

Pacific Journal of Mathematics

SOME n -ARC THEOREMS

EDWARD D. TYMCHATYN

SOME n -ARC THEOREMS

E. D. TYMCHATYN

G. T. Whyburn gave an inductive proof of the n -arc theorem for complete, locally connected, metric spaces. In this note Whyburn's proof is modified to generalize this theorem to the class of regular, T_1 , locally connected spaces. This result is then used to obtain an affirmative solution to a conjecture of J. H. V. Hunt.

Our notation will follow that of Whyburn [3] and Hunt [1].

Let X be a topological space and let P and Q be disjoint closed sets in X . A set C is said to *separate* P and Q in the broad sense in X if $X \setminus C = A \cup B$ where A is separated from B , $P \setminus C \subset A$ and $Q \setminus C \subset B$. The space X is said to be *n -point strongly connected between P and Q* if no subset of X with fewer than n points separates P and Q in the broad sense in X . A subset of X is said to *join* P and Q if some component of the set meets both P and Q .

1. The second n -arc theorem.

THEOREM 1. *The locally connected, regular, T_1 space X is n -point strongly connected between two disjoint closed sets P and Q if and only if there exist n disjoint open sets in X which join P and Q .*

Proof. The sufficiency is obvious. We shall prove necessity by induction on n . The case $n = 1$ follows from the fact that the components of X are open (as X is locally connected) and hence some component of X meets both P and Q . Suppose the theorem holds for all positive integers less than n .

Suppose X is n -point strongly connected between the disjoint closed sets P and Q . Let S denote the set of all $x \in X$ such that there exists a set S_x which is the union of n disjoint open connected sets $n - 1$ of which join P and Q and the n th one joins P and x . Then S is clearly open in X . If $y \in X$ then $X \setminus \{y\}$ is $(n - 1)$ -point strongly connected between $P \setminus \{y\}$ and $Q \setminus \{y\}$. By induction $X \setminus \{y\}$ contains a set U_1, \dots, U_{n-1} of disjoint open connected sets joining P and Q . Since X is regular and locally connected there exist by the chaining lemma open connected sets V_1, \dots, V_{n-1} such that for each i $\bar{V}_i \subset U_i$ and V_i joins P and Q . The sets V_1, \dots, V_{n-1} have closures which are disjoint from y and from each other. If $y \in P$ then y is clearly in S so $P \subset S$.

The set S is also closed in X . For let $y \in \bar{S}$. We may suppose $y \notin P$. Let A be the union of $(n-1)$ connected open sets V_1, \dots, V_{n-1} with disjoint closures joining P and Q such that $y \notin \bar{A}$. Let R be a connected region containing y such that $\bar{R} \cap (P \cup \bar{A}) = \emptyset$. Let $x \in R \cap S$. Let S_x be the union of n open connected sets U_1, \dots, U_n with pairwise disjoint closures such that U_1, \dots, U_{n-1} join P and Q and U_n joins P and x . For each $i = 1, \dots, n-1$ let $\alpha_i = \alpha_{i,1} \cup \dots \cup \alpha_{i,n_i}$ be an irreducible chain of open connected sets in V_i such that $\alpha_{i,1} \cap Q \neq \emptyset$, $\alpha_{i,n_i} \cap P \neq \emptyset$ and each $\alpha_{i,j}$ meets at most one of $\bar{U}_1, \dots, \bar{U}_n$. For each $i = 1, \dots, n$ let $\beta_i = \beta_{i,1} \cup \dots \cup \beta_{i,m_i}$ be an irreducible chain of open connected sets in U_i with $\beta_{i,1} \cap P \neq \emptyset$, $\beta_{i,m_i} \cap (Q \cup R) \neq \emptyset$ and each $\beta_{i,j}$ meets at most one of the disjoint closed sets $\bar{V}_1, \dots, \bar{V}_n, \bar{R}$. Let $B = \beta_1 \cup \dots \cup \beta_n$.

In α_i let $n_{i,1}$ be the smallest integer such that $\alpha_{i,n_{i,1}}$ meets $B \cup P$. Let ${}_1\alpha_i = \alpha_{i,1} \cup \dots \cup \alpha_{i,n_{i,1}}$. Let $A_1 = {}_1\alpha_1 \cup \dots \cup {}_1\alpha_{n-1}$. In β_i let $m_{i,1}$ be the smallest integer such that $\beta_{i,m_{i,1}}$ meets $Q \cup R \cup A_1$. For each i let ${}_1\beta_i = \beta_{i,1} \cup \dots \cup \beta_{i,m_{i,1}}$ and let $B_1 = {}_1\beta_1 \cup \dots \cup {}_1\beta_n$. In α_i let $n_{i,2}$ be the smallest integer such that $\alpha_{i,n_{i,2}}$ meets $B_1 \cup P$. Let ${}_2\alpha_i = \alpha_{i,1} \cup \dots \cup \alpha_{i,n_{i,2}}$ and let $A_2 = {}_2\alpha_1 \cup \dots \cup {}_2\alpha_{n-1}$. In β_i let $m_{i,2}$ be the smallest integer such that $\beta_{i,m_{i,2}}$ meets $Q \cup R \cup A_2$. Let ${}_2\beta_i = \beta_{i,1} \cup \dots \cup \beta_{i,m_{i,2}}$ and let $B_2 = {}_2\beta_1 \cup \dots \cup {}_2\beta_n$. We can continue this process indefinitely. For each i $1 \leq m_{r+1,i} \leq m_{r,i}$ and $n_{r,i} \leq n_{r+1,i} \leq n_i$. It follows that there exists a positive integer s such that $A_j = A_s$ and $B_j = B_s$ for all $j \geq s$.

Now A_s and B_s are unions of $n-1$ and n respectively disjoint chains of open connected sets. For each $j = 1, \dots, n$ ${}_s\beta_j$ meets at most one ${}_s\alpha_i$ and ${}_s\beta_j \cap {}_s\alpha_i \subset \alpha_{i,n_i}$. Also, for each $i = 1, \dots, n-1$ ${}_s\alpha_i$ meets at most one ${}_s\beta_j$. For each $i = 1, \dots, n-1$ let $e_i = {}_s\alpha_i$ if ${}_s\alpha_i$ meets P and let $e_i = {}_s\alpha_i \cup {}_s\beta_j$ where j is the unique integer such that ${}_s\beta_j$ meets ${}_s\alpha_i$ if ${}_s\alpha_i$ does not meet P . The sets e_1, \dots, e_{n-1} are disjoint chains of connected open sets such that e_i joins P to Q . Note that each e_i is disjoint from R . Since each ${}_s\alpha_i$ for $i = 1, \dots, n-1$ meets at most one ${}_s\beta_j$ for $j = 1, \dots, n$ there exists an ${}_s\beta_j$ which is disjoint from each of e_1, \dots, e_{n-1} . If ${}_s\beta_j$ meets Q then $e_1, \dots, e_{n-1}, {}_s\beta_j$ are n disjoint open sets which join P and Q and the theorem is true for X . If ${}_s\beta_j$ is disjoint from Q then ${}_s\beta_j$ meets R and so $e_1, \dots, e_{n-1}, {}_s\beta_j \cup R$ are n disjoint open connected sets such that e_1, \dots, e_{n-1} join P and Q and ${}_s\beta_j \cup R$ joins P and y . Hence $y \in S$ and S is closed. It follows that S is a union of components of X . Since $P \subset S$ and X is n -point strongly connected between P and Q some component of X meets both P and Q . Hence $Q \cap S \neq \emptyset$. If $x \in Q \cap S$ then S_x satisfies the theorem.

The following result is called the second n -arc theorem by Menger [2]. It was first proved in the form given below by Whyburn [3].

COROLLARY 2. *If X is a complete, locally connected, metric space*

that is n -point strongly connected between the two disjoint closed sets P and Q , then X contains n disjoint arcs joining P and Q .

Proof. The corollary follows immediately from Theorem 1 since an open connected set in a complete, locally connected, metric space is arcwise connected.

2. n -large point connectedness. Let \mathcal{C} be a family of disjoint closed subsets of a topological space X . Following Hunt [1], we call a subset S of X a *large point* of X with respect to \mathcal{C} if S is a point or S is a member of \mathcal{C} . We shall say that X is *n -large point strongly connected between* two disjoint closed sets A and B with respect to \mathcal{C} provided no set of fewer than n large points with respect to \mathcal{C} separates A and B in the broad sense in X .

If A_1, \dots, A_n and B are disjoint closed subsets of a topological space X we say that a set of n disjoint sets $\alpha_1, \dots, \alpha_n$ in X *joins* A_1, \dots, A_n and B if each α_i joins $A_1 \cup \dots \cup A_n$ and B , each α_i meets exactly one A_j and each A_j meets exactly one α_i .

The following theorem was proved by Hunt [1] for the case X a locally compact, locally connected, metric space. It is obtained here as an easy corollary of our Theorem 1.

COROLLARY (Hunt) 3. *Let X be a normal, T_1 , locally connected space and let A_1, \dots, A_n and B be disjoint closed sets in X . Let $\mathcal{C} = \{A_1, \dots, A_n\}$. A necessary and sufficient condition that there be n disjoint open sets in X joining A_1, \dots, A_n and B is that X be n -large point strongly connected between $A_1 \cup \dots \cup A_n$ and B with respect to \mathcal{C} .*

Proof. Define an equivalence relation \sim on X by setting $x \sim y$ if and only if $x = y$ or $x, y \in A_i$ for some $i \in \{1, \dots, n\}$. Then \sim is a closed equivalence relation on X . Let $\pi: X \rightarrow X/\sim$ be the natural projection of X onto the quotient space X/\sim . Then X/\sim is T_1 . Since X is normal and \sim has only a finite number of nondegenerate equivalence classes it follows that X/\sim is regular. It is well-known (and easy to prove) that the quotient space of a locally connected space is locally connected. It is easy to check that X/\sim is n -point strongly connected between $A = \pi(A_1 \cup \dots \cup A_n)$ and B . By Theorem 1 there exist n -disjoint open connected sets U_1, \dots, U_n joining A and B . If $U_i \cap A = \pi(A_{j_i})$ then it is easy to see that $\pi^{-1}(U_i)$ joins A_{j_i} to B .

If A_1, \dots, A_n and B_1, \dots, B_n are disjoint closed sets in a topological space X , a family of n disjoint open connected sets U_1, \dots, U_n in X is said to *join* A_1, \dots, A_n and B_1, \dots, B_n if each U_i joins $A_1 \cup \dots \cup A_n$ and

$B_1 \cup \cdots \cup B_n$, each U_i meets exactly one A_j and exactly one B_k , each B_i meets exactly one U_j and each A_i meets exactly one U_j .

The following corollary gives an affirmative solution to a conjecture posed by Hunt in [1].

COROLLARY 4. *Let $A_1, \cdots, A_n, B_1, \cdots, B_n$ be disjoint closed subsets of a normal, T_1 , locally connected space X . Let $\mathcal{C} = \{A_1, \cdots, A_n, B_1, \cdots, B_n\}$. A necessary and sufficient condition that there be n disjoint open connected sets in X joining A_1, \cdots, A_n and B_1, \cdots, B_n is that X be n large point strongly connected between A_1, \cdots, A_n and B_1, \cdots, B_n with respect to \mathcal{C} .*

Proof. The proof is similar to that of Theorem 3 and is omitted.

3. A question. It seems natural to ask if the preceding results have analogues for non locally connected spaces.

Question. If X is a regular, T_1 space and P and Q are disjoint closed sets in X such that X is n -point strongly connected between P and Q , do there exist disjoint open sets U_1, \cdots, U_n such that U_i cannot be separated between P and Q ?

REFERENCES

1. J. H. V. Hunt, *An n -arc theorem for Peano spaces*, Pacific J. Math., **36** (1971), 351–356.
2. K. Menger, *Kurventheorie*, Teubner, Berlin–Leipzig, 1932.
3. G. T. Whyburn, *On n -arc connectedness*, Trans. Amer. Math. Soc., **63** (1948), 452–456.

Received April 2, 1976 and in revised form May 31, 1976. This work was supported in part by a grant from the National Research Council of Canada.

UNIVERSITY OF SASKATCHEWAN

Helen Elizabeth Adams, <i>Factorization-prime ideals in integral domains</i>	1
Patrick Robert Ahern and Robert Bruce Schneider, <i>The boundary behavior of Henkin's kernel</i>	9
Daniel D. Anderson, Jacob R. Matijevic and Warren Douglas Nichols, <i>The Krull intersection theorem. II</i>	15
Efraim Pacillas Armendariz, <i>On semiprime P.I.-algebras over commutative regular rings</i>	23
Robert H. Bird and Charles John Parry, <i>Integral bases for bicyclic biquadratic fields over quadratic subfields</i>	29
Tae Ho Choe and Young Hee Hong, <i>Extensions of completely regular ordered spaces</i>	37
John Dauns, <i>Generalized monofrom and quasi injective modules</i>	49
F. S. De Blasi, <i>On the differentiability of multifunctions</i>	67
Paul M. Eakin, Jr. and Avinash Madhav Sathaye, <i>R-endomorphisms of $R[[X]]$ are essentially continuous</i>	83
Larry Quin Eifler, <i>Open mapping theorems for probability measures on metric spaces</i>	89
Garret J. Etgen and James Pawlowski, <i>Oscillation criteria for second order self adjoint differential systems</i>	99
Ronald Fintushel, <i>Local S^1 actions on 3-manifolds</i>	111
Kenneth R. Goodearl, <i>Choquet simplexes and σ-convex faces</i>	119
John R. Graef, <i>Some nonoscillation criteria for higher order nonlinear differential equations</i>	125
Charles Henry Heiberg, <i>Norms of powers of absolutely convergent Fourier series: an example</i>	131
Les Andrew Karlovitz, <i>Existence of fixed points of nonexpansive mappings in a space without normal structure</i>	153
Gangaram S. Ladde, <i>Systems of functional differential inequalities and functional differential systems</i>	161
Joseph Michael Lambert, <i>Conditions for simultaneous approximation and interpolation with norm preservation in $C[a, b]$</i>	173
Ernest Paul Lane, <i>Insertion of a continuous function</i>	181
Robert F. Lax, <i>Weierstrass points of products of Riemann surfaces</i>	191
Dan McCord, <i>An estimate of the Nielsen number and an example concerning the Lefschetz fixed point theorem</i>	195
Paul Milnes and John Sydney Pym, <i>Counterexample in the theory of continuous functions on topological groups</i>	205
Peter Johanna I. M. De Paepe, <i>Homomorphism spaces of algebras of holomorphic functions</i>	211
Judith Ann Palagallo, <i>A representation of additive functionals on L^p-spaces, $0 < p < 1$</i>	221
S. M. Patel, <i>On generalized numerical ranges</i>	235
Thomas Thornton Read, <i>A limit-point criterion for expressions with oscillatory coefficients</i>	243
Elemer E. Rosinger, <i>Division of distributions</i>	257
Peter S. Shoenfeld, <i>Highly proximal and generalized almost finite extensions of minimal sets</i>	265
R. Sirois-Dumais and Stephen Willard, <i>Quotient-universal sequential spaces</i>	281
Robert Charles Thompson, <i>Convex and concave functions of singular values of matrix sums</i>	285
Edward D. Tymchatyn, <i>Some n-arc theorems</i>	291
Jang-Mei Gloria Wu, <i>Variation of Green's potential</i>	295