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TAX STRUCTURES WHOSE PROGRESSIVITY IS INFLATION NEUTRAL

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It is the purpose of this paper to formalize mathematically the effect of inflation on the progressivity or vertical equity of individual income tax using some standard measures of vertical equity. As an application, we produce tax structures whose progressivity is inflation neutral.

Suppose that $T(y)$ is the tax liability for an individual with income y. Then the effective tax rate $A_T(y)$ is defined by the formula $A_T(y) = T(y)/y$. A tax is generally recognized as progressive if the effective rate of taxation is an increasing nonconstant function of income. If the tax function T is differentiable, then A_T is strictly increasing if and only if the marginal tax rate $T'(y)$ exceeds the effective tax rate at each income level. If the marginal tax rate is an increasing function of income, then the tax is progressive. The converse is of course invalid. A differentiable tax function T is called *confiscatory* at income level y if $T'(y) \geqq 1$.

Musgrave and Tun Thin [4] present several methods of describing the degree of progressivity of a tax, but none are universally accepted. Frequently, the vertical equity of a tax is measured by the steepness of its effective tax rate curve, and tax function T_1 is called more progressive than tax function T_2 if at each income level y,

$$A'_{T_1}(y) > A'_{T_2}(y).$$

Surely it is not the size of the effective rate but its rate of increase which determines the relative progressivity of the tax. For example, a 75 percent effective tax rate on each taxpayer is not progressive at all, although the effective rate is high. Analogously, comparative progressivity can be gauged with reference to the steepness of the marginal tax rate curve: tax function T_1 is more progressive than T_2 if $T''_1(y) > T''_2(y)$ for all y. Alternatively, one can measure progressivity in terms of the elasticity of tax liability to pre-tax income at each level of income or the elasticity of post-tax income to pre-tax income.

Since the basic goal of progressive taxation is to ensure an equitable distribution of income, many economists favor indices of vertical equity that indicate the tax structure's performance of this task. In general, a numerical index is assigned to an income distribution representing the inequality inherent therein. The progressivity of a tax with respect to this particular income distribution is then gauged by the improvement of this index after the tax burden has been deducted. The pioneering treatise on the measurement of inequality in income is due to Dalton [2]. Dalton preferred the relative standard deviation (coefficient of variation) and the relative mean difference (Gini ratio) to the other measures which he considered. In this article we will use the Gini ratio exclusively to measure the dispersion of income. The Gini ratio G for a population of n individuals with incomes $y_1, y_2, \cdots, y_n$ is defined by

$$G = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j| \Big/ \frac{1}{n} \sum_{i=1}^{n} y_i.$$

If we list these individuals in ascending order of income, we may write

$$G = \frac{2}{n} \frac{\sum_{i=1}^{n} \sum_{j<i} y_i - y_j}{\sum_{i=1}^{n} y_i}.$$

It is well known that this index is exactly four times the value of the Gini coefficient, defined as the area between the Lorenz curve for the distribution of incomes and the line of perfect equality [5].

Clearly a proportional tax transforms a fixed income distribution into a different one without altering the original Gini ratio, since the relative disparity in incomes is not changed. We shall call a tax *redistributive* with respect to a particular income population if the Gini ratio for the post-tax income distribution is smaller than the Gini ratio for the pre-tax income distribution. A tax may be redistributive with respect to some income distributions, but not with respect to others. It appears that a progressive tax can only improve the Gini ratio, but this is not trivial to verify. It suffices to establish the following result.

THEOREM. *Let $y_1, \cdots, y_n$ and $r_1, \cdots, r_n$ be positive numbers having the following properties whenever $i > j$:* (1) $r_i \leqq r_j$ (2) $y_i \geqq y_j$ (3) $r_i y_i \geqq r_j y_j$. *Then the Gini ratio for the income distribution $y_1, \cdots, y_n$ is not less than the Gini ratio for the income distribution $r_1 y_1, \cdots, r_n y_n$.*

The following lemma is crucial in the proof.

LEMMA. *Let $r_1, \cdots, r_n$ be a sequence of positive numbers satisfying $r_i \leqq r_j$ whenever $i > j$. Let $b_1, \cdots, b_n$ be a sequence of positive numbers and $a_1, \cdots, a_n$ be a sequence of nonnegative numbers satisfying $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and $a_i / b_i \geqq a_k / b_k$ whenever $i > k$. Then $\sum_{i=1}^{n} a_i r_i \leqq \sum_{i=1}^{n} b_i r_i$.*

Proof. If $a_i \leqq b_i$, the last condition implies that $a_k \leqq b_k$ whenever $k \leqq i$. Suppose the nonempty set $\{i: a_i \leqq b_i\}$ is $\{1, \cdots, m\}$. Since $\Sigma_{i=1}^n a_i = \Sigma_{i=1}^n b_i$, we have $\Sigma_{i=1}^m b_i - a_i = \Sigma_{i=m+1}^n a_i - b_i$. From the first condition, we conclude that $\Sigma_{i=1}^m (b_i - a_i) r_i \geqq \Sigma_{i=m+1}^n (a_i - b_i) r_i$, so that $\Sigma_{i=1}^n b_i r_i \geqq \Sigma_{i=1}^n a_i r_i$.

Proof of the theorem. We must show that

$$\text{(1)} \qquad \frac{\sum_{i=1}^n \sum_{j<i} r_i y_i - r_j y_j}{\sum_{i=1}^n r_i y_i} \leqq \frac{\sum_{i=1}^n \sum_{j<i} y_i - y_j}{\sum_{i=1}^n y_i}.$$

Since $r_i y_j \leqq r_j y_j$ whenever $i > j$, for $i = 1, \cdots, n$ we have $\Sigma_{j<i} r_i y_i - r_j y_j \leqq r_i \Sigma_{j<i} y_i - y_j$. Hence to establish (1), we need only show that

$$\text{(2)} \qquad \frac{\sum_{i=1}^n \left(\sum_{j<i} y_i - y_j \right) r_i}{\sum_{i=1}^n \sum_{j<i} y_i - y_j} \leqq \frac{\sum_{i=1}^n y_i r_i}{\sum_{i=1}^n y_i}.$$

Both sides of (2) are convex combinations of $r_1, r_2, \cdots, r_n$. This suggests applying the lemma with

$$a_k = \frac{\sum_{j<k} y_k - y_j}{\sum_{i=1}^n \sum_{j<i} y_i - y_j} \quad \text{and} \quad b_k = \frac{y_k}{\sum_{i=1}^n y_i}$$

provided we can show that

$$\text{(3)} \qquad \frac{\sum_{j<i} y_i - y_j}{y_i} \geqq \frac{\sum_{j<k} y_k - y_j}{y_k}$$

whenever $k < i$. Crossmultiplying and adding $y_k \Sigma_{j<i} y_j$ to both sides we easily see that (3) is equivalent to

$$\text{(4)} \qquad (i - k) y_i y_k \geqq (y_k - y_i) \sum_{j=1}^{k-1} y_j + y_k \sum_{j=k}^{i-1} y_j$$

The first expression on the right side of (4) is nonpositive, whereas $(i - k) y_i y_k \geqq y_k \Sigma_{j=k}^{i-1} y_j$ since $y_i \geqq y_j$ for $j = k, \cdots, i - 1$. Hence (3) is established and the theorem is proved.

2. Tax progressivity and inflation. Tax liability is a function of *nominal income*, income measured in dollars rather than the ability to command goods and services. Let us refer to the nominal dollars of a fixed base year as *real dollars.* To analyze the impact of pure inflation on a progressive personal income tax, we make certain restrictive assumptions about our population and the characteristics of our tax. We assume an unchanged distribution of real income from year to year. In particular, we insist that the relative distribution of nominal income is not altered by the inflation and that the economy experiences no real growth. We assume that the treatment of a particular nominal income class by the tax structure throughout the inflationary period is time invariant. Thus if an average income earner with nominal income y pays a tax $T(y)$ in a fixed year, an income earner with the same nominal (but lower real) income would pay the same tax in subsequent years.

Suppose that the price level has risen by a factor of $c > 1$, and $T(y)$ is the tax liability for an individual with nominal income y determined by a progressive tax operative since the base year. At this time cy nominal dollars command the same economic power as y real dollars; such an income determines a tax liability of $(1/c)T(cy)$ real dollars. Since A_T is an increasing function, we have

$$\frac{T(cy)}{cy} \geqq \frac{T(y)}{y} \quad \text{so that} \quad \frac{1}{c}\,T(cy) \geqq T(y).$$

Hence, each taxpayer's real tax liability can only increase. After inflation, real post-tax income corresponding to real pre-tax income y is simply $y-(1/c)T(cy)$. This may be expressed as

$$[y - T(y)] - \frac{1}{c}\left[\frac{T(cy) - cT(y)}{y - T(y)}\right][y - T(y)]. \tag{5}$$

This formulation allows one to view inflation coupled with a progressive tax as a composition of taxes applied to real pre-tax income. Initially, the tax structure independent of inflation reduces each income y to $y - T(y)$. Then, inflation in conjunction with the tax structure reduces this new income figure by a fraction of itself as displayed in (5). *A priori*, one cannot say how this fraction varies with $y - T(y)$.

If the original tax were geared to produce a post-tax income distribution with a prescribed Gini ratio in the base year, it is desirable that this property be inflation neutral. If $T(cy) - cT(y)/y - T(y)$ does not depend on income but only on the value of the dollar determined by c, then the inflation induced component of the tax will be proportional. Thus the redistributivity of the tax in the base year as measured by the Gini ratio of the post-tax income distribution is unchanged by inflation. In this case we can write

$$h(c) = \frac{T(cy) - cT(y)}{y - T(y)}.$$

We now characterize those progressive tax functions of this form assuming the differentiability of h and T. Differentiating with respect to c, we obtain

$$h'(c) = \frac{yT'(cy) - T(y)}{y - T(y)}$$

which can be rewritten as

$$h'(c) - 1 = \frac{yT'(cy) - y}{y - T(y)}.$$

Setting $c = 1$ yields

$$h'(1) - 1 = \frac{yT'(y) - y}{y - T(y)}. \tag{6}$$

If $h'(1) = 1$, then it is easy to see that $T(y) = y + k$, whence $h(c) = c - 1$. If $h'(1) \neq 1$, set $k_1 = h'(1) - 1$. The differential equation (6) can be rewritten as

$$yT'(y) + k_1 T(y) = (k_1 + 1)y.$$

This first order linear differential equation has the general solution

$$T(y) = y + k_2 y^{-k_1}$$

and $h(c) = c - c^{-k_1}$. The constant k_2 must be negative, or else $A_T(y) \geqq 1$ for all y. If $k_1 < -1$, then the tax is regressive; if $k_1 = -1$, then it is proportional. If k_1 is nonnegative, then the tax will be confiscatory at all levels of income, for $T'(y) \geqq 1$ for all y. Only when $-1 < k_1 < 0$ is the tax progressive and nonconfiscatory. In this case the tax function yields a negative income tax on sufficiently low incomes. As one might expect, these tax structures are those for which the elasticity of post-tax income with respect to pre-tax income is constant.

If the steepness of the effective tax rate curve is used to measure progressivity, then we can again characterize those tax functions whose progressivity is inflation neutral. Let S denote the tax function arising from the tax function T distorted by inflation. At each level of real income y, it is obvious that $S(y)=(1/c)T(cy)$. Denoting the corresponding effective tax rate function by A_S, we have $A_S(y)=A_T(cy)$, so that $A'_S(y)=cA'_T(cy)$. The quantity $A'_S(y)$ indicates the steepness of the distorted effective tax rate curve at real income y.

If the progressivity of T is to be invariant under a change in the price level in the above sense, we must have for all positive c and y

$$A'_T(y)=A'_S(y)=cA'_T(cy)$$

Setting $y=1$ yields $A'_T(y)=(1/y)A'_T(1)$. We can now solve for the tax function T in the following equation:

$$\frac{yT'(y)-T(y)}{y^2}=A'_T(y)=\frac{k_1}{y}. \tag{7}$$

Rewriting this in the form $T'(y)-T(y)/y=k_1$ and using the integrating factor $1/y$ we obtain

$$T(y)=k_1 y \ln y + k_2 y.$$

From (7) we note that T is progressive if and only if k_1 is positive, and in the present context k_1 measures the progressivity of T.

The steepness of the marginal tax rate curve for the tax function S is given by $S''(y)=cT''(cy)$. Thus the steepness of the original marginal tax rate curve is not altered if $T''(y)=cT''(cy)$, and it follows that $T''(y)=k_1/y$. The general solution for T is now easily produced:

$$T(y)=k_1 y \ln y + k_2 y + k_3.$$

Notice that if $k_3=0$, then such a tax function's progressivity is inflation neutral with respect to both the effective and marginal tax rate curves. For small incomes a negative income tax is obtained, but $\lim_{y\to 0^+} T(y)=0$. Unfortunately, a far more aesthetically unpleasant observation must be made: the tax becomes confiscatory at high income levels.

3. The progressivity of an exponential tax function. Mishan and Dicks–Mireaux [3] and later Blackburn [1] present strong statistical evidence that the federal individual income tax can be reasonably represented by the exponential model, $T(y)=ay^b$ where b is near 1.4. Each measure of progressivity developed in this article reveals that such a tax becomes more progressive through inflation. Since $T(y)=ay^b$ we quickly obtain

$$\frac{T(cy) - cT(y)}{y - T(y)} = \frac{acy^{b-1}(c^{b-1} - 1)}{1 - ay^{b-1}}. \tag{8}$$

For fixed c the right side of (8) is a strictly increasing function of income, assuming that the tax is never confiscatory. From (5) it follows that the inflation modified exponential tax function can be viewed as the composition of T followed by a progressive tax function. Thus, for each pre-tax income distribution in the base year, the Gini ratio for the corresponding post-tax income distribution before inflation exceeds the Gini ratio for the post-tax income distribution after inflation. Thus the inflation distorted tax does a better job of redistributing income.

In addition, the steepness of the effective tax rate curve and the marginal tax rate curve attributed to T is increased at each level of real income. If S again denotes T transformed by inflation by a factor of c, then at each value of real income y

$$A_S'(y) = cA_T'(cy) = a(b-1)c^{b-1}y^{b-2} > a(b-1)y^{b-2} = A_T'(y)$$

and

$$S''(y) = cT''(cy) = ab(b-1)c^{b-1}y^{b-2} > ab(b-1)y^{b-2} = T''(y).$$

The exponential tax curve does not approximate the actual tax curve particularly well at very high income levels; indeed, it must become confiscatory at the highest levels. Thus, the above reasoning probably is inappropriate for gauging the impact of inflation on progressivity over a long period since a sizeable segment of the population will secure these large nominal incomes. Nevertheless, the claim that the rich are more heavily penalized by an existing progressive tax in an inflationary period is well-founded, for a large percentage of the taxable income that accrues to the rich vis-a-vis capital gains and interest income is simply illusory.

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