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ON A CLASS OF UNBOUNDED OPERATOR ALGEBRAS. II

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In this paper we continue our study of unbounded operator algebras. On the basis of the space $L^\infty[0, 1]$ introduced by R. Arens [1] we define and investigate unbounded Hilbert algebras. The primary purpose of this paper is to investigate the relation between unbounded Hilbert algebras and EW^* -algebras and the structure of some EW^* -algebras.

1. Introduction. In a previous paper [10] we began our study of EW^* -algebras. For the definitions and the basic properties concerning EW^* -algebras is referred to [10]. It is well known that semifinite von Neumann algebras are related to Hilbert algebras. That is, if \mathcal{D}_0 is a Hilbert algebra, then the left von Neumann algebra $\mathcal{U}_0(\mathcal{D}_0)$ is defined and $\mathcal{U}_0(\mathcal{D}_0)$ is a semifinite von Neumann algebra and conversely if \mathfrak{A} is a semifinite von Neumann algebra, then there exists a Hilbert algebra \mathcal{D}_0 such that \mathfrak{A} is isomorphic to the left von Neumann algebra $\mathcal{U}_0(\mathcal{D}_0)$. In this paper we study the above facts about EW^* -algebras. So, our starting point will be the extension of Hilbert algebras.

DEFINITION 1.1. Let \mathcal{D} be a pre-Hilbert space with inner product (\mid) and a $*$ -algebra. If \mathcal{D} satisfies the following conditions (1) ~ (3);

- (1) $(\xi \mid \eta) = (\eta^* \mid \xi^*), \quad \xi, \eta \in \mathcal{D};$
- (2) $(\xi\eta \mid \zeta) = (\eta \mid \xi^*\zeta), \quad \xi, \eta, \zeta \in \mathcal{D};$

By (2) we define $\pi(\xi)$ and $\pi'(\eta)$ by;

$$\pi(\xi)\eta = \pi'(\eta)\xi = \xi\eta, \quad \xi, \eta \in \mathcal{D}.$$

Then $\pi(\xi)$ and $\pi'(\eta)$ are closable operators on \mathcal{D} and we have $\pi(\xi)^* \supset \pi(\xi^*)$ and $\pi'(\eta)^* \supset \pi'(\eta^*)$. We call π (resp. π') the left (resp. right) regular representation of \mathcal{D} .

(3) Putting

$$\mathcal{D}_0 = \{\xi \in \mathcal{D}; \pi(\xi) \text{ is continuous}\},$$

\mathcal{D}_0^2 is dense in \mathcal{D} , then \mathcal{D} is called an unbounded Hilbert algebra over \mathcal{D}_0 . In particular, if $\mathcal{D}_0 \neq \mathcal{D}$, then \mathcal{D} is called a pure unbounded Hilbert algebra over \mathcal{D}_0 .

In §2 we investigate the properties of unbounded Hilbert algebras and we introduce examples of such unbounded Hilbert algebras $(L^\infty[0, 1],$

$L^\omega(-\infty, \infty)$, $L_1^\omega(-\infty, \infty)$, $L_2^\omega(-\infty, \infty)$, $L_1^\omega(G)$, $L_2^\omega(G)$ (G ; unimodular locally compact group)).

In §3 we consider the noncommutative integration with respect to a von Neumann algebra as constructed by Segal in [14]. Let \mathcal{D} be a pure unbounded Hilbert algebra over \mathcal{D}_0 . Then $L^\omega(\mathcal{D}_0)$ and $L_2^\omega(\mathcal{D}_0)$ are defined and they are pure unbounded Hilbert algebras. In particular, $L_2^\omega(\mathcal{D}_0)$ is maximal in pure unbounded Hilbert algebras containing \mathcal{D}_0 . Furthermore \mathcal{D}^2 (resp. \mathcal{D}) is a $*$ -subalgebra of pure unbounded Hilbert algebra $L^\omega(\mathcal{D}_0)$ (resp. $L_2^\omega(\mathcal{D}_0)$) (Theorem 3.9.). We can define a left EW^* -algebra $\mathcal{U}(\mathcal{D})$ of a pure unbounded Hilbert algebra \mathcal{D} over \mathcal{D}_0 , i.e., $\mathcal{U}(\mathcal{D})$ is a minimal EW^* -algebra on $L_2^\omega(\mathcal{D}_0)$ over $\mathcal{U}_0(\mathcal{D}_0)$ and $\mathcal{U}(\mathcal{D}) \supset \pi(\mathcal{D})$, where we denote by \bar{A} the smallest closed extension of a closable operator A and we put $\bar{\mathfrak{A}} = \{\bar{A}; A \in \mathfrak{A}\}$ (Theorem 3.10.).

In §4 we define traces on EW^* -algebras and we investigate the structure of some EW^* -algebras.

DEFINITION 1.2. Let \mathfrak{A} be an EW^* -algebra and let φ be a map of \mathfrak{A}^+ into $[0, \infty]$. If φ satisfies the following conditions (1) ~ (3), then φ is called a trace on \mathfrak{A}^+ ;

- (1) $\varphi(S + T) = \varphi(S) + \varphi(T)$, $S, T \in \mathfrak{A}^+$;
- (2) $\varphi(\lambda S) = \lambda \varphi(S)$, $\lambda \geq 0$, $S \in \mathfrak{A}^+$;
- (3) $\varphi(S^*S) = \varphi(SS^*)$, $S \in \mathfrak{A}$.

If the conditions $\varphi(S) = 0$, $S \in \mathfrak{A}^+$ implies $S = 0$, then φ is called faithful. If, for each increasing net $\{T_\alpha\}$ of \mathfrak{A}^+ that converges σ -weakly to $S \in \mathfrak{A}^+$ (hereafter we denote $T_\alpha \uparrow S$), we have $\varphi(T_\alpha) \uparrow \varphi(S)$, then φ is called normal. If $\varphi(S) < \infty$ for every $S \in \mathfrak{A}^+$, then φ is called finite. If, for each $S \in \mathfrak{A}^+$, there exists a net $\{T_\alpha\}$ such that $T_\alpha \uparrow S$ and $\varphi(T_\alpha) < \infty$, then φ is called semifinite.

Let $\mathcal{U}(\mathcal{D})$ be the left EW^* -algebra of a pure unbounded Hilbert algebra \mathcal{D} over \mathcal{D}_0 . Then there exists a faithful normal semifinite trace φ on $\mathcal{U}(\mathcal{D})^+$ such that $\varphi/\mathcal{U}(\mathcal{D})_b^+$ equals the natural trace on $\mathcal{U}_0(\mathcal{D}_0)^+$ and $\mathcal{U}(\mathcal{D})(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi$ (we note $\mathfrak{N}_\varphi = \{T \in \mathcal{U}(\mathcal{D}); \varphi(T^*T) < \infty\}$ and $(\mathfrak{N}_\varphi)_b = \mathfrak{N}_\varphi \cap \mathcal{U}(\mathcal{D})_b$) (Theorem 4.2.). Conversely if \mathfrak{A} is an EW^* -algebra with a faithful normal semifinite trace φ satisfying $\mathfrak{A}(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi$, then \mathfrak{N}_φ is a pure unbounded Hilbert algebra over $(\mathfrak{N}_\varphi)_b$ and \mathfrak{A} is isomorphic to the left EW^* -algebra $\mathcal{U}(\mathfrak{N}_\varphi)$ of \mathfrak{N}_φ (Theorem 4.11.).

2. Unbounded Hilbert algebras. In this section let \mathcal{D} be a pure unbounded Hilbert algebra over \mathcal{D}_0 and let \mathfrak{H} be the completion of \mathcal{D} . Clearly \mathcal{D}_0 is a Hilbert algebra and the completion of \mathcal{D}_0 is a Hilbert space \mathfrak{H}_0 . For each $x \in \mathfrak{H}$ we define $\pi_0(x)$ and $\pi'_0(x)$ by;

$$\begin{aligned} \pi_0(x)\xi &= \overline{\pi'_0(\xi)}x, & \xi \in \mathcal{D}_0 \\ \pi'_0(x)\xi &= \overline{\pi_0(\xi)}x, & \xi \in \mathcal{D}_0, \end{aligned}$$

where π_0 (resp. π'_0) is the left (resp. right) regular representation of the Hilbert algebra \mathfrak{D}_0 . Then $\pi_0(x)$ and $\pi'_0(x)$ are linear operators on \mathfrak{H} with domain \mathcal{D}_0 . By ([12] Theorem 3) we have

$$\overline{\pi_0(Jx)} = \pi_0(x)^*, \quad \overline{\pi'_0(Jx)} = \pi'_0(x)^*$$

for all $x \in \mathfrak{H}$, where J denotes the involution of \mathfrak{H} .

LEMMA 2.1. *For each $\xi \in \mathcal{D}$ we have*

- (1) $\overline{\pi(\xi)} = \pi_0(\xi)$, $\overline{\pi'(\xi)} = \pi'_0(\xi)$;
- (2) $\overline{\pi(\xi^*)} = \pi(\xi)^*$, $\overline{\pi'(\xi^*)} = \pi'(\xi)^*$.

Proof. (1); Clearly we get $\pi_0(\xi) \subset \pi(\xi)$. Hence $\pi_0(\xi)^* \supset \pi(\xi)^*$. Since $\pi_0(\xi)^* = \pi_0(\xi^*)$ and $\pi(\xi)^* \supset \pi(\xi^*)$, we have

$$\overline{\pi(\xi)} = \pi_0(\xi^*)^* \supset \pi(\xi^*)^* \supset \overline{\pi(\xi)}.$$

Therefore we get $\overline{\pi_0(\xi)} = \overline{\pi(\xi)}$.

(2); By (1) we have

$$\overline{\pi(\xi^*)} = \overline{\pi_0(\xi^*)} = \pi_0(\xi)^* = \pi(\xi)^*.$$

LEMMA 2.2. *For each $\lambda, \mu \in \mathfrak{C}$ (the field of complex numbers) and $\xi, \xi_i, \eta, \eta_i \in \mathcal{D}$ ($i = 1, 2$) we have*

$$\begin{aligned} \pi(\lambda\xi_1 + \mu\xi_2) &= \lambda\pi(\xi_1) + \mu\pi(\xi_2); \\ \pi(\xi_1\xi_2) &= \pi(\xi_1)\pi(\xi_2); \\ \pi(\xi^*) &\subset \pi(\xi)^*; \\ \pi'(\lambda\eta_1 + \mu\eta_2) &= \lambda\pi'(\eta_1) + \mu\pi'(\eta_2); \\ \pi'(\eta_1\eta_2) &= \pi'(\eta_2)\pi'(\eta_1); \\ \pi'(\eta^*) &\subset \pi'(\eta)^*. \end{aligned}$$

Putting

$$\pi(\xi)^\# = \pi(\xi^*), \quad \pi'(\eta)^\# = \pi'(\eta^*),$$

$\pi(\mathcal{D})$ and $\pi'(\mathcal{D})$ are $\#$ -algebras on \mathcal{D} and we have the following properties;

$$(1) \quad \pi(\mathcal{D})_b = \pi(\mathcal{D}_0), \quad \pi'(\mathcal{D})_b = \pi'(\mathcal{D}_0);$$

- (2) $J\pi(\xi)J = \pi'(\xi)^{\#}, \quad J\pi'(\xi)J = \pi(\xi)^{\#}, \quad \xi \in \mathcal{D};$
 (3) $\pi(\xi)\pi'(\eta) = \pi'(\eta)\pi(\xi), \quad \xi, \eta \in \mathcal{D};$
 (4) $\overline{\pi(\xi)^{\#}} = \pi(\xi)^*, \quad \overline{\pi'(\xi)^{\#}} = \pi'(\xi)^*, \quad \xi \in \mathcal{D}.$

Hence we get

$$\overline{\pi(\mathcal{D})}_b'' = \mathcal{U}_0(\mathcal{D}_0), \quad \overline{\pi'(\mathcal{D})}_b'' = \mathcal{V}_0(\mathcal{D}_0),$$

where $\mathcal{U}_0(\mathcal{D}_0)$ (resp. $\mathcal{V}_0(\mathcal{D}_0)$) is the left (resp. right) von Neumann algebra of \mathcal{D}_0 .

PROPOSITION 2.3. For each $\lambda \in \mathfrak{G}$ and $\xi, \eta \in \mathcal{D}$ we have

$$\begin{aligned} \overline{\pi(\xi) + \pi(\eta)} &= \overline{\pi(\xi + \eta)}, \quad \overline{\pi(\xi) \cdot \pi(\eta)} = \overline{\pi(\xi\eta)}, \\ \lambda \cdot \overline{\pi(\xi)} &= \overline{\pi(\lambda\xi)}, \quad \overline{\pi(\xi)^*} = \overline{\pi(\xi^*)}. \end{aligned}$$

Therefore $\overline{\pi(\mathcal{D})}$ is a $*$ -algebra of closed operators on \mathfrak{H} under the operations of strong sum, strong product, adjoint and strong scalar multiplication. Similarly $\overline{\pi'(\mathcal{D})}$ is a $*$ -algebra of closed operators on \mathfrak{H} . Furthermore we have

$$\overline{J\pi(\xi)J} = \pi'(\xi)^*, \quad \overline{J\pi'(\xi)J} = \pi(\xi)^*, \quad \xi \in \mathcal{D}.$$

Proof. By Lemma 2.1. we have $\overline{\pi(\xi)} = \pi(\xi^*)^*$ for every $\xi \in \mathcal{D}$ and hence

$$\begin{aligned} \overline{\pi(\xi) + \pi(\eta)} &= \overline{\overline{\pi(\xi)} + \overline{\pi(\eta)}} = \overline{\pi(\xi^*)^* + \pi(\eta^*)^*} \\ &\subset (\pi(\xi^*) + \pi(\eta^*))^* = \pi((\xi + \eta)^*)^* \\ &= \overline{\pi(\xi + \eta)}, \end{aligned}$$

and so $\overline{\pi(\xi) + \pi(\eta)} = \overline{\pi(\xi + \eta)}$. Similarly $\overline{\pi(\xi) \cdot \pi(\eta)} = \overline{\pi(\xi)\pi(\eta)} = \overline{\pi(\xi\eta)}$ and $\lambda \cdot \overline{\pi(\xi)} = \overline{\pi(\lambda\xi)}$ are showed. By Lemma 2.2 (2) we have $J\pi(\xi)J = \pi'(\xi)^{\#}$, $\xi \in \mathcal{D}$ and hence $\overline{J\pi(\xi)J} = \overline{\pi'(\xi)^{\#}} = \pi'(\xi)^*$ by Lemma 2.1. On the other hand we can easily show $\overline{J\pi(\xi)J} = \overline{J\pi(\xi)J}$. Therefore we have $\overline{J\pi(\xi)J} = \pi'(\xi)^*$.

Problem. Does there exist an EW^* -algebra \mathfrak{A} such that $\overline{\mathfrak{A}}_b = \mathcal{U}_0(\mathcal{D}_0)$ and $\overline{\mathfrak{A}} \supset \overline{\pi(\mathcal{D})}$?

In §3 we show that there exist such EW^* -algebras. In particular, there exists an EW^* -algebra such that is minimal in such EW^* -algebras and we call it the left EW^* -algebra of \mathcal{D} .

We introduce examples of unbounded Hilbert algebras.

(1) $L^\omega[0, 1]$. Let $L^\omega[0, 1]$ be the set of all complex-valued measurable functions f on $[0, 1]$ such that $f \in L^p[0, 1]$, $p = 1, 2, \dots$. By the whole collection of L^p -norms

$$\|f\|_p = \left[\int_0^1 |f(t)|^p dt \right]^{1/p}, \quad p = 1, 2, \dots$$

and by pointwise multiplication and involution ($f^*(t) = \overline{f(t)}$, $t \in [0, 1]$) the space $L^\omega[0, 1]$ is a complete metrizable locally convex $*$ -algebra with jointly continuous multiplication. R. Arens [1] showed $L^\omega[0, 1]$ is not a locally m -convex algebra. However, G. R. Allan [2] showed that $L^\omega[0, 1]$ is a GB^* -algebra with $(L^\omega[0, 1])_0 = L^\infty[0, 1]$. We introduce the inner product into $L^\omega[0, 1]$ by;

$$(f | g) = \int_0^1 f(t) \overline{g(t)} dt, \quad f, g \in L^\omega[0, 1].$$

Then $L^\omega[0, 1]$ is regarded as a pure unbounded Hilbert algebra over $L^\infty[0, 1]$.

(2) $L^\omega(-\infty, \infty)$. Let $L^\omega(-\infty, \infty)$ be the set of all complex-valued measurable functions f on $(-\infty, \infty)$ such that $f \in L^p(-\infty, \infty)$ for every real number $p \geq 1$. Under the following operations

$$(fg)(x) = f(x)g(x), \quad (\lambda f)(x) = \lambda f(x),$$

$$f^*(x) = \overline{f(x)}$$

and inner product $(f | g) = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx$, we can show that $L^\omega(-\infty, \infty)$ is a pure unbounded Hilbert algebra.

(3) $L_1^\omega(G)$ and $L_2^\omega(G)$. Let G be a unimodular locally compact group and let dx be a Haar measure on G . Let $L^p(G)$ be the Banach space of measurable functions f on G for which the norm

$$\|f\|_p = \left[\int_G |f(x)|^p dx \right]^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty = \text{ess sup } |f(x)|$$

is finite. We note

$L(G)$; the space of complex-valued continuous functions with compact supports,

$$L^\omega(G) = \bigcap_{1 \leq p \leq \infty} L^p(G), \quad L_1^\omega(G) = \bigcap_{1 < p \leq \infty} L^p(G),$$

$$L_2^\omega(G) = \bigcap_{1 < p \leq 2} L^p(G).$$

Under the convolution $f * g$ as multiplication, involution f^* ($f^*(x) = \overline{f(x^{-1})}$) and inner product $(f | g) = \int_G f(x) \overline{g(x)} dx$ on $L^2(G)$, $L^\omega(G)$ is a Hilbert algebra and $L_1^\omega(G)$ and $L_2^\omega(G)$ are unbounded Hilbert algebras. In fact, suppose $f \in L^p(G)$ and $g \in L^q(G)$ ($1/p + 1/q \geq 1$). Then by Young's inequality $f * g$ exists and $\|f * g\|_r \leq \|f\|_p \|g\|_q$ where $1/r = 1/p + 1/q - 1$. Furthermore, for each $f \in L^p(G)$ ($1 \leq p < \infty$) we have $\|f^*\|_p = \|f\|_p$. Therefore we can easily show that $L^\omega(G)$, $L_1^\omega(G)$ and $L_2^\omega(G)$ are $*$ -algebras. Since $L(G) \subset L^\omega(G) \subset L^1(G) \cap L^2(G)$ and $L(G)$, $L^1(G) \cap L^2(G)$ are Hilbert algebras, $L^\omega(G)$ is clearly a Hilbert algebra. We can easily show that $(f | g) = (g^* | f^*)$ and $(f * g | h) = (g | f^* * h)$ for every $f, g, h \in L_1^\omega(G)$ (resp. $L_2^\omega(G)$). Furthermore we have

$$L^\omega(G) \subset (L_1^\omega(G))_0 \quad (\text{resp. } L_2^\omega(G)_0) \subset L^2(G),$$

and so $(L_1^\omega(G)_0)^2$ (resp. $(L_2^\omega(G)_0)^2$) is dense in $L^2(G)$. Therefore $L_1^\omega(G)$ and $L_2^\omega(G)$ are unbounded Hilbert algebras.

Problem. Is an unbounded Hilbert algebra $L_1^\omega(G)$ (or $L_2^\omega(G)$) pure?

If G is a compact group, then $L^2(G)$ is an H^* -algebra, and so $L_1^\omega(G)$ and $L_2^\omega(G)$ are Hilbert algebras.

If $G = (-\infty, \infty)$, then

$$L_1^\omega(-\infty, \infty) = \bigcap_{1 < p \leq \infty} L^p(-\infty, \infty)$$

and

$$L_2^\omega(-\infty, \infty) = \bigcap_{1 < p \leq 2} L^p(-\infty, \infty)$$

are pure unbounded Hilbert algebras under the following operations and inner product

$$\begin{aligned}
(f * g)(x) &= \int_{-\infty}^{\infty} f(y)g(x-y)dy, \\
(\lambda f)(x) &= \lambda f(x), \quad f^*(x) = \overline{f(-x)}, \\
(f \mid g) &= \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.
\end{aligned}$$

In fact, we note

$$\pi(f)g = f * g, \quad f, g \in L_{1^*}^{\omega}(-\infty, \infty)$$

and

$$(L_{1^*}^{\omega}(-\infty, \infty))_0 = \{f \in L_{1^*}^{\omega}(-\infty, \infty); \pi(f) \text{ is continuous}\}.$$

We have only to show $(L_{1^*}^{\omega}(-\infty, \infty))_0 \neq L_{1^*}^{\omega}(-\infty, \infty)$. By the theory of Hilbert algebras we have

$$\begin{aligned}
(L^1(-\infty, \infty) \cap L^2(-\infty, \infty))_b &= \{f \in L^2(-\infty, \infty); \overline{\pi(f)} \text{ is a bounded} \\
&\quad \text{linear operator on } L^2(-\infty, \infty)\} \\
&= \{f \in L^2(-\infty, \infty); \hat{f} \in L^{\infty}(-\infty, \infty)\},
\end{aligned}$$

where \hat{f} denotes the Fourier transform of f . Clearly we have

$$(L_{1^*}^{\omega}(-\infty, \infty))_0 \subset \{f \in L^2(-\infty, \infty); \hat{f} \in L^{\infty}(-\infty, \infty)\}.$$

Putting

$$f(x) = \begin{cases} 0, & x < 1 \\ 1/x, & x \geq 1 \end{cases}$$

we can show $f \in L_{1^*}^{\omega}(-\infty, \infty)$ and $\hat{f} \notin L^{\infty}(-\infty, \infty)$, and so $L_{1^*}^{\omega}(-\infty, \infty)_0 \neq L_{1^*}^{\omega}(-\infty, \infty)$. Consequently $L_{1^*}^{\omega}(-\infty, \infty)$ is pure.

3. L^{ω} -spaces with respect to noncommutative integration. Our starting point for the construction of L^{ω} -space will be the algebras of operators measurable with respect to a von Neumann algebra as constructed by Segal in [14]. Let \mathfrak{A} be a semifinite von Neumann algebra on a Hilbert space \mathfrak{H} and let φ be a faithful normal semifinite trace on \mathfrak{A}^+ . Let \mathfrak{A}_p and \mathfrak{A}_u , respectively, denote the set of all projections and that of unitary operators in \mathfrak{A} .

DEFINITION 3.1. A linear set \mathfrak{D} in \mathfrak{H} is said to be strongly dense (resp. φ -restrictedly strongly dense) provided

- (a) $U'\mathfrak{D} \subset \mathfrak{D}$ for every $U' \in \mathfrak{A}'_u$;
- (b) there exists a sequence of projections $P_n \in \mathfrak{A}$ such that $P_n\mathfrak{H} \subset \mathfrak{D}$, $P_n^\perp \downarrow 0$ and P_n^\perp is a finite projection (resp. $\varphi(P_n^\perp) < \infty$). An operator $T\eta\mathfrak{A}$ is called essentially measurable (resp. φ -restrictedly essentially measurable) if T has a strongly dense (resp. φ -restrictedly strongly dense) domain and a closed extension. Moreover if T is closed, T is called measurable (resp. φ -restrictedly measurable).

LEMMA 3.2. ([11] Lemma 1.1.) *Let T be a closed densely defined operator $\eta\mathfrak{A}$. Then;*

- (1) *T is measurable (resp. φ -restrictedly measurable) if and only if so is $|T|$.*

- (2) *Let $T \geq 0$ and let $T = \int_0^\infty \lambda dE(\lambda)$ be its spectral resolution. T is measurable (resp. φ -restrictedly measurable) if and only if $E(\lambda)^\perp (= I - E(\lambda))$ is a finite projection (resp. $\varphi(E(\lambda)^\perp) < \infty$) for a positive λ .*

We denote the set of all operators on \mathfrak{H} measurable (resp. φ -restrictedly measurable) with respect to \mathfrak{A} by $\mathfrak{M}(\mathfrak{A})$ (resp. $\mathfrak{M}(\varphi)$).

PROPOSITION 3.3. ([7] Prop. 4.3.) *The sets $\mathfrak{M}(\mathfrak{A})$ and $\mathfrak{M}(\varphi)$ form EW^* -algebras over \mathfrak{A} under the operations of strong sum, strong product, adjoint and strong scalar multiplication.*

Let \mathfrak{M}_φ be the maximal ideal associated with φ , that is, the set of $A \in \mathfrak{A}$ with $\varphi(|A|) < \infty$. For every $T \in \mathfrak{M}(\mathfrak{A})^+$ we put

$$\mu(T) = \sup_{A \in \mathfrak{M}_\varphi, A \leq T} \varphi(A).$$

DEFINITION 3.4. A measurable operator $T\eta\mathfrak{A}$ is said to be p th power integrable with respect to φ if $\mu(|T|^p) < \infty$. Let $L^p(\varphi)$ ($1 \leq p < \infty$) stand for the set of p th power integrable operators $\eta\mathfrak{A}$. The L^p -norm of $T \in L^p(\varphi)$ is defined as $\mu(|T|^p)^{1/p}$ and designated by $\|T\|_p$. If $p = \infty$, we shall identify \mathfrak{A} with $L^\infty(\varphi)$.

A measurable operator T belongs to $L^p(\varphi)$ ($1 \leq p < \infty$) if and only if T is φ -restrictedly measurable and $-\int_0^\infty \lambda^p d\varphi(E(\lambda)^\perp) < \infty$, where $\int_0^\infty \lambda dE(\lambda)$ is the spectral resolution of $|T|$.

THEOREM 3.5. [11] (1) *For $1 \leq p < \infty$ $L^p(\varphi)$ is a Banach space with norm $\|T\|_p$ and the following properties are satisfied.*

- (a) $\|T\|_p = \|T^*\|_p = \|U \cdot T \cdot U^*\|_p$ for $T \in L^p(\varphi)$ and $U \in \mathfrak{A}_u$.
 (b) For $S, T \in L^p(\varphi)$ such that $|T| \leq |S|$ we have $\|T\|_p \leq \|S\|_p$.
 (c) For $A \in \mathfrak{A}$ and $T \in L^p(\varphi)$ we have $\|A \cdot T\|_p \leq \|A\| \|T\|_p$.
 (d) If $0 \leq T_1 \leq T_2 \leq \dots$ is a sequence of elements of $L^p(\varphi)$ such that $\{\|T_n\|_p\}$ is bounded, then there exists $T := \sup T_n$ and $\lim_{n \rightarrow \infty} \|T - T_n\|_p = 0$.
 (2) Let $1/p + 1/q = 1$ where $1 \leq p, q \leq \infty$. Then
 (a) $\mu(S \cdot T) = \mu(T \cdot S)$ for $S \in L^p(\varphi)$ and $T \in L^q(\varphi)$. If furthermore, $S, T \geq 0$, then $\mu(S \cdot T) \geq 0$; and conversely, if $\mu(S \cdot T) \geq 0$ for every $T \geq 0$, then $S \geq 0$.
 (b) $|\mu(T_1 \cdot T_2 \cdot \dots \cdot T_n)| \leq \mu(|T_1 \cdot T_2 \cdot \dots \cdot T_n|) \leq \|T_1\|_{p_1} \|T_2\|_{p_2} \dots \|T_n\|_{p_n}$ for $T_i \in L^{p_i}(\varphi)$ with $\sum_{i=1}^n 1/p_i = 1$, $p_i \geq 1$ ($i = 1, 2, \dots, n$).

$$(c) \quad \|S\|_p = \sup_{T \in L^q(\varphi), \|T\|_q \leq 1} |\mu(S \cdot T)|$$

for $S \in L^p(\varphi)$ where the sup is attained by some T if $1 \leq p < \infty$.

$$(d) \quad |\mu(S \cdot T)|^2 \leq \mu(|S^*| \cdot |T|) \mu(|S| \cdot |T^*|) \leq \mu(|S \cdot T|) \mu(|T \cdot S|)$$

for $S \in L^p(\varphi)$ and $T \in L^q(\varphi)$.

- (3) Let $1/p + 1/q = 1/r$ where $1 \leq p, q, r \leq \infty$.
 (a) If $T \in L^p(\varphi)$ and $S \in L^q(\varphi)$, then $T \cdot S \in L^r(\varphi)$ and we have $\|T \cdot S\|_r \leq \|T\|_p \|S\|_q$.
 (b) Let T be a φ -restrictedly measurable operator $\eta \mathfrak{A}$. If $T \cdot S \in L^r(\varphi)$ for every $S \in L^q(\varphi)^+$, then $T \in L^p(\varphi)$.

Let \mathcal{D}_0 be a Hilbert algebra. Let $\mathcal{U}_0(\mathcal{D}_0)$ be the left von Neumann algebra of \mathcal{D}_0 and let φ_0 be the natural trace on $\mathcal{U}_0(\mathcal{D}_0)^+$. The completion \mathfrak{H} of \mathcal{D}_0 is equivalent to an H -system [3]. Putting

$$(\mathcal{D}_0)_b = \{x \in \mathfrak{H}; \overline{\pi_0(x)} \text{ is bounded}\},$$

$(\mathcal{D}_0)_b$ is a maximal Hilbert algebra containing \mathcal{D}_0 and $\mathcal{U}_0(\mathcal{D}_0)(\mathcal{D}_0)_b \subset (\mathcal{D}_0)_b$. For every $x \in \mathfrak{H}$ $\overline{\pi_0(x)}$ is φ_0 -restrictedly measurable ([11] Lemma 2.3.). We can easily show that $L^2(\varphi_0) = \{\pi_0(x); x \in \mathfrak{H}\}$ and $L^2(\varphi_0)$ is a Hilbert space isometric with \mathfrak{H} . Moreover we remark that $L^2(\varphi_0)$ is an H -system isomorphic with \mathfrak{H} by the map. $x \rightarrow \pi_0(x)$. This follows from the facts that (1) if xy is defined and equals z , then $\pi_0(x) \cdot \pi_0(y) = \pi_0(xy)$ and (2) if $\pi_0(x) \cdot \pi_0(y)$ equals $\pi_0(z)$, then xy is defined and equals z . We have

$$L^1(\varphi_0) = \left\{ \sum_{i=1}^m \overline{\pi_0(x_i)} \cdot \overline{\pi_0(y_i)}; x_i, y_i \in \mathfrak{H} \right\}$$

and the integral $\mu(T)$ of $T = \sum_{i=1}^m \overline{\pi_0(x_i)} \cdot \overline{\pi_0(y_i)}$ equals $\sum_{i=1}^m (y_i | x_i^*)$.

DEFINITION 3.5. We define the L^ω -spaces with respect to the natural trace φ_0 as follows;

$$L^\omega(\varphi_0) = \bigcap_{1 \leq p < \infty} L^p(\varphi_0),$$

$$L_2^\omega(\varphi_0) = \bigcap_{2 \leq p < \infty} L^p(\varphi_0).$$

Similarly we define the L^ω -spaces with respect to the Hilbert algebra \mathcal{D}_0 as follows;

$$L^\omega(\mathcal{D}_0) = \{x \in \mathfrak{H}; \overline{\pi_0(x)} \in L^\omega(\varphi_0)\},$$

$$L_2^\omega(\mathcal{D}_0) = \{x \in \mathfrak{H}; \overline{\pi_0(x)} \in L_2^\omega(\varphi_0)\}.$$

PROPOSITION 3.6. *The space $L^\omega(\mathcal{D}_0)$ (resp. $L_2^\omega(\mathcal{D}_0)$) is an unbounded Hilbert algebra containing $(\mathcal{D}_0)_b^2$ (resp. $(\mathcal{D}_0)_b$).*

Proof. For $1 \leq p < \infty$ and $S, T \in L^\omega(\varphi_0)$

$$\|S \cdot T\|_p \leq \|S\|_{2p} \|T\|_{2p}$$

and hence $S \cdot T \in L^\omega(\varphi_0)$. Therefore, for each x and y in $L^\omega(\mathcal{D}_0)$ xy is defined and equals $\pi_0(x)y$. Furthermore for each $T \in L^p(\varphi_0)$ ($1 \leq p < \infty$) $\|T\|_p = \|T^*\|_p$ and hence $x^* \in L^\omega(\mathcal{D}_0)$ for every $x \in L^\omega(\mathcal{D}_0)$. Consequently $L^\omega(\mathcal{D}_0)$ is a $*$ -algebra. We can easily show $L^\omega(\mathcal{D}_0) \supset (\mathcal{D}_0)_b^2$, and so $L^\omega(\mathcal{D}_0)$ is a pre-Hilbert space and its completion is $L^2(\mathcal{D}_0) = \mathfrak{H}$. For every x, y and z in $L^\omega(\mathcal{D}_0)$ we have

$$(x \mid y) = (y^* \mid x^*)$$

and

$$(xy \mid z) = (\overline{\pi_0(x)}y \mid z) = (y \mid \pi_0(x)^*z) = (y \mid \overline{\pi_0(x^*)}z) = (y \mid x^*z).$$

Consequently $L^\omega(\mathcal{D}_0)$ is an unbounded Hilbert algebra. Similarly we can show that $L_2^\omega(\mathcal{D}_0)$ is an unbounded Hilbert algebra containing $(\mathcal{D}_0)_b$.

PROPOSITION 3.7. *The space $L^\omega(\varphi_0)$ (resp. $L_2^\omega(\varphi_0)$) is an unbounded Hilbert algebra containing $\pi_0((\mathcal{D}_0)_b)^2$ (resp. $\pi_0((\mathcal{D}_0)_b)$) under the strong sum, strong product, adjoint, strong scalar multiplication and inner product on $L^2(\varphi_0)$.*

Proof. We can easily show that the map $x \in \mathfrak{H} \rightarrow \overline{\pi_0(x)} \in L^2(\varphi_0)$ is an isometric isomorphism of $L^\omega(\mathcal{D}_0)$ onto $L^\omega(\varphi_0)$. By Proposition 3.6. $L^\omega(\varphi_0)$ is an unbounded Hilbert algebra.

Problem. Is $L^\omega(\mathcal{D}_0)$ a pure unbounded Hilbert algebra? Does there exist a pure unbounded Hilbert algebra containing \mathcal{D}_0 ?

PROPOSITION 3.8. *The following conditions are equivalent.*

- (1) *There exists a pure unbounded Hilbert algebra \mathcal{D} containing \mathcal{D}_0 .*
- (2) *$L_2^\omega(\mathcal{D}_0)$ is a pure unbounded Hilbert algebra.*
- (3) *$L^\omega(\mathcal{D}_0)$ is a pure unbounded Hilbert algebra.*
- (4) *There exists a positive element x in \mathfrak{H} (i.e., $\overline{\pi_0(x)} \geq 0$) such that $x \notin (\mathcal{D}_0)_b$ and $x^n \in \mathfrak{H}$, $n = 1, 2, \dots$.*

Proof. (1) \Rightarrow (4); There exists an element $\xi \in \mathcal{D}$ such that $\overline{\pi(\xi)}$ is an unbounded operator on \mathfrak{H} . Clearly $\xi^* \xi \notin (\mathcal{D}_0)_b$ and $(\xi^* \xi)^n \in \mathcal{D} \subset \mathfrak{H}$, $n = 1, 2, \dots$.

(4) \Rightarrow (3); Let $y = x^2$. Then we can easily show that $y \notin (\mathcal{D}_0)_b$ and for each positive integer n $\overline{\pi_0(y)} \in L^n(\varphi_0)$. Let $\overline{\pi_0(y)} = \int_0^\infty \lambda dE(\lambda)$ be the spectral resolution. For each p with $1 \leq p < \infty$ there is a positive integer n such that $n \leq p < n+1$. Then we have

$$\begin{aligned} - \int_0^\infty \lambda^p d\varphi_0(E(\lambda)^+) &\leq - \int_0^1 \lambda^n d\varphi_0(E(\lambda)^+) - \int_1^\infty \lambda^{n+1} d\varphi_0(E(\lambda)^+) \\ &\leq - \int_0^\infty \lambda^n d\varphi_0(E(\lambda)^+) - \int_0^\infty \lambda^{n+1} d\varphi_0(E(\lambda)^+) \\ &< \infty. \end{aligned}$$

Therefore $\overline{\pi_0(y)} \in L^p(\varphi_0)$, i.e., $y \in L^p(\mathcal{D}_0)$ for every $1 \leq p < \infty$, and so $y \in L^\omega(\mathcal{D}_0)$ and $\overline{\pi_0(y)}$ is unbounded. Consequently $L^\omega(\mathcal{D}_0)$ is a pure unbounded Hilbert algebra.

(3) \Rightarrow (2); Since $L^\omega(\mathcal{D}_0) \subset L_2^\omega(\mathcal{D}_0)$, the assertion (3) \Rightarrow (2) is obvious.

(2) \Rightarrow (1); $L_2^\omega(\mathcal{D}_0)$ is a pure unbounded Hilbert algebra containing \mathcal{D}_0 .

THEOREM 3.9. *Let \mathcal{D} be a pure unbounded Hilbert algebra over \mathcal{D}_0 . Then \mathcal{D}^2 (resp. \mathcal{D}) is a $*$ -subalgebra of the pure unbounded Hilbert algebra $L^\omega(\mathcal{D}_0)$ (resp. $L_2^\omega(\mathcal{D}_0)$). In particular, $L_2^\omega(\mathcal{D}_0)$ is maximal in pure unbounded Hilbert algebras containing \mathcal{D}_0 .*

Proof. By Proposition 3.8 $L^\omega(\mathcal{D}_0)$ and $L_2^\omega(\mathcal{D}_0)$ are pure unbounded Hilbert algebras. In the same way as the proof (4) \Rightarrow (3) of Proposition 3.8 we can easily show $L^\omega(\mathcal{D}_0) \supset \mathcal{D}^2$ and $L_2^\omega(\mathcal{D}_0) \supset \mathcal{D}$.

Problem. Let \mathcal{D} be a pure unbounded Hilbert algebra over \mathcal{D}_0 . Does there exist an EW^* -algebra \mathfrak{A} such that $\overline{\mathfrak{A}}_b = \mathcal{U}_0(\mathcal{D}_0)$ and $\overline{\mathfrak{A}} \supset \pi(\mathcal{D})$?

Let \mathcal{D} be a pure unbounded Hilbert algebra over \mathcal{D}_0 . By Proposition 3.8 $L_2^{\omega}(\mathcal{D}_0)$ is a pure unbounded Hilbert algebra such that

$$\mathcal{D}_0 \subset \mathcal{D} \subset L_2^{\omega}(\mathcal{D}_0) \subset \mathfrak{H}, \quad \text{and} \quad L^{\infty}(\varphi_0)L_2^{\omega}(\mathcal{D}_0) \subset L_2^{\omega}(\mathcal{D}_0).$$

Let π (resp. π_2^{ω}) be the left regular representation of \mathcal{D} (resp. $L_2^{\omega}(\mathcal{D}_0)$). By Lemma 2.1 we have $\pi_2^{\omega}(\mathcal{D}) = \pi(\mathcal{D}) = \pi_0(\mathcal{D})$.

Then $\pi_2^{\omega}(\mathcal{D})$ is a $\#$ -algebra on $L_2^{\omega}(\mathcal{D}_0)$ under $\pi_2^{\omega}(\xi)^{\#} = \pi_2^{\omega}(\xi^*)$ and $L^{\infty}(\varphi_0)/L_2^{\omega}(\mathcal{D}_0) := \{T/L_2^{\omega}(\mathcal{D}_0); T \in L^{\infty}(\varphi_0)\}$ is a $\#$ -algebra on $L_2^{\omega}(\mathcal{D}_0)$ under $(T/L_2^{\omega}(\mathcal{D}_0))^{\#} = T^*/L_2^{\omega}(\mathcal{D}_0)$, where $T/L_2^{\omega}(\mathcal{D}_0)$ is the restriction of T onto $L_2^{\omega}(\mathcal{D}_0)$.

NOTATION. We denote by $\mathcal{U}(\mathcal{D})$ a $\#$ -algebra on $L_2^{\omega}(\mathcal{D}_0)$ generated by $\pi_2^{\omega}(\mathcal{D})$ and $L^{\infty}(\varphi_0)/L_2^{\omega}(\mathcal{D}_0)$.

THEOREM 3.10. *Let \mathcal{D} be a pure unbounded Hilbert algebra over \mathcal{D}_0 . Then $\mathcal{U}(\mathcal{D})$ and $\mathcal{U}(L_2^{\omega}(\mathcal{D}_0))$ are EW^* -algebras on $L_2^{\omega}(\mathcal{D}_0)$ such that $\mathcal{U}(\mathcal{D})_b = \mathcal{U}(L_2^{\omega}(\mathcal{D}_0))_b = \mathcal{U}_0(\mathcal{D}_0)$ and $\mathcal{U}(L_2^{\omega}(\mathcal{D}_0)) \supset \mathcal{U}(\mathcal{D}) \supset \pi(\mathcal{D})$.*

DEFINITION 3.11. Let \mathcal{D} be a pure unbounded Hilbert algebra over \mathcal{D}_0 . $\mathcal{U}(\mathcal{D})$ is called the left EW^* -algebra of \mathcal{D} .

4. Traces on EW^* -algebras. Let \mathfrak{A} be an EW^* -algebra and let φ be a trace on \mathfrak{A}^+ . We note

$$\mathfrak{N}_{\varphi} = \{T \in \mathfrak{A}; \varphi(T^*T) < \infty\}$$

and let \mathfrak{M}_{φ} be a linear combination of $\{ST^{\#}; S, T \in \mathfrak{N}_{\varphi}\}$. Then, clearly, \mathfrak{N}_{φ} (resp. \mathfrak{M}_{φ}) is a $\#$ -subspace of \mathfrak{A} satisfying $\mathfrak{A}_b\mathfrak{N}_{\varphi} \subset \mathfrak{N}_{\varphi}$ and $\mathfrak{N}_{\varphi}\mathfrak{A}_b \subset \mathfrak{N}_{\varphi}$ (resp. $\mathfrak{A}_b\mathfrak{M}_{\varphi} \subset \mathfrak{M}_{\varphi}$ and $\mathfrak{M}_{\varphi}\mathfrak{A}_b \subset \mathfrak{M}_{\varphi}$). We can easily show that the positive part \mathfrak{M}_{φ}^+ of \mathfrak{M}_{φ} equals $\{T \in \mathfrak{A}^+; \varphi(T) < \infty\}$ and \mathfrak{M}_{φ} is a linear combination of \mathfrak{M}_{φ}^+ . We define $\dot{\varphi}$ by;

$$\begin{aligned} \dot{\varphi}(S) &= \lambda_1\varphi(S_1) + \cdots + \lambda_n\varphi(S_n), \quad S = \lambda_1S_1 + \cdots + \lambda_nS_n, \\ \lambda_i &\in \mathbb{C}, \quad S_i \in \mathfrak{M}_{\varphi}^+. \end{aligned}$$

Then it is not difficult to show that $\dot{\varphi}$ is a well-defined linear form on \mathfrak{M}_{φ} and it satisfies

- (1) $\dot{\varphi}(S) = \varphi(S), \quad S \in \mathfrak{M}_\varphi^+;$
- (2) $\dot{\varphi}(S^*T) = \dot{\varphi}(TS^*), \quad S, T \in \mathfrak{N}_\varphi;$
- (3) $\dot{\varphi}(ST) = \dot{\varphi}(TS), \quad S \in \mathfrak{M}_\varphi, \quad T \in \mathfrak{A}_b.$

We note

$$\bar{\varphi}(\bar{T}) = \varphi(T), \quad T \in \mathfrak{A}_b^+.$$

Then $\bar{\varphi}$ is a trace on $\bar{\mathfrak{A}}_b^+$ and we have

$$\overline{(\mathfrak{N}_\varphi)_b} = \mathfrak{N}_{\bar{\varphi}} \quad \text{and} \quad \overline{(\mathfrak{M}_\varphi)_b} = \mathfrak{M}_{\bar{\varphi}}.$$

DEFINITION 4.1. Let \mathfrak{A} be an EW^* -algebra and let φ be a trace on \mathfrak{A}^+ . If every $\bar{A} \in \bar{\mathfrak{A}}$ is $\bar{\varphi}$ -restrictedly measurable, then \mathfrak{A} is called φ -measurable.

Let \mathcal{D} be a pure unbounded Hilbert algebra over \mathcal{D}_0 and let \mathfrak{H} be the completion of \mathcal{D} . Let \mathcal{E} be a pure unbounded Hilbert algebra over $(\mathcal{D}_0)_b$ containing \mathcal{D} . Let \mathfrak{A} be a φ_0 -measurable (merely measurable) EW^* -algebra on \mathcal{E} such that $\bar{\mathfrak{A}}_b = \mathcal{U}_0(\mathcal{D}_0)$ and $\bar{\mathfrak{A}} \supset \overline{\pi(\mathcal{D})}$ ($\mathcal{U}(\mathcal{D})$ and $\mathcal{U}(L_2^\omega(\mathcal{D}_0))$ are examples of such EW^* -algebras), where φ_0 is the natural trace on $\mathcal{U}_0(\mathcal{D}_0)^+$.

NOTATION. For each $S \in \mathfrak{A}^+$ we define φ as follows;

$$\varphi(S) = \begin{cases} (x | x), & \text{if } \overline{S^{1/2}} = \overline{\pi_0(x)}, \quad x \in L_2^\omega(\mathcal{D}_0); \\ \infty, & \text{if otherwise.} \end{cases}$$

THEOREM 4.2. (1) φ is a faithful normal semifinite trace on \mathfrak{A}^+ .
 (2) We have

$$\bar{\mathfrak{N}}_\varphi = \bar{\mathfrak{A}} \cap L_2^\omega(\varphi_0) \quad \text{and} \quad \bar{\mathfrak{M}}_\varphi = \bar{\mathfrak{A}} \cap L^\omega(\varphi_0).$$

(3) Putting

$$\mathfrak{N}(\mathcal{D}_0) = \{x \in \mathfrak{H}; \overline{\pi_0(x)} \in \bar{\mathfrak{N}}_\varphi\} \quad \text{and} \quad \mathfrak{M}(\mathcal{D}_0) = \{x \in \mathfrak{H}; \overline{\pi_0(x)} \in \bar{\mathfrak{M}}_\varphi\},$$

$\mathfrak{N}(\mathcal{D}_0)$ (resp. $\mathfrak{M}(\mathcal{D}_0)$) is a pure unbounded Hilbert algebra over $(\mathcal{D}_0)_b$ (resp. $(\mathcal{D}_0)_b^2$) containing \mathcal{D} (resp. \mathcal{D}^2).

- (4) $\bar{\varphi}$ equals the natural trace φ_0 on $\mathcal{U}_0(\mathcal{D}_0)^+$.
- (5) Let μ be the integral on $L^1(\varphi_0)$. Then

$$\dot{\varphi}(T) = \mu(\bar{T}), \quad T \in \mathfrak{M}_\varphi.$$

In particular, for every $x, y \in \mathfrak{N}(\mathcal{D}_0)$

$$\dot{\varphi}(\overline{\pi_0(y)^* \cdot \pi_0(x)}) = (x \mid y).$$

(6) $\mathfrak{A}(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi$ and $\mathfrak{A}(\mathfrak{M}_\varphi)_b \subset \mathfrak{M}_\varphi$.

(7) Every element T of \mathfrak{A} is represented by

$$T = T_0 + T_1, \quad T_0 \in \mathfrak{A}_b, \quad T_1 \in \mathfrak{M}_\varphi.$$

(8) If $T \in \mathfrak{A}$, then we have $\bar{T} = \overline{(T/\mathcal{D}_0)}$.

Proof. (2); Let $T \in \mathfrak{N}_\varphi$ and let $T = U|T|$ be the polar decomposition of T . Since $\varphi(T^*T) = \varphi(|T|^2) < \infty$, $|T| = \pi_0(x)$, $x \in L_2^\omega(\mathcal{D}_0)$, and so $|T| \in L_2^\omega(\varphi_0)$ and hence $\bar{T} = \bar{U} \cdot |T| \in L_2^\omega(\varphi_0) \cap \bar{\mathfrak{A}}$. The converse is obvious. Moreover we get

$$\overline{\mathfrak{M}_\varphi} = \overline{\mathfrak{N}_\varphi}^2 = (\bar{\mathfrak{A}} \cap L_2^\omega(\varphi_0))^2 = \bar{\mathfrak{A}} \cap L^\omega(\varphi_0).$$

(3); By (2) we can easily show (3).

(4); Let $T \in \mathfrak{A}_b^+$. Since $\mathfrak{A}_b \cap L_2^\omega(\varphi_0) = \overline{\pi_0((\mathcal{D}_0)_b)}$,

$$\begin{aligned} \bar{\varphi}(\bar{T}) &= \varphi(T) = \begin{cases} (x \mid x), & \text{if } \overline{T^{1/2}} = \overline{\pi_0(x)}, \ x \in L_2^\omega(\mathcal{D}_0); \\ \infty, & \text{if otherwise} \end{cases} \\ &= \begin{cases} (x \mid x), & \text{if } \overline{T^{1/2}} = \overline{\pi_0(x)}, \ x \in (\mathcal{D}_0)_b; \\ \infty, & \text{if otherwise} \end{cases} \\ &= \varphi_0(\bar{T}). \end{aligned}$$

(5); Let $T \in \mathfrak{M}_\varphi^+$. By (2) there exists an element x of $L_2^\omega(\mathcal{D}_0)$ such that $\overline{T^{1/2}} = \overline{\pi_0(x)}$. Then we have $\varphi(T) = (x \mid x) = \mu(\bar{T})$, and so $\bar{\varphi}(T) = \mu(\bar{T})$, $T \in \mathfrak{M}_\varphi$.

(6); Let π be the left regular representation of \mathcal{E} . We can easily show that

$$T\pi(\xi) = \pi(T\xi), \quad T \in \mathfrak{A}, \quad \xi \in (\mathcal{D}_0)_b \subset \mathcal{E}.$$

Therefore $\pi(T\xi) = T\pi(\xi) \in \mathfrak{A}$ and $\overline{\pi(T\xi)} = \overline{\pi_0(T\xi)}$, $T\xi \in \mathcal{E} \subset L_2^\omega(\mathcal{D}_0)$, and so $T\pi(\xi) \in \mathfrak{N}_\varphi$.

(7); Let $T \in \mathfrak{A}$ and let $T = U|T|$ be the polar decomposition of T . Let $|T| = \int_0^\infty \lambda dE_T(\lambda)$ be the spectral resolution of $|T|$. Since $|T|$ is

a φ_0 -restrictedly measurable operator, $\overline{E_T(\lambda_0)^+} \in \overline{(\mathfrak{M}_\varphi)_b^+}$ for some $\lambda_0 > 0$. By (6) $\mathfrak{A}(\mathfrak{M}_\varphi)_b \subset \mathfrak{M}_\varphi$, and so putting

$$T_1 = TE_T(\lambda_0)^+ = U|T|E_T(\lambda_0)^+ \text{ and } T_0 = TE_T(\lambda_0),$$

$T_0 \in \mathfrak{A}_b$, $T_1 \in \mathfrak{M}_\varphi$ and $T = T_0 + T_1$.

(8); Let $T \in \mathfrak{A}$. By (7) we have

$$\begin{aligned} \bar{T} &= \bar{T}_0 + \bar{T}_1, \quad T_0 \in \mathfrak{A}_b, \quad T_1 \in \mathfrak{M}_\varphi \\ &= \bar{T}_0 + \overline{\pi_0(x)}, \quad x \in L^\infty(\mathcal{D}_0) \\ &= \overline{(T_0/\mathcal{D}_0)} + \overline{\pi_0(x)} = \overline{T_0/\mathcal{D}_0 + \pi_0(x)} = \overline{T/\mathcal{D}_0}. \end{aligned}$$

(1); We shall show that φ is a trace on \mathfrak{A}^+ , i.e.,

(a) $\varphi(S + T) = \varphi(S) + \varphi(T)$, $S, T \in \mathfrak{A}^+$;

(b) $\varphi(\lambda S) = \lambda\varphi(S)$, $\lambda \geq 0$, $S \in \mathfrak{A}^+$;

(c) $\varphi(S^*S) = \varphi(SS^*)$, $S \in \mathfrak{A}$.

(a); Let $S, T \in \mathfrak{A}^+$. Suppose $\varphi(S + T) < \infty$. Since \bar{S} (or \bar{T}) $\leq \bar{S} + \bar{T}$ and $\bar{S} + \bar{T} \in \overline{\mathfrak{M}_\varphi^+}$, \bar{S} and \bar{T} in $\overline{\mathfrak{M}_\varphi^+}$, and so $\varphi(S) = \mu(\bar{S}) < \infty$ and $\varphi(T) = \mu(\bar{T}) < \infty$ by (5). Suppose $\varphi(S) < \infty$ and $\varphi(T) < \infty$. Since \bar{S} and \bar{T} in $L^1(\varphi_0)^+$, by Theorem 3.5. we have $\bar{S} + \bar{T} \in L^1(\varphi_0)^+$ and

$$\varphi(S) + \varphi(T) = \mu(\bar{S}) + \mu(\bar{T}) = \mu(\bar{S} + \bar{T}) = \mu(\overline{S + T}) = \varphi(S + T).$$

(b); clear.

(c); Let $S \in \mathfrak{A}$. Suppose $\varphi(S^*S) < \infty$. Let $S = U|S|$ be the polar decomposition of S . Then $|S| = \pi_0(x)$, $x \in L_2^\infty(\mathcal{D}_0)$ and $|S^*| = |S^*| = \pi_0(x^*)$, and so we get

$$\varphi(S^*S) = (x | x) = (x^* | x^*) = \varphi(SS^*).$$

Consequently φ is a trace on \mathfrak{A}^+ . Since $\bar{\varphi} = \varphi_0$ by (4), $\bar{\varphi}$ is a faithful normal semifinite trace on $\overline{\mathfrak{A}_b^+}$. We can easily show that φ is faithful. We shall show that φ is normal. Let $T_\alpha \uparrow T$, $T_\alpha, T \in \mathfrak{A}^+$. Suppose $\varphi(T) < \infty$. Then there exist $\{x_\alpha\} \subset L_2^\infty(\mathcal{D}_0)$ and $x \in L_2^\infty(\mathcal{D}_0)$ such that $\overline{T_\alpha^{1/2}} = \overline{\pi_0(x_\alpha)}$ and $\overline{T^{1/2}} = \overline{\pi_0(x)}$. We can easily show that $\varphi(T_\alpha) = \|x_\alpha\|^2 \uparrow \varphi(T) = \|x\|^2$. Suppose $\varphi(T) = \infty$ and $\sup_\alpha \varphi(T_\alpha) < \infty$. There exists a net $\{x_\alpha\}$ of $L_2^\infty(\mathcal{D}_0)$ such that $\overline{T_\alpha^{1/2}} = \overline{\pi_0(x_\alpha)}$. Let $\bar{T} = \int_0^\infty \lambda dE_T(\lambda)$ be the spectral resolution of \bar{T} . Since \bar{T} is φ_0 -restrictedly measurable, $\overline{E_T(\lambda_0)^+} \in \overline{(\mathfrak{M}_\varphi)_b^+}$ for some $\lambda_0 > 0$, and so by (5) we get

$$TE_T(\lambda_0)^\perp \in \mathfrak{M}_\varphi^+ \quad \text{and} \quad \bar{T} = \int_0^{\lambda_0} \lambda d\overline{E_T(\lambda)} + \overline{\bar{T}E_T(\lambda_0)}^\perp.$$

From $\varphi(T) = \infty$, we have $\bar{\varphi}\left(\int_0^{\lambda_0} \lambda d\overline{E_T(\lambda)}\right) = \infty$. Since $T_\alpha \uparrow T$, we get $E_T(\lambda_0)T_\alpha E_T(\lambda_0) \in \mathfrak{A}_b$ and

$$E_T(\lambda_0)T_\alpha E_T(\lambda_0) \uparrow E_T(\lambda_0)TE_T(\lambda_0) = \int_0^{\lambda_0} \lambda dE_T(\lambda).$$

Then we can show that

$$\overline{E_T(\lambda_0)T_\alpha E_T(\lambda_0)} \uparrow \int_0^{\lambda_0} \lambda d\overline{E_T(\lambda)},$$

and so by the normality of $\bar{\varphi}$

$$\bar{\varphi}(\overline{E_T(\lambda_0)T_\alpha E_T(\lambda_0)}) \uparrow \bar{\varphi}\left(\int_0^{\lambda_0} \lambda d\overline{E_T(\lambda)}\right) = \infty.$$

On the other hand we have

$$\begin{aligned} \bar{\varphi}\left(\int_0^{\lambda_0} \lambda d\overline{E_T(\lambda)}\right) &= \sup_\alpha \bar{\varphi}(\overline{E_T(\lambda_0)T_\alpha E_T(\lambda_0)}) \\ &= \sup_\alpha \bar{\varphi}(\overline{E_T(\lambda_0)} \cdot \overline{\pi_0(x_\alpha)^2} \cdot \overline{E_T(\lambda_0)}) \\ &= \sup_\alpha \bar{\varphi}(\overline{\pi_0(\overline{E_T(\lambda_0)}x_\alpha) \cdot \pi_0(\overline{E_T(\lambda_0)}x_\alpha^*)^*}) \\ &= \sup_\alpha \overline{(E_T(\lambda_0)x_\alpha \mid E_T(\lambda_0)x_\alpha^*)} \\ &\leq \sup_\alpha \|x_\alpha\|^2 = \sup_\alpha \varphi(T_\alpha) < \infty. \end{aligned}$$

This contradicts $\bar{\varphi}\left(\int_0^{\lambda_0} \lambda d\overline{E_T(\lambda)}\right) = \infty$. Consequently φ is normal. Finally we shall show that φ is semifinite. Since $\bar{\varphi}$ is semifinite, there exists a net $\{T_\alpha\}$ of $(\mathfrak{M}_\varphi)_b^+$ such that $\bar{T}_\alpha \uparrow \bar{I}$. Let $T \in \mathfrak{A}^+$. By (6) we have

$$T^{\frac{1}{2}}T_\alpha T^{\frac{1}{2}} \in \mathfrak{M}_\varphi^+ \quad \text{and} \quad T^{\frac{1}{2}}T_\alpha T^{\frac{1}{2}} \uparrow T,$$

and so φ is semifinite.

DEFINITION 4.3. The trace φ of Theorem 4.2. is called the natural trace on \mathfrak{A}^+ .

COROLLARY 4.4. *For every $A \in \mathfrak{A}$ and $x \in L_2^{\omega}(\mathcal{D}_0)$ we have*

$$\mathfrak{A}L_2^{\omega}(\mathcal{D}_0) \subset L_2^{\omega}(\mathcal{D}_0) \quad \text{and} \quad \bar{A} \cdot \overline{\pi_0(x)} = \overline{\pi_0(\bar{A}x)}.$$

In particular, we have

$$\mathfrak{A}\mathfrak{N}_{\varphi} \subset \mathfrak{N}_{\varphi} \quad \text{and} \quad \mathfrak{A}\mathfrak{M}_{\varphi} \subset \mathfrak{M}_{\varphi}.$$

Proof. By Theorem 4.2.(7) we get $A = A_0 + A_1$, $A_0 \in \mathfrak{A}_b$, $A_1 \in \mathfrak{M}_{\varphi}$, and so $\bar{A} = \bar{A}_0 + \overline{\pi_0(y)}$, $y \in L^{\omega}(\mathcal{D}_0)$. Hence $\mathfrak{D}(\bar{A}) = \mathfrak{D}(\pi_0(y)) \supset L_2^{\omega}(\mathcal{D}_0)$ and we have

$$\begin{aligned} \bar{A}L_2^{\omega}(\mathcal{D}_0) &= \overline{A_0}L_2^{\omega}(\mathcal{D}_0) + \overline{A_1}L_2^{\omega}(\mathcal{D}_0) \\ &\subset L_2^{\omega}(\mathcal{D}_0), \end{aligned}$$

and

$$\begin{aligned} \bar{A} \cdot \overline{\pi_0(x)} &= (\overline{A_0} + \overline{\pi_0(y)}) \cdot \overline{\pi_0(x)} \\ &= \overline{A_0 \pi_0(x)} + \overline{\pi_0(y) \cdot \pi_0(x)} \\ &= \overline{\pi_0(\bar{A}_0 x)} + \overline{\pi_0(\pi_0(y)x)} \\ &= \overline{\pi_0(\bar{A}_0 x + \bar{A}_1 x)} \\ &= \overline{\pi_0(\bar{A}x)}. \end{aligned}$$

Moreover, since $\overline{\mathfrak{N}_{\varphi}} = \mathfrak{A} \cap L_2^{\omega}(\varphi_0)$ and $\overline{\mathfrak{M}_{\varphi}} = \mathfrak{A} \cap L^{\omega}(\varphi_0)$, we have $\mathfrak{A}\mathfrak{N}_{\varphi} \subset \mathfrak{N}_{\varphi}$ and $\mathfrak{A}\mathfrak{M}_{\varphi} \subset \mathfrak{M}_{\varphi}$.

For every $A \in \mathfrak{A}$ putting

$$\tilde{A}x = \bar{A}x, \quad x \in L_2^{\omega}(\mathcal{D}_0),$$

\tilde{A} is a linear operator on $L_2^{\omega}(\mathcal{D}_0)$ by Corollary 4.4.. Let $\tilde{\mathfrak{A}} = \{\tilde{A}; A \in \mathfrak{A}\}$. Then we have

$$\tilde{A}\tilde{B} = \widetilde{AB}, \quad \lambda\tilde{A} = \widetilde{\lambda A} \quad \text{and} \quad \tilde{A}^{\#} = A^*/L_2^{\omega}(\mathcal{D}_0) = \widetilde{A^{\#}}$$

for every $A, B \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$. We can easily show that $\tilde{\mathfrak{A}}$ equals the left $EW^{\#}$ -algebra $\mathcal{U}(\mathfrak{N}(\mathcal{D}_0))$ of a pure unbounded Hilbert algebra $\mathfrak{N}(\mathcal{D}_0)$. So, we obtain the following theorem.

THEOREM 4.5. *Let \mathcal{D} be a pure unbounded Hilbert algebra over \mathcal{D}_0*

and let \mathcal{E} be a pure unbounded Hilbert algebra over $(\mathcal{D}_0)_b$ containing \mathcal{D} . Let \mathfrak{A} be a measurable EW^* -algebra on \mathcal{E} such that $\mathfrak{A}_b = \mathcal{U}_0(\mathcal{D}_0)$ and $\mathfrak{A} \supset \pi(\mathcal{D})$. Then \mathfrak{A} is regarded as the left EW^* -algebra $\mathfrak{A}(\mathfrak{N}(\mathcal{D}_0))$ of a pure unbounded Hilbert algebra $\mathfrak{N}(\mathcal{D}_0)$ over $(\mathcal{D}_0)_b$ containing \mathcal{D} .

Finally we shall show that an EW^* -algebra with a faithful normal semifinite trace is isomorphic to a left EW^* -algebra of a pure unbounded Hilbert algebra (Theorem 4.11). Let \mathfrak{A} be an EW^* -algebra on \mathcal{D} and let φ be a faithful trace on \mathfrak{A}^+ . For each $S, T \in \mathfrak{N}_\varphi$ putting

$$(\lambda(S) \mid \lambda(T)) = \dot{\varphi}(T^*S),$$

(\mid) is an inner product on $\lambda(\mathfrak{N}_\varphi)$ and by, for each $S, T \in \mathfrak{N}_\varphi$ and $\alpha \in \mathfrak{E}$,

$$\lambda(S) + \lambda(T) = \lambda(S + T), \quad \alpha\lambda(S) = \lambda(\alpha S),$$

$\lambda(\mathfrak{N}_\varphi)$ is a pre-Hilbert space. Let \mathfrak{H}_φ be the completion of $\lambda(\mathfrak{N}_\varphi)$. Let \mathfrak{A} be a φ -measurable EW^* -algebra on \mathcal{D} and let φ be a faithful normal semifinite trace on \mathfrak{A}^+ satisfying $\mathfrak{A}(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi$.

LEMMA 4.6. *The property “ $\mathfrak{A}(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi$ ” leads the property “ $\mathfrak{A}\mathfrak{N}_\varphi \subset \mathfrak{N}_\varphi$ ”.*

Proof. Let $A \in \mathfrak{A}$ and $S \in \mathfrak{N}_\varphi$. Let $S = U|S|$ be the polar decomposition of S and let $\overline{|S|} = \int_0^\infty \lambda d\overline{E_S(\lambda)}$ be the spectral resolution of $\overline{|S|}$. Since $\overline{|S|}$ is a $\bar{\varphi}$ -restrictedly measurable operator, $\overline{E_S(\lambda_0)^+} \in \overline{(\mathfrak{M}_\varphi)_b^+}$ for some $\lambda_0 > 0$, and so we have

$$\begin{aligned} AS &= AU|S| = AU \left(\int_0^{\lambda_0} \lambda dE_S(\lambda) + |S|E_S(\lambda_0)^+ \right) \\ &= AU \int_0^{\lambda_0} \lambda dE_S(\lambda) + ASE_S(\lambda_0)^+ \\ &\in \mathfrak{A}(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi. \end{aligned}$$

LEMMA 4.7. *Let $A \in \mathfrak{A}$. Then there exist $A_0 \in \mathfrak{A}_b$ and $A_1 \in \mathfrak{M}_\varphi$ such that*

$$A = A_0 + A_1.$$

Proof. Let $A = U|A|$ be the polar decomposition of A and let $\overline{|A|} = \int_0^\infty \lambda d\overline{E_A(\lambda)}$ be the spectral resolution. Since $\overline{|A|}$ is $\bar{\varphi}$ -restrictedly measurable, $\overline{E_A(\lambda_0)^+} \in \overline{(\mathfrak{M}_\varphi)_b^+}$ for some $\lambda_0 > 0$. Putting

$$A_0 = U \left(\int_0^{\lambda_0} \lambda dE_A(\lambda) \right) \quad \text{and} \quad A_1 = AE_A(\lambda_0)^\perp,$$

$$A_0 \in \mathfrak{A}_b, \quad A_1 \in \mathfrak{A}(\mathfrak{M}_\varphi)_b \subset \mathfrak{M}_\varphi \quad \text{and} \quad A = A_0 + A_1.$$

LEMMA 4.8. *The pre-Hilbert space $\lambda(\mathfrak{N}_\varphi)$ is a pure unbounded Hilbert algebra over $\lambda((\mathfrak{N}_\varphi)_b)$.*

Proof. We shall show that $\lambda((\mathfrak{N}_\varphi)_b)$ is dense in $\lambda(\mathfrak{N}_\varphi)$. For each $T \in \mathfrak{N}_\varphi$ let $T = U|T|$ be the polar decomposition of T . Then $|T| = U^*T \in \mathfrak{N}_\varphi^+$. Let $\overline{|T|} = \int_0^\infty \lambda d\overline{E_T}(\lambda)$ be the spectral resolution of $\overline{|T|}$. Putting

$$\overline{S_n} = \int_0^n \lambda d\overline{E_T}(\lambda),$$

$S_n \in (\mathfrak{N}_\varphi)_b^+$ and $\{S_n\}$ converges σ -strongly to $|T|$, and so $S_n^2 \uparrow |T|^2$ and since φ is normal, we get

$$\|\lambda(S_n)\|^2 = \varphi(S_n^2) \uparrow \varphi(|T|^2) = \|\lambda(|T|)\|^2$$

and

$$\begin{aligned} (\lambda(|T|) \mid \lambda(S_n)) &= \dot{\varphi}(|T|S_n) \\ &= \varphi(|T|^{\frac{1}{2}}S_n|T|^{\frac{1}{2}}) \uparrow \varphi(|T|^2) = \|\lambda(|T|)\|^2, \end{aligned}$$

and hence

$$\lim_{n \rightarrow \infty} \|\lambda(US_n) - \lambda(T)\| \leq \lim_{n \rightarrow \infty} \|\lambda(S_n) - \lambda(|T|)\| = 0.$$

Therefore $\lambda((\mathfrak{N}_\varphi)_b)$ is dense in $\lambda(\mathfrak{N}_\varphi)$. Since $\bar{\varphi}$ is a faithful normal semifinite trace on $\overline{\mathfrak{A}_b^+}$, $\lambda(\overline{(\mathfrak{N}_\varphi)_b}) = \lambda(\mathfrak{N}_\varphi)$ is a maximal Hilbert algebra, and so we can easily show that $\lambda((\mathfrak{N}_\varphi)_b)$ is a maximal Hilbert algebra. For every $S, T \in \mathfrak{N}_\varphi$ we define the operations on $\lambda(\mathfrak{N}_\varphi)$ as follows;

$$\begin{aligned} \lambda(S)\lambda(T) &= \lambda(ST), & \alpha\lambda(S) &= \lambda(\alpha S), \\ \lambda(S)^* &= \lambda(S^*), & (\lambda(S) \mid \lambda(T)) &= \dot{\varphi}(T^*S). \end{aligned}$$

Then it is not difficult to show that $\lambda(\mathfrak{N}_\varphi)$ is an unbounded Hilbert algebra over $\lambda((\mathfrak{N}_\varphi)_b)$. Finally we shall show that $\lambda(\mathfrak{N}_\varphi)$ is pure. By

Lemma 4.7. every element A of \mathfrak{A} is represented by $A = A_0 + A_1$, $A_0 \in \mathfrak{A}_b$, $A_1 \in \mathfrak{M}_\varphi$. If $A \in \mathfrak{A} - \mathfrak{A}_b$, then $A_1 \in \mathfrak{M}_\varphi - (\mathfrak{M}_\varphi)_b$, and so $\lambda((\mathfrak{M}_\varphi)_b) \neq \lambda(\mathfrak{M}_\varphi)$ and $\lambda((\mathfrak{M}_\varphi)_b)$ is a maximal Hilbert algebra. Therefore $\lambda(\mathfrak{M}_\varphi)$ is pure.

LEMMA 4.9. *For every $A \in \mathfrak{A}$ putting*

$$\Psi(A)\lambda(T) = \lambda(AT), \quad T \in \mathfrak{M}_\varphi,$$

$\Psi(A)$ is a linear operator on $\lambda(\mathfrak{M}_\varphi)$. $\Psi(\mathfrak{A})$ is a measurable EW^ -algebra on $\lambda(\mathfrak{M}_\varphi)$ such that $\overline{\Psi(\mathfrak{A})_b} = \overline{\Psi(\mathfrak{A}_b)} = \mathcal{U}_0(\lambda((\mathfrak{M}_\varphi)_b))$ and $\overline{\Psi(\mathfrak{A})} \supset \pi(\lambda(\mathfrak{M}_\varphi))$ and Ψ is an isomorphism of \mathfrak{A} onto $\Psi(\mathfrak{A})$.*

Proof. By Lemma 4.6. $\mathfrak{A}\mathfrak{M}_\varphi \subset \mathfrak{M}_\varphi$, and so $\Psi(A)$ is a linear operator on $\lambda(\mathfrak{M}_\varphi)$. For every $S \in \mathfrak{M}_\varphi$ we have $\Psi(S) = \pi(\lambda(S))$, where π is the left regular representation of the pure unbounded Hilbert algebra $\lambda(\mathfrak{M}_\varphi)$. We shall show $\Psi(\mathfrak{A})_b = \Psi(\mathfrak{A}_b)$. Clearly we have $\Psi(\mathfrak{A}_b) \subset \Psi(\mathfrak{A})_b$. Conversely let $\Psi(A) \in \Psi(\mathfrak{A})_b$. By Lemma 4.7. $A = A_0 + A_1$, $A_0 \in \mathfrak{A}_b$, $A_1 \in \mathfrak{M}_\varphi$, and so $\Psi(A_1) = \pi(\lambda(A_1)) \in \Psi(\mathfrak{M}_\varphi)_b$. Since $\lambda((\mathfrak{M}_\varphi)_b)$ is a maximal Hilbert algebra, $\lambda(A_1) \in \lambda((\mathfrak{M}_\varphi)_b)$, i.e., $A_1 \in (\mathfrak{M}_\varphi)_b$. Therefore $A = A_0 + A_1 \in \mathfrak{A}_b$, and so $\Psi(A) \in \Psi(\mathfrak{A}_b)$. By the theory of von Neumann algebras, $\overline{\Psi(\mathfrak{A}_b)} = \mathcal{U}_0(\lambda((\mathfrak{M}_\varphi)_b))$. Moreover it is easy to show that $\Psi(\mathfrak{A}) \supset \Psi(\mathfrak{M}_\varphi) = \pi(\lambda(\mathfrak{M}_\varphi))$ and Ψ is an isomorphism of \mathfrak{A} onto $\Psi(\mathfrak{A})$. Since \mathfrak{A} is φ -measurable, we can easily show that $\Psi(\mathfrak{A})$ is measurable.

LEMMA 4.10. *Let ψ be the natural trace on $\Psi(\mathfrak{A})^+$. Then we have*

$$\varphi(A) = \psi(\Psi(A)), \quad A \in \mathfrak{A}^+.$$

Proof. By the definition of the natural trace ψ we get

$$\mathfrak{M}_\psi^+ = \pi(\lambda(\mathfrak{M}_\varphi^+)) = \Psi(\mathfrak{M}_\varphi^+)$$

and moreover for every $A \in \mathfrak{M}_\varphi^+$

$$\varphi(A) = \|\lambda(A^{\frac{1}{2}})\|^2 = \psi(\pi(\lambda(A))) = \psi(\Psi(A)).$$

By Lemma 4.6. ~ 4.10. and Theorem 4.5. we obtain the following theorem.

THEOREM 4.11. *Let \mathfrak{A} be an EW^* -algebra and let φ be a faithful normal semifinite trace on \mathfrak{A}^+ . Suppose that \mathfrak{A} is a φ -measurable*

EW^* -algebra and $\mathfrak{A}(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi$. Then $\lambda(\mathfrak{N}_\varphi)$ is a pure unbounded Hilbert algebra over $\lambda((\mathfrak{N}_\varphi)_b)$ and putting

$$\Psi(A)\lambda(S) = \lambda(AS), \quad S \in \mathfrak{N}_\varphi$$

for every $A \in \mathfrak{A}$, $\Psi(A)$ is a linear operator on $\lambda(\mathfrak{N}_\varphi)$. The isomorphism Ψ is extended to an isomorphism Φ of \mathfrak{A} onto the left EW^* -algebra $\mathcal{U}(\lambda(\mathfrak{N}_\varphi))$ of $\lambda(\mathfrak{N}_\varphi)$. Let ψ be the natural trace on $\Phi(\mathfrak{A})^+$. Then $\varphi = \psi \circ \Phi$.

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Pacific Journal of Mathematics

Vol. 66, No. 2

December, 1976

Gerald A. Beer, <i>Tax structures whose progressivity is inflation neutral</i>	305
William M. Cornette, <i>A generalization of the unit interval</i>	313
David E. Evans, <i>Unbounded completely positive linear maps on C^*-algebras</i>	325
Hector O. Fattorini, <i>Some remarks on convolution equations for vector-valued distributions</i>	347
Amassa Courtney Fauntleroy, <i>Automorphism groups of unipotent groups of Chevalley type</i>	373
Christian C. Fenske and Heinz-Otto Peitgen, <i>On fixed points of zero index in asymptotic fixed point theory</i>	391
Atsushi Inoue, <i>On a class of unbounded operator algebras. II</i>	411
Herbert Meyer Kamowitz, <i>The spectra of endomorphisms of algebras of analytic functions</i>	433
Jimmie Don Lawson, <i>Embeddings of compact convex sets and locally compact cones</i>	443
William Lindgren and Peter Joseph Nyikos, <i>Spaces with bases satisfying certain order and intersection properties</i>	455
Emily Mann Peck, <i>Lattice projections on continuous function spaces</i>	477
Morris Marden and Peter A. McCoy, <i>Level sets of polynomials in n real variables</i>	491
Francis Joseph Narcowich, <i>An imbedding theorem for indeterminate Hermitian moment sequences</i>	499
John Dacey O'Neill, <i>Rings whose additive subgroups are subrings</i>	509
Chull Park and David Lee Skoug, <i>Wiener integrals over the sets bounded by sectionally continuous barriers</i>	523
Vladimir Scheffer, <i>Partial regularity of solutions to the Navier-Stokes equations</i>	535
Eugene Spiegel and Allan Trojan, <i>On semi-simple group algebras. II</i>	553
Katsuo Takano, <i>On Cameron and Storvick's operator valued function space integral</i>	561