j_i} to B.

If $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$ are disjoint closed sets in a topological space X, a family of n disjoint open connected sets $U_1, \cdots, U_n$ in X is said to *join* $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$ if each U_i joins $A_1 \cup \cdots \cup A_n$ and

$B_1 \cup \cdots \cup B_n$, each U_i meets exactly one A_j and exactly one B_k, each B_i meets exactly one U_j and each A_i meets exactly one U_j.

The following corollary gives an affirmative solution to a conjecture posed by Hunt in **[1]**.

COROLLARY 4. *Let $A_1, \cdots, A_n$, $B_1, \cdots, B_n$ be disjoint closed subsets of a normal, T_1, locally connected space X. Let $\mathscr{C} = \{A_1, \cdots, A_n, B_1, \cdots, B_n\}$. A necessary and sufficient condition that there be n disjoint open connected sets in X joining $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$ is that X be n large point strongly connected between $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$ with respect to $\mathscr{C}$.*

Proof. The proof is similar to that of Theorem 3 and is omitted.

3. A question. It seems natural to ask if the preceding results have analogues for non locally connected spaces.

Question. If X is a regular, T_1 space and P and Q are disjoint closed sets in X such that X is n-point strongly connected between P and Q, do there exist disjoint open sets $U_1, \cdots, U_n$ such that U_i cannot be separated between P and Q?

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