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Atsushi Inoue

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ATSUSHI INOUE

In this paper we continue our study of unbounded operator algebras. On the basis of the space $L^\omega[0,1]$ introduced by R. Arens [1] we define and investigate unbounded Hilbert algebras. The primary purpose of this paper is to investigate the relation between unbounded Hilbert algebras and $EW^\#$-algebras and the structure of some $EW^\#$-algebras.

1. Introduction. In a previous paper **[10]** we began our study of $EW^\#$-algebras. For the definitions and the basic properties concerning $EW^\#$-algebras is referred to **[10]**. It is well known that semifinite von Neumann algebras are related to Hilbert algebras. That is, if $\mathscr{D}_0$ is a Hilbert algebra, then the left von Neumann algebra $\mathscr{U}_0(\mathscr{D}_0)$ is defined and $\mathscr{U}_0(\mathscr{D}_0)$ is a semifinite von Neumann algebra and conversely if $\mathfrak{A}$ is a semifinite von Neumann algebra, then there exists a Hilbert algebra $\mathscr{D}_0$ such that $\mathfrak{A}$ is isomorphic to the left von Neumann algebra $\mathscr{U}_0(\mathscr{D}_0)$. In this paper we study the above facts about $EW^\#$-algebras. So, our starting point will be the extension of Hilbert algebras.

DEFINITION 1.1. Let $\mathscr{D}$ be a pre-Hilbert space with inner product $(\ |\ )$ and a $*$-algebra. If $\mathscr{D}$ satisfies the following conditions (1) ~ (3);

(1) $(\xi \mid \eta) = (\eta^* \mid \xi^*)$, $\xi, \eta \in \mathscr{D}$;

(2) $(\xi\eta \mid \zeta) = (\eta \mid \xi^*\zeta)$, $\xi, \eta, \zeta \in \mathscr{D}$;

By (2) we define $\pi(\xi)$ and $\pi'(\eta)$ by;

$$\pi(\xi)\eta = \pi'(\eta)\xi = \xi\eta, \quad \xi, \eta \in \mathscr{D}.$$

Then $\pi(\xi)$ and $\pi'(\eta)$ are closable operators on $\mathscr{D}$ and we have $\pi(\xi)^* \supset \pi(\xi^*)$ and $\pi'(\eta)^* \supset \pi'(\eta^*)$. We call π (resp. π') the left (resp. right) regular representation of $\mathscr{D}$.

(3) Putting

$$\mathscr{D}_0 = \{\xi \in \mathscr{D};\ \pi(\xi) \text{ is continuous}\},$$

$\mathscr{D}_0^2$ is dense in $\mathscr{D}$, then $\mathscr{D}$ is called an unbounded Hilbert algebra over $\mathscr{D}_0$. In particular, if $\mathscr{D}_0 \neq \mathscr{D}$, then $\mathscr{D}$ is called a pure unbounded Hilbert algebra over $\mathscr{D}_0$.

In §2 we investigate the properties of unbounded Hilbert algebras and we introduce examples of such unbounded Hilbert algebras ($L^\omega[0,1]$,

$L^\omega(-\infty,\infty)$, $L_{1^*}^\omega(-\infty,\infty)$, $L_{2^*}^\omega(-\infty,\infty)$, $L_1^\omega(G)$, $L_2^\omega(G)$ (G; unimodular locally compact group)).

In §3 we consider the noncommutative integration with respect to a von Neumann algebra as constructed by Segal in **[14]**. Let $\mathscr{D}$ be a pure unbounded Hilbert algebra over $\mathscr{D}_0$. Then $L^\omega(\mathscr{D}_0)$ and $L_2^\omega(\mathscr{D}_0)$ are defined and they are pure unbounded Hilbert algebras. In particular, $L_2^\omega(\mathscr{D}_0)$ is maximal in pure unbounded Hilbert algebras containing $\mathscr{D}_0$. Furthermore $\mathscr{D}^2$ (resp. $\mathscr{D}$) is a $*$-subalgebra of pure unbounded Hilbert algebra $L^\omega(\mathscr{D}_0)$ (resp. $L_2^\omega(\mathscr{D}_0)$) (Theorem 3.9.). We can define a left $EW^{\#}$-algebra $\mathscr{U}(\mathscr{D})$ of a pure unbounded Hilbert algebra $\mathscr{D}$ over $\mathscr{D}_0$, i.e., $\mathscr{U}(\mathscr{D})$ is a minimal $EW^{\#}$-algebra on $L_2^\omega(\mathscr{D}_0)$ over $\mathscr{U}_0(\mathscr{D}_0)$ and $\mathscr{U}(\mathscr{D}) \supset \overline{\pi(\mathscr{D})}$, where we denote by $\bar{A}$ the smallest closed extension of a closable operator A and we put $\bar{\mathfrak{A}} = \{\bar{A}\,;\, A \in \mathfrak{A}\}$ (Theorem 3.10.).

In §4 we define traces on $EW^{\#}$-algebras and we investigate the structure of some $EW^{\#}$-algebras.

DEFINITION 1.2. Let $\mathfrak{A}$ be an $EW^{\#}$-algebra and let φ be a map of $\mathfrak{A}^+$ into $[0,\infty]$. If φ satisfies the following conditions (1) ~ (3), then φ is called a trace on $\mathfrak{A}^+$;

(1) $\varphi(S+T) = \varphi(S) + \varphi(T), \quad S, T \in \mathfrak{A}^+;$

(2) $\varphi(\lambda S) = \lambda\varphi(S), \quad \lambda \geqq 0,\ S \in \mathfrak{A}^+;$

(3) $\varphi(S^{\#}S) = \varphi(SS^{\#}), \quad S \in \mathfrak{A}.$

If the conditions $\varphi(S) = 0$, $S \in \mathfrak{A}^+$ implies $S = 0$, then φ is called faithful. If, for each increasing net $\{T_\alpha\}$ of $\mathfrak{A}^+$ that converges σ-weakly to $S \in \mathfrak{A}^+$ (hereafter we denote $T_\alpha \uparrow S$), we have $\varphi(T_\alpha) \uparrow \varphi(S)$, then φ is called normal. If $\varphi(S) < \infty$ for every $S \in \mathfrak{A}^+$, then φ is called finite. If, for each $S \in \mathfrak{A}^+$, there exists a net $\{T_\alpha\}$ such that $T_\alpha \uparrow S$ and $\varphi(T_\alpha) < \infty$, then φ is called semifinite.

Let $\mathscr{U}(\mathscr{D})$ be the left $EW^{\#}$-algebra of a pure unbounded Hilbert algebra $\mathscr{D}$ over $\mathscr{D}_0$. Then there exists a faithful normal semifinite trace φ on $\mathscr{U}(\mathscr{D})^+$ such that $\varphi/\mathscr{U}(\mathscr{D})_b^+$ equals the natural trace on $\mathscr{U}_0(\mathscr{D}_0)^+$ and $\mathscr{U}(\mathscr{D})(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi$ (we note $\mathfrak{N}_\varphi = \{T \in \mathscr{U}(\mathscr{D});\ \varphi(T^{\#}T) < \infty\}$ and $(\mathfrak{N}_\varphi)_b = \mathfrak{N}_\varphi \cap \mathscr{U}(\mathscr{D})_b$) (Theorem 4.2.). Conversely if $\mathfrak{A}$ is an $EW^{\#}$-algebra with a faithful normal semifinite trace φ satisfying $\mathfrak{A}(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi$, then $\mathfrak{N}_\varphi$ is a pure unbounded Hilbert algebra over $(\mathfrak{N}_\varphi)_b$ and $\mathfrak{N}$ is isomorphic to the left $EW^{\#}$-algebra $\mathscr{U}(\mathfrak{N}_\varphi)$ of $\mathfrak{N}_\varphi$ (Theorem 4.11.).

2. Unbounded Hilbert algebras. In this section let $\mathscr{D}$ be a pure unbounded Hilbert algebra over $\mathscr{D}_0$ and let $\mathfrak{H}$ be the completion of $\mathscr{D}$. Clearly $\mathscr{D}_0$ is a Hilbert algebra and the completion of $\mathscr{D}_0$ is a Hilbert space $\mathfrak{H}$. For each $x \in \mathfrak{H}$ we define $\pi_0(x)$ and $\pi_0'(x)$ by;

$$\pi_0(x)\xi = \overline{\pi_0'(\xi)}\,x, \qquad \xi \in \mathscr{D}_0$$

$$\pi_0'(x)\xi = \overline{\pi_0(\xi)}\,x, \qquad \xi \in \mathscr{D}_0,$$

where π_0 (resp. π'_0) is the left (resp. right) regular representation of the Hilbert algebra $\mathfrak{D}_0$. Then $\pi_0(x)$ and $\pi'_0(x)$ are linear operators on $\mathfrak{H}$ with domain $\mathscr{D}_0$. By ([12] Theorem 3) we have

$$\overline{\pi_0(Jx)} = \pi_0(x)^*, \qquad \overline{\pi'_0(Jx)} = \pi'_0(x)^*$$

for all $x \in \mathfrak{H}$, where J denotes the involution of $\mathfrak{H}$.

LEMMA 2.1. *For each $\xi \in \mathscr{D}$ we have*
(1) $\overline{\pi(\xi)} = \overline{\pi_0(\xi)}$, $\overline{\pi'(\xi)} = \overline{\pi'_0(\xi)}$;
(2) $\overline{\pi(\xi^*)} = \pi(\xi)^*$, $\overline{\pi'(\xi^*)} = \pi'(\xi)^*$.

Proof. (1); Clearly we get $\pi_0(\xi) \subset \pi(\xi)$. Hence $\pi_0(\xi)^* \supset \pi(\xi)^*$. Since $\pi_0(\xi)^* = \overline{\pi_0(\xi^*)}$ and $\pi(\xi)^* \supset \pi(\xi^*)$, we have

$$\overline{\pi_0(\xi)} = \pi_0(\xi^*)^* \supset \pi(\xi^*)^* \supset \overline{\pi(\xi)}.$$

Therefore we get $\overline{\pi_0(\xi)} = \overline{\pi(\xi)}$.
(2); By (1) we have

$$\overline{\pi(\xi^*)} = \overline{\pi_0(\xi^*)} = \pi_0(\xi)^* = \pi(\xi)^*.$$

LEMMA 2.2. *For each* $\lambda, \mu \in \mathfrak{C}$ (the field of complex numbers) *and* $\xi, \xi_i, \eta, \eta_i \in \mathscr{D}$ $(i = 1, 2)$ *we have*

$$\pi(\lambda\xi_1 + \mu\xi_2) = \lambda\pi(\xi_1) + \mu\pi(\xi_2);$$
$$\pi(\xi_1\xi_2) = \pi(\xi_1)\pi(\xi_2);$$
$$\pi(\xi^*) \subset \pi(\xi)^*;$$
$$\pi'(\lambda\eta_1 + \mu\eta_2) = \lambda\pi'(\eta_1) + \mu\pi'(\eta_2);$$
$$\pi'(\eta_1\eta_2) = \pi'(\eta_2)\pi'(\eta_1);$$
$$\pi'(\eta^*) \subset \pi'(\eta)^*.$$

Putting

$$\pi(\xi)^{\#} = \pi(\xi^*), \qquad \pi'(\eta)^{\#} = \pi'(\eta^*),$$

$\pi(\mathscr{D})$ and $\pi'(\mathscr{D})$ are #-algebras on $\mathscr{D}$ and we have the following properties;

(1) $\pi(\mathscr{D})_b = \pi(\mathscr{D}_0)$, $\pi'(\mathscr{D})_b = \pi'(\mathscr{D}_0)$;

(2) $J\pi(\xi)J = \pi'(\xi)^{\#}, \quad J\pi'(\xi)J = \pi(\xi)^{\#}, \quad \xi \in \mathscr{D};$

(3) $\pi(\xi)\pi'(\eta) = \pi'(\eta)\pi(\xi), \quad \xi, \eta \in \mathscr{D};$

(4) $\overline{\pi(\xi)^{\#}} = \pi(\xi)^{*}, \quad \overline{\pi'(\xi)^{\#}} = \pi'(\xi)^{*}, \quad \xi \in \mathscr{D}.$

Hence we get

$$\overline{\pi(\mathscr{D})''_b} = \mathscr{U}_0(\mathscr{D}_0), \quad \overline{\pi'(\mathscr{D})''_b} = \mathscr{V}_0(\mathscr{D}_0),$$

where $\mathscr{U}_0(\mathscr{D}_0)$ (resp. $\mathscr{V}_0(\mathscr{D}_0)$) *is the left* (resp. right) *von Neumann algebra of* $\mathscr{D}_0$.

PROPOSITION 2.3. *For each* $\lambda \in \mathfrak{C}$ *and* $\xi, \eta \in \mathscr{D}$ *we have*

$$\overline{\pi(\xi)} + \overline{\pi(\eta)} = \overline{\pi(\xi+\eta)}, \quad \overline{\pi(\xi)} \cdot \overline{\pi(\eta)} = \overline{\pi(\xi\eta)},$$

$$\lambda \cdot \overline{\pi(\xi)} = \overline{\pi(\lambda\xi)}, \quad \overline{\pi(\xi)}^{*} = \overline{\pi(\xi^{*})}.$$

Therefore $\overline{\pi(\mathscr{D})}$ *is a* $*$*-algebra of closed operators on* $\mathfrak{H}$ *under the operations of strong sum, strong product, adjoint and strong scalar multiplication. Similarly* $\overline{\pi'(\mathscr{D})}$ *is a* $*$*-algebra of closed operators on* $\mathfrak{H}$. *Furthermore we have*

$$J\overline{\pi(\xi)}J = \pi'(\xi)^{*}, \quad J\overline{\pi'(\xi)}J = \pi(\xi)^{*}, \quad \xi \in \mathscr{D}.$$

Proof. By Lemma 2.1. we have $\overline{\pi(\xi)} = \pi(\xi^{*})^{*}$ for every $\xi \in \mathscr{D}$ and hence

$$\begin{aligned}\overline{\pi(\xi)} + \overline{\pi(\eta)} = \overline{\overline{\pi(\xi)} + \overline{\pi(\eta)}} &= \overline{\pi(\xi^{*})^{*} + \pi(\eta^{*})^{*}} \\ &\subset (\pi(\xi^{*}) + \pi(\eta^{*}))^{*} = \pi((\xi+\eta)^{*})^{*} \\ &= \overline{\pi(\xi+\eta)},\end{aligned}$$

and so $\overline{\pi(\xi)} + \overline{\pi(\eta)} = \overline{\pi(\xi+\eta)}$. Similarly $\overline{\pi(\xi)} \cdot \overline{\pi(\eta)} = \overline{\overline{\pi(\xi)}\,\overline{\pi(\eta)}} = \overline{\pi(\xi\eta)}$ and $\lambda \cdot \overline{\pi(\xi)} = \overline{\pi(\lambda\xi)}$ are showed. By Lemma 2.2 (2) we have $J\pi(\xi)J = \pi'(\xi)^{\#}$, $\xi \in \mathscr{D}$ and hence $\overline{J\pi(\xi)J} = \overline{\pi'(\xi)^{\#}} = \pi'(\xi)^{*}$ by Lemma 2.1. On the other hand we can easily show $\overline{J\pi(\xi)J} = J\overline{\pi(\xi)}J$. Therefore we have $J\overline{\pi(\xi)}J = \pi'(\xi)^{*}$.

Problem. Does there exist an $EW^{\#}$-algebra $\mathfrak{A}$ such that $\overline{\mathfrak{A}}_b = \mathscr{U}_0(\mathscr{D}_0)$ and $\bar{\mathfrak{A}} \supset \overline{\pi(\mathscr{D})}$?

In §3 we show that there exist such $EW^{\#}$-algebras. In particular, there exists an $EW^{\#}$-algebra such that is minimal in such $EW^{\#}$-algebras and we call it the left $EW^{\#}$-algebra of $\mathscr{D}$.

We introduce examples of unbounded Hilbert algebras.

(1) $L^{\omega}[0,1]$. Let $L^{\omega}[0,1]$ be the set of all complex-valued measurable functions f on $[0,1]$ such that $f \in L^p[0,1]$, $p = 1, 2, \cdots$. By the whole collection of L^p-norms

$$\|f\|_p = \left[\int_0^1 |f(t)|^p\, dt\right]^{1/p}, \qquad p = 1, 2, \cdots$$

and by pointwise multiplication and involution ($f^*(t) = \overline{f(t)}$, $t \in [0,1]$) the space $L^{\omega}[0,1]$ is a complete metrizable locally convex $*$-algebra with jointly continuous multiplication. R. Arens **[1]** showed $L^{\omega}[0,1]$ is not a locally m-convex algebra. However, G. R. Allan **[2]** showed that $L^{\omega}[0,1]$ is a GB^*-algebra with $(L^{\omega}[0,1])_0 = L^{\infty}[0,1]$. We introduce the inner product into $L^{\omega}[0,1]$ by;

$$(f \mid g) = \int_0^1 f(t)\overline{g(t)}\,dt, \quad f, g \in L^{\omega}[0,1].$$

Then $L^{\omega}[0,1]$ is regarded as a pure unbounded Hilbert algebra over $L^{\infty}[0,1]$.

(2) $L^{\omega}(-\infty,\infty)$. Let $L^{\omega}(-\infty,\infty)$ be the set of all complex-valued measurable functions f on $(-\infty,\infty)$ such that $f \in L^p(-\infty,\infty)$ for every real number $p \geqq 1$. Under the following operations

$$(fg)(x) = f(x)g(x), \quad (\lambda f)(x) = \lambda f(x),$$
$$f^*(x) = \overline{f(x)}$$

and inner product $(f \mid g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$, we can show that $L^{\omega}(-\infty,\infty)$ is a pure unbounded Hilbert algebra.

(3) $L_1^{\omega}(G)$ and $L_2^{\omega}(G)$. Let G be a unimodular locally compact group and let dx be a Haar measure on G. Let $L^p(G)$ be the Banach space of measurable functions f on G for which the norm

$$\|f\|_p = \left[\int_G |f(x)|^p\, dx\right]^{1/p}, \quad 1 \leqq p < \infty,$$
$$\|f\|_{\infty} = \operatorname{ess\,sup} |f(x)|$$

is finite. We note

$L(G)$; the space of complex-valued continuous functions with compact supports,

$$L^{\omega}(G)=\bigcap_{1\leqq p\leqq\infty} L^{p}(G),\quad L_{1}^{\omega}(G)=\bigcap_{1<p\leqq\infty} L^{p}(G),$$

$$L_{2}^{\omega}(G)=\bigcap_{1<p\leqq 2} L^{p}(G).$$

Under the convolution $f*g$ as multiplication, involution f^* ($f^*(x)=\overline{f(x^{-1})}$) and inner product $(f\mid g)=\int_G f(x)\overline{g(x)}dx$ on $L^2(G)$, $L^{\omega}(G)$ is a Hilbert algebra and $L_1^{\omega}(G)$ and $L_2^{\omega}(G)$ are unbounded Hilbert algebras. In fact, suppose $f\in L^p(G)$ and $g\in L^q(G)$ $(1/p+1/q\geqq 1)$. Then by Young's inequality $f*g$ exists and $\|f*g\|_r\leqq\|f\|_p\|g\|_q$ where $1/r=1/p+1/q-1$. Furthermore, for each $f\in L^p(G)$ $(1\leqq p<\infty)$ we have $\|f^*\|_p=\|f\|_p$. Therefore we can easily show that $L^{\omega}(G)$, $L_1^{\omega}(G)$ and $L_2^{\omega}(G)$ are $*$-algebras. Since $L(G)\subset L^{\omega}(G)\subset L^1(G)\cap L^2(G)$ and L(G), $L^1(G)\cap L^2(G)$ are Hilbert algebras, $L^{\omega}(G)$ is clearly a Hilbert algebra. We can easily show that $(f\mid g)=(g^*\mid f^*)$ and $(f*g\mid h)=(g\mid f^**h)$ for every $f,g,h\in L_1^{\omega}(G)$ (resp. $L_2^{\omega}(G)$). Furthermore we have

$$L^{\omega}(G)\subset(L_1^{\omega}(G))_0\quad(\text{resp. } L_2^{\omega}(G)_0)\subset L^2(G),$$

and so $(L_1^{\omega}(G)_0)^2$ (resp. $(L_2^{\omega}(G)_0)^2$) is dense in $L^2(G)$. Therefore $L_1^{\omega}(G)$ and $L_2^{\omega}(G)$ are unbounded Hilbert algebras.

Problem. Is an unbounded Hilbert algebra $L_1^{\omega}(G)$ (or $L_2^{\omega}(G)$) pure?

If G is a compact group, then $L^2(G)$ is an H^*-algebra, and so $L_1^{\omega}(G)$ and $L_2^{\omega}(G)$ are Hilbert algebras.

If $G=(-\infty,\infty)$, then

$$L_{1*}^{\omega}(-\infty,\infty)=\bigcap_{1<p\leqq\infty} L^{p}(-\infty,\infty)$$

and

$$L_{2*}^{\omega}(-\infty,\infty)=\bigcap_{1<p\leqq 2} L^{p}(-\infty,\infty)$$

are pure unbounded Hilbert algebras under the following operations and inner product

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy,$$

$$(\lambda f)(x) = \lambda f(x), \quad f^*(x) = \overline{f(-x)},$$

$$(f \mid g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.$$

In fact, we note

$$\pi(f)g = f * g, \quad f, g \in L_{1^*}^{\omega}(-\infty, \infty)$$

and

$$(L_{1^*}^{\omega}(-\infty, \infty))_0 = \{f \in L_{1^*}^{\omega}(-\infty, \infty);\ \pi(f) \text{ is continuous}\}.$$

We have only to show $(L_{1^*}^{\omega}(-\infty, \infty))_0 \neq L_{1^*}^{\omega}(-\infty, \infty)$. By the theory of Hilbert algebras we have

$$\begin{aligned}(L^1(-\infty,\infty) \cap L^2(-\infty,\infty))_b &= \{f \in L^2(-\infty,\infty);\ \overline{\pi(f)} \text{ is a bounded} \\ &\qquad \text{linear operator on } L^2(-\infty,\infty)\} \\ &= \{f \in L^2(-\infty,\infty);\ \hat{f} \in L^\infty(-\infty,\infty)\},\end{aligned}$$

where $\hat{f}$ denotes the Fourier transform of f. Clearly we have

$$(L_{1^*}^{\omega}(-\infty, \infty))_0 \subset \{f \in L^2(-\infty,\infty);\ \hat{f} \in L^\infty(-\infty,\infty)\}.$$

Putting

$$f(x) = \begin{cases} 0, & x < 1 \\ 1/x, & x \geqq 1 \end{cases}$$

we can show $f \in L_{1^*}^{\omega}(-\infty, \infty)$ and $\hat{f} \notin L^\infty(-\infty,\infty)$, and so $L_{1^*}^{\omega}(-\infty, \infty)_0 \neq L_{1^*}^{\omega}(-\infty, \infty)$. Consequently $L_{1^*}^{\omega}(-\infty, \infty)$ is pure.

3. L^ω-spaces with respect to noncommutative integration. Our starting point for the construction of L^ω-space will be the algebras of operators measurable with respect to a von Neumann algebra as constructed by Segal in **[14]**. Let $\mathfrak{A}$ be a semifinite von Neumann algebra on a Hilbert space $\mathfrak{H}$ and let φ be a faithful normal semifinite trace on $\mathfrak{A}^+$. Let $\mathfrak{A}_p$ and $\mathfrak{A}_u$, respectively, denote the set of all projections and that of unitary operators in $\mathfrak{A}$.

DEFINITION 3.1. A linear set $\mathfrak{D}$ in $\mathfrak{H}$ is said to be strongly dense (resp. φ-restrictedly strongly dense) provided

(a) $U'\mathfrak{D} \subset \mathfrak{D}$ for every $U' \in \mathfrak{A}'_u$;

(b) there exists a sequence of projections $P_n \in \mathfrak{A}$ such that $P_n\mathfrak{H} \subset \mathfrak{D}$, $P_n^{\perp} \downarrow 0$ and $P_n^{\perp}$ is a finite projection (resp. $\varphi(P_n^{\perp}) < \infty$). An operator $T\eta\mathfrak{A}$ is called essentially measurable (resp. φ-restrictedly essentially measurable) if T has a strongly dense (resp. φ-restrictedly strongly dense) domain and a closed extension. Moreover if T is closed, T is called measurable (resp. φ-restrictedly measurable).

LEMMA 3.2. (**[11]** Lemma 1.1.) *Let T be a closed densely defined operator $\eta\mathfrak{A}$. Then;*

(1) *T is measurable* (resp. φ-restrictedly measurable) *if and only if so is $|T|$.*

(2) *Let $T \geqq 0$ and let $T = \int_0^\infty \lambda dE(\lambda)$ be its spectral resolution. T is measurable* (resp. φ-restrictedly measurable) *if and only if $E(\lambda)^{\perp}$ $(= I - E(\lambda))$ is a finite projection* (resp. $\varphi(E(\lambda)^{\perp}) < \infty$) *for a positive λ.*

We denote the set of all operators on $\mathfrak{H}$ measurable (resp. φ-restrictedly measurable) with respect to $\mathfrak{A}$ by $\mathfrak{M}(\mathfrak{A})$ (resp. $\mathfrak{M}(\varphi)$).

PROPOSITION 3.3. (**[7]** Prop. 4.3.) *The sets $\mathfrak{M}(\mathfrak{A})$ and $\mathfrak{M}(\varphi)$ form EW^*-algebras over $\mathfrak{A}$ under the operations of strong sum, strong product, adjoint and strong scalar multiplication.*

Let $\mathfrak{M}_\varphi$ be the maximal ideal associated with φ, that is, the set of $A \in \mathfrak{A}$ with $\varphi(|A|) < \infty$. For every $T \in \mathfrak{M}(\mathfrak{A})^+$ we put

$$\mu(T) = \sup_{A \in \mathfrak{M}_\varphi, A \leqq T} \varphi(A).$$

DEFINITION 3.4. A measurable operator $T\eta\mathfrak{A}$ is said to be pth power integrable with respect to φ if $\mu(|T|^p) < \infty$. Let $L^p(\varphi)$ $(1 \leqq p < \infty)$ stand for the set of pth power integrable operators $\eta\mathfrak{A}$. The L^p-norm of $T \in L^p(\varphi)$ is defined as $\mu(|T|^p)^{1/p}$ and designated by $\|T\|_p$. If $p = \infty$, we shall identify $\mathfrak{A}$ with $L^\infty(\varphi)$.

A measurable operator T belongs to $L^p(\varphi)$ $(1 \leqq p < \infty)$ if and only if T is φ-restrictedly measurable and $-\int_0^\infty \lambda^p d\varphi(E(\lambda)^{\perp}) < \infty$, where $\int_0^\infty \lambda dE(\lambda)$ is the spectral resolution of $|T|$.

THEOREM 3.5. **[11]** (1) *For $1 \leqq p < \infty$ $L^p(\varphi)$ is a Banach space with norm $\|T\|_p$ and the following properties are satisfied.*

(a) $\|T\|_p = \|T^*\|_p = \|U \cdot T \cdot U^*\|_p$ *for* $T \in L^p(\varphi)$ *and* $U \in \mathfrak{A}_u$.

(b) *For* $S, T \in L^p(\varphi)$ *such that* $|T| \leqq |S|$ *we have* $\|T\|_p \leqq \|S\|_p$.

(c) *For* $A \in \mathfrak{A}$ *and* $T \in L^p(\varphi)$ *we have* $\|A \cdot T\|_p \leqq \|A\| \, \|T\|_p$.

(d) *If* $0 \leqq T_1 \leqq T_2 \leqq \cdots$ *is a sequence of elements of* $L^p(\varphi)$ *such that* $\{\|T_n\|_p\}$ *is bounded, then there exists* $T := \sup T_n$ *and* $\lim_{n\to\infty}\|T - T_n\|_p = 0$.

(2) *Let* $1/p + 1/q = 1$ *where* $1 \leqq p, q \leqq \infty$. *Then*

(a) $\mu(S \cdot T) = \mu(T \cdot S)$ *for* $S \in L^p(\varphi)$ *and* $T \in L^q(\varphi)$. *If furthermore,* $S, T \geqq 0$, *then* $\mu(S \cdot T) \geqq 0$; *and conversely, if* $\mu(S \cdot T) \geqq 0$ *for every* $T \geqq 0$, *then* $S \geqq 0$.

(b) $|\mu(T_1 \cdot T_2 \cdot \cdots \cdot T_n)| \leqq \mu(|T_1 \cdot T_2 \cdot \cdots \cdot T_n|) \leqq \|T_1\|_{p_1} \|T_2\|_{p_2} \cdots \|T_n\|_{p_n}$ *for* $T_i \in L^{p_i}(\varphi)$ *with* $\sum_{i=1}^n 1/p_i = 1$, $p_i \geqq 1$ $(i = 1, 2, \cdots, n)$.

(c)
$$\|S\|_p = \sup_{T \in L^q(\varphi), \|T\|_q \leqq 1} |\mu(S \cdot T)|$$

for $S \in L^p(\varphi)$ *where the sup is attained by some* T *if* $1 \leqq p < \infty$.

(d)
$$|\mu(S \cdot T)|^2 \leqq \mu(|S^*| \cdot |T|)\mu(|S| \cdot |T^*|) \leqq \mu(|S \cdot T|)\mu(|T \cdot S|)$$

for $S \in L^p(\varphi)$ *and* $T \in L^q(\varphi)$.

(3) *Let* $1/p + 1/q = 1/r$ *where* $1 \leqq p, q, r \leqq \infty$.

(a) *If* $T \in L^p(\varphi)$ *and* $S \in L^q(\varphi)$, *then* $T \cdot S \in L^r(\varphi)$ *and we have* $\|T \cdot S\|_r \leqq \|T\|_p \|S\|_q$.

(b) *Let* T *be a* φ*-restrictedly measurable operator* $\eta \mathfrak{A}$. *If* $T \cdot S \in L^r(\varphi)$ *for every* $S \in L^q(\varphi)^+$, *then* $T \in L^p(\varphi)$.

Let $\mathscr{D}_0$ be a Hilbert algebra. Let $\mathscr{U}_0(\mathscr{D}_0)$ be the left von Neumann algebra of $\mathscr{D}_0$ and let φ_0 be the natural trace on $\mathscr{U}_0(\mathscr{D}_0)^+$. The completion $\mathfrak{H}$ of $\mathscr{D}_0$ is equivalent to an H-system **[3]**. Putting

$$(\mathscr{D}_0)_b = \{x \in \mathfrak{H};\ \overline{\pi_0(x)} \text{ is bounded}\},$$

$(\mathscr{D}_0)_b$ is a maximal Hilbert algebra containing $\mathscr{D}_0$ and $\mathscr{U}_0(\mathscr{D}_0)(\mathscr{D}_0)_b \subset (\mathscr{D}_0)_b$. For every $x \in \mathfrak{H}$ $\overline{\pi_0(x)}$ is φ_0-restrictedly measurable (**[11]** Lemma 2.3.). We can easily show that $L^2(\varphi_0) = \{\overline{\pi_0(x)};\ x \in \mathfrak{H}\}$ and $L^2(\varphi_0)$ is a Hilbert space isometric with $\mathfrak{H}$. Moreover we remark that $L^2(\varphi_0)$ is an H-system isomorphic with $\mathfrak{H}$ by the map. $x \to \overline{\pi_0(x)}$. This follows from the facts that (1) if xy is defined and equals z, then $\overline{\pi_0(x)} \cdot \overline{\pi_0(y)} = \overline{\pi_0(xy)}$ and (2) if $\overline{\pi_0(x)} \cdot \overline{\pi_0(y)}$ equals $\overline{\pi_0(z)}$, then xy is defined and equals z. We have

$$L^1(\varphi_0) = \{\sum_{i=1}^m \overline{\pi_0(x_i)} \cdot \overline{\pi_0(y_i)};\ x_i, y_i \in \mathfrak{H}\}$$

and the integral $\mu(T)$ of $T = \sum_{i=1}^m \overline{\pi_0(x_i)} \cdot \overline{\pi_0(y_i)}$ equals $\sum_{i=1}^m (y_i \mid x_i^*)$.

DEFINITION 3.5. We define the L^ω-spaces with respect to the natural trace φ_0 as follows;

$$L^\omega(\varphi_0) = \bigcap_{1 \leqq p < \infty} L^p(\varphi_0),$$

$$L_2^\omega(\varphi_0) = \bigcap_{2 \leqq p < \infty} L^p(\varphi_0).$$

Similarly we define the L^ω-spaces with respect to the Hilbert algebra $\mathscr{D}_0$ as follows;

$$L^\omega(\mathscr{D}_0) = \{x \in \mathfrak{H};\ \overline{\pi_0(x)} \in L^\omega(\varphi_0)\},$$

$$L_2^\omega(\mathscr{D}_0) = \{x \in \mathfrak{H};\ \overline{\pi_0(x)} \in L_2^\omega(\varphi_0)\}.$$

PROPOSITION 3.6. *The space $L^\omega(\mathscr{D}_0)$ (resp. $L_2^\omega(\mathscr{D}_0)$) is an unbounded Hilbert algebra containing $(\mathscr{D}_0)_b^2$ (resp. $(\mathscr{D}_0)_b$).*

Proof. For $1 \leqq p < \infty$ and $S, T \in L^\omega(\varphi_0)$

$$\|S \cdot T\|_p \leqq \|S\|_{2p} \|T\|_{2p}$$

and hence $S \cdot T \in L^\omega(\varphi_0)$. Therefore, for each x and y in $L^\omega(\mathscr{D}_0)$ xy is defined and equals $\overline{\pi_0(x)}y$. Furthermore for each $T \in L^p(\varphi_0)$ $(1 \leqq p < \infty)$ $\|T\|_p = \|T^*\|_p$ and hence $x^* \in L^\omega(\mathscr{D}_0)$ for every $x \in L^\omega(\mathscr{D}_0)$. Consequently $L^\omega(\mathscr{D}_0)$ is a $*$-algebra. We can easily show $L^\omega(\mathscr{D}_0) \supset (\mathscr{D}_0)_b^2$, and so $L^\omega(\mathscr{D}_0)$ is a pre-Hilbert space and its completion is $L^2(\mathscr{D}_0) = \mathfrak{H}$. For every x, y and z in $L^\omega(\mathscr{D}_0)$ we have

$$(x \mid y) = (y^* \mid x^*)$$

and

$$(xy \mid z) = (\overline{\pi_0(x)}y \mid z) = (y \mid \pi_0(x)^* z) = (y \mid \overline{\pi_0(x^*)}z) = (y \mid x^* z).$$

Consequently $L^\omega(\mathscr{D}_0)$ is an unbounded Hilbert algebra. Similarly we can show that $L_2^\omega(\mathscr{D}_0)$ is an unbounded Hilbert algebra containing $(\mathscr{D}_0)_b$.

PROPOSITION 3.7. *The space $L^\omega(\varphi_0)$ (resp. $L_2^\omega(\varphi_0)$) is an unbounded Hilbert algebra containing $\pi_0((\mathscr{D}_0)_b)^2$ (resp. $\pi_0((\mathscr{D}_0)_b)$) under the strong sum, strong product, adjoint, strong scalar multiplication and inner product on $L^2(\varphi_0)$.*

Proof. We can easily show that the map $x \in \mathfrak{H} \to \overline{\pi_0(x)} \in L^2(\varphi_0)$ is an isometric isomorphism of $L^\omega(\mathscr{D}_0)$ onto $L^\omega(\varphi_0)$. By Proposition 3.6. $L^\omega(\varphi_0)$ is an unbounded Hilbert algebra.

Problem. Is $L^\omega(\mathscr{D}_0)$ a pure unbounded Hilbert algebra? Does there exist a pure unbounded Hilbert algebra containing $\mathscr{D}_0$?

PROPOSITION 3.8. *The following conditions are equivalent.*

(1) *There exists a pure unbounded Hilbert algebra* $\mathscr{D}$ *containing* $\mathscr{D}_0$.

(2) $L_2^\omega(\mathscr{D}_0)$ *is a pure unbounded Hilbert algebra.*

(3) $L^\omega(\mathscr{D}_0)$ *is a pure unbounded Hilbert algebra.*

(4) *There exists a positive element* x *in* $\mathfrak{H}$ (i.e., $\overline{\pi_0(x)} \geqq 0$) *such that* $x \notin (\mathscr{D}_0)_b$ *and* $x^n \in \mathfrak{H}$, $n = 1, 2, \cdots$.

Proof. (1) $\Rightarrow$ (4); There exists an element $\xi \in \mathscr{D}$ such that $\overline{\pi(\xi)}$ is an unbounded operator on $\mathfrak{H}$. Clearly $\xi^*\xi \notin (\mathscr{D}_0)_b$ and $(\xi^*\xi)^n \in \mathscr{D} \subset \mathfrak{H}$, $n = 1, 2, \cdots$.

(4) $\Rightarrow$ (3); Let $y = x^2$. Then we can easily show that $y \notin (\mathscr{D}_0)_b$ and for each positive integer n $\overline{\pi_0(y)} \in L^n(\varphi_0)$. Let $\overline{\pi_0(y)} = \int_0^\infty \lambda dE(\lambda)$ be the spectral resolution. For each p with $1 \leqq p < \infty$ there is a positive integer n such that $n \leqq p < n+1$. Then we have

$$\begin{aligned} -\int_0^\infty \lambda^p d\varphi_0(E(\lambda)^\perp) &\leqq -\int_0^1 \lambda^n d\varphi_0(E(\lambda)^\perp) - \int_1^\infty \lambda^{n+1} d\varphi_0(E(\lambda)^\perp) \\ &\leqq -\int_0^\infty \lambda^n d\varphi_0(E(\lambda)^\perp) - \int_0^\infty \lambda^{n+1} d\varphi_0(E(\lambda)^\perp) \\ &< \infty. \end{aligned}$$

Therefore $\overline{\pi_0(y)} \in \underline{L^p(\varphi_0)}$, i.e., $y \in L^p(\mathscr{D}_0)$ for every $1 \leqq p < \infty$, and so $y \in L^\omega(\mathscr{D}_0)$ and $\pi_0(y)$ is unbounded. Consequently $L^\omega(\mathscr{D}_0)$ is a pure unbounded Hilbert algebra.

(3) $\Rightarrow$ (2); Since $L^\omega(\mathscr{D}_0) \subset L_2^\omega(\mathscr{D}_0)$, the assertion (3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1); $L_2^\omega(\mathscr{D}_0)$ is a pure unbounded Hilbert algebra containing $\mathscr{D}_0$.

THEOREM 3.9. *Let* $\mathscr{D}$ *be a pure unbounded Hilbert algebra over* $\mathscr{D}_0$. *Then* $\mathscr{D}^2$ (resp. $\mathscr{D}$) *is a* $*$*-subalgebra of the pure unbounded Hilbert algebra* $L^\omega(\mathscr{D}_0)$ (resp. $L_2^\omega(\mathscr{D}_0)$). *In particular,* $L_2^\omega(\mathscr{D}_0)$ *is maximal in pure unbounded Hilbert algebras containing* $\mathscr{D}_0$.

Proof. By Proposition 3.8 $L^\omega(\mathscr{D}_0)$ and $L_2^\omega(\mathscr{D}_0)$ are pure unbounded Hilbert algebras. In the same way as the proof (4) $\Rightarrow$ (3) of Proposition 3.8 we can easily show $L^\omega(\mathscr{D}_0) \supset \mathscr{D}^2$ and $L_2^\omega(\mathscr{D}_0) \supset \mathscr{D}$.

Problem. Let $\mathscr{D}$ be a pure unbounded Hilbert algebra over $\mathscr{D}_0$. Does there exist an $EW^{\#}$-algebra $\mathfrak{A}$ such that $\overline{\mathfrak{A}_b} = \overline{\mathscr{U}_0(\mathscr{D}_0)}$ and $\bar{\mathfrak{A}} \supset \overline{\pi(\mathscr{D})}$?

Let $\mathscr{D}$ be a pure unbounded Hilbert algebra over $\mathscr{D}_0$. By Proposition 3.8 $L_2^\omega(\mathscr{D}_0)$ is a pure unbounded Hilbert algebra such that

$$\mathscr{D}_0 \subset \mathscr{D} \subset L_2^\omega(\mathscr{D}_0) \subset \mathfrak{H}, \quad \text{and} \quad L^\infty(\varphi_0) L_2^\omega(\mathscr{D}_0) \subset L_2^\omega(\mathscr{D}_0).$$

Let π (resp. π_2^ω) be the left regular representation of $\mathscr{D}$ (resp. $L_2^\omega(\mathscr{D}_0)$). By Lemma 2.1 we have $\overline{\pi_2^\omega(\mathscr{D})} = \overline{\pi(\mathscr{D})} = \overline{\pi_0(\mathscr{D})}$.

Then $\pi_2^\omega(\mathscr{D})$ is a $\#$-algebra on $L_2^\omega(\mathscr{D}_0)$ under $\pi_2^\omega(\xi)^{\#} = \pi_2^\omega(\xi^*)$ and $L^\infty(\varphi_0)/L_2^\omega(\mathscr{D}_0) := \{T/L_2^\omega(\mathscr{D}_0);\ T \in L^\infty(\varphi_0)\}$ is a $\#$-algebra on $L_2^\omega(\mathscr{D}_0)$ under $(T/L_2^\omega(\mathscr{D}_0))^{\#} = T^*/L_2^\omega(\mathscr{D}_0)$, where $T/L_2^\omega(\mathscr{D}_0)$ is the restriction of T onto $L_2^\omega(\mathscr{D}_0)$.

NOTATION. We denote by $\mathscr{U}(\mathscr{D})$ a $\#$-algebra on $L_2^\omega(\mathscr{D}_0)$ generated by $\pi_2^\omega(\mathscr{D})$ and $L^\infty(\varphi_0)/L_2^\omega(\mathscr{D}_0)$.

THEOREM 3.10. *Let $\mathscr{D}$ be a pure unbounded Hilbert algebra over $\mathscr{D}_0$. Then $\overline{\mathscr{U}(\mathscr{D})}$ and $\overline{\mathscr{U}(L_2^\omega(\mathscr{D}_0))}$ are $EW^{\#}$-algebras on $\overline{L_2^\omega(\mathscr{D}_0)}$ such that $\overline{\mathscr{U}(\mathscr{D})_b} = \overline{\mathscr{U}(L_2^\omega(\mathscr{D}_0))_b} = \mathscr{U}_0(\mathscr{D}_0)$ and $\overline{\mathscr{U}(L_2^\omega(\mathscr{D}_0))} \supset \overline{\mathscr{U}(\mathscr{D})} \supset \overline{\pi(\mathscr{D})}$.*

DEFINITION 3.11. Let $\mathscr{D}$ be a pure unbounded Hilbert algebra over $\mathscr{D}_0$. $\overline{\mathscr{U}(\mathscr{D})}$ is called the left $EW^{\#}$-algebra of $\mathscr{D}$.

4. Traces on $EW^{\#}$-algebras. Let $\mathfrak{A}$ be an $EW^{\#}$-algebra and let φ be a trace on $\mathfrak{A}^+$. We note

$$\mathfrak{N}_\varphi = \{T \in \mathfrak{A};\ \varphi(T^{\#}T) < \infty\}$$

and let $\mathfrak{M}_\varphi$ be a linear combination of $\{ST^{\#};\ S, T \in \mathfrak{N}_\varphi\}$. Then, clearly, $\mathfrak{N}_\varphi$ (resp. $\mathfrak{M}_\varphi$) is a $\#$-subspace of $\mathfrak{A}$ satisfying $\mathfrak{A}_b \mathfrak{N}_\varphi \subset \mathfrak{N}_\varphi$ and $\mathfrak{N}_\varphi \mathfrak{A}_b \subset \mathfrak{N}_\varphi$ (resp. $\mathfrak{A}_b \mathfrak{M}_\varphi \subset \mathfrak{M}_\varphi$ and $\mathfrak{M}_\varphi \mathfrak{A}_b \subset \mathfrak{M}_\varphi$). We can easily show that the positive part $\mathfrak{M}_\varphi^+$ of $\mathfrak{M}_\varphi$ equals $\{T \in \mathfrak{A}^+;\ \varphi(T) < \infty\}$ and $\mathfrak{M}_\varphi$ is a linear combination of $\mathfrak{M}_\varphi^+$. We define $\dot{\varphi}$ by;

$$\dot{\varphi}(S) = \lambda_1 \varphi(S_1) + \cdots + \lambda_n \varphi(S_n), \quad S = \lambda_1 S_1 + \cdots + \lambda_n S_n,$$
$$\lambda_i \in \mathfrak{C}, \qquad S_i \in \mathfrak{M}_\varphi^+.$$

Then it is not difficult to show that $\dot{\varphi}$ is a well-defined linear form on $\mathfrak{M}_\varphi$ and it satisfies

(1) $\dot{\varphi}(S)=\varphi(S), \qquad S\in\mathfrak{M}_\varphi^+$;
(2) $\dot{\varphi}(S^\# T)=\dot{\varphi}(TS^\#), \qquad S, T\in\mathfrak{N}_\varphi$;
(3) $\dot{\varphi}(ST)=\dot{\varphi}(TS), \qquad S\in\mathfrak{M}_\varphi, \quad T\in\mathfrak{A}_b$.

We note

$$\bar{\varphi}(\bar{T})=\varphi(T), \qquad T\in\mathfrak{A}_b^+.$$

Then $\bar{\varphi}$ is a trace on $\bar{\mathfrak{A}}_b^+$ and we have

$$\overline{(\mathfrak{N}_\varphi)_b}=\mathfrak{N}_{\bar{\varphi}} \quad \text{and} \quad \overline{(\mathfrak{M}_\varphi)_b}=\mathfrak{M}_{\bar{\varphi}}.$$

DEFINITION 4.1. Let $\mathfrak{A}$ be an $EW^\#$-algebra and let φ be a trace on $\mathfrak{A}^+$. If every $\bar{A}\in\bar{\mathfrak{A}}$ is $\bar{\varphi}$-restrictedly measurable, then $\mathfrak{A}$ is called φ-measurable.

Let $\mathscr{D}$ be a pure unbounded Hilbert algebra over $\mathscr{D}_0$ and let $\mathfrak{H}$ be the completion of $\mathscr{D}$. Let $\mathscr{E}$ be a pure unbounded Hilbert algebra over $(\mathscr{D}_0)_b$ containing $\mathscr{D}$. Let $\mathfrak{A}$ be a φ_0-measurable (merely measurable) $EW^\#$-algebra on $\mathscr{E}$ such that $\bar{\mathfrak{A}}_b=\mathscr{U}_o(\mathscr{D}_0)$ and $\bar{\mathfrak{A}}\supset\overline{\pi(\mathscr{D})}$ ($\mathscr{U}(\mathscr{D})$ and $\mathscr{U}(L_2^\omega(\mathscr{D}_0))$ are examples of such $EW^\#$-algebras), where φ_0.is the natural trace on $\mathscr{U}_0(\mathscr{D}_0)^+$.

NOTATION. For each $S\in\mathfrak{A}^+$ we define φ as follows;

$$\varphi(S)=\begin{cases}(x\mid x), & \text{if } \overline{S^{1/2}}=\overline{\pi_0(x)},\ x\in L_2^\omega(\mathscr{D}_0);\\ \infty, & \text{if otherwise.}\end{cases}$$

THEOREM 4.2. (1) *φ is a faithful normal semifinite trace on $\mathfrak{A}^+$.*
(2) *We have*

$$\bar{\mathfrak{N}}_\varphi=\bar{\mathfrak{A}}\cap L_2^\omega(\varphi_0) \text{ and } \bar{\mathfrak{M}}_\varphi=\bar{\mathfrak{A}}\cap L^\omega(\varphi_0).$$

(3) *Putting*

$$\mathfrak{N}(\mathscr{D}_0)=\{x\in\mathfrak{H};\ \overline{\pi_0(x)}\in\overline{\mathfrak{N}_\varphi}\} \text{ and } \mathfrak{M}(\mathscr{D}_0)=\{x\in\mathfrak{H};\ \overline{\pi_0(x)}\in\overline{\mathfrak{M}_\varphi}\},$$

$\mathfrak{N}(\mathscr{D}_0)$ (resp. $\mathfrak{M}(\mathscr{D}_0)$) *is a pure unbounded Hilbert algebra over* $(\mathscr{D}_0)_b$ (resp. $(\mathscr{D}_0)_b^2$) *containing* $\mathscr{D}$ (resp. $\mathscr{D}^2$).
(4) *$\bar{\varphi}$ equals the natural trace φ_0 on $\mathscr{U}_0(\mathscr{D}_0)^+$.*
(5) *Let μ be the integral on $L^1(\varphi_0)$. Then*

$$\dot{\varphi}(T)=\mu(\bar{T}), \qquad T\in\mathfrak{M}_\varphi.$$

In particular, for every $x, y \in \mathfrak{N}(\mathscr{D}_0)$

$$\dot{\varphi}(\overline{\pi_0(y)}^* \cdot \overline{\pi_0(x)}) = (x \mid y).$$

(6) $\mathfrak{A}(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi$ *and* $\mathfrak{A}(\mathfrak{M}_\varphi)_b \subset \mathfrak{M}_\varphi$.

(7) *Every element* T *of* $\mathfrak{A}$ *is represented by*

$$T = T_0 + T_1, \qquad T_0 \in \mathfrak{A}_b, \qquad T_1 \in \mathfrak{M}_\varphi.$$

(8) *If* $T \in \mathfrak{A}$, *then we have* $\bar{T} = \overline{(T/\mathscr{D}_0)}$.

Proof. (2); Let $T \in \mathfrak{N}_\varphi$ and let $T = U|T|$ be the polar decomposition of T. Since $\varphi(T^\# T) = \varphi(|T|^2) < \infty$, $\overline{|T|} = \overline{\pi_0(x)}$, $x \in L_2^\omega(\mathscr{D}_0)$, and so $\overline{|T|} \in L_2^\omega(\varphi_0)$ and hence $\bar{T} = \bar{U} \cdot \overline{|T|} \in L_2^\omega(\varphi_0) \cap \bar{\mathfrak{A}}$. The converse is obvious. Moreover we get

$$\overline{\mathfrak{M}_\varphi} = \overline{\mathfrak{N}_\varphi}^2 = (\bar{\mathfrak{A}} \cap L_2^\omega(\varphi_0))^2 = \bar{\mathfrak{A}} \cap L^\omega(\varphi_0).$$

(3); By (2) we can easily show (3).

(4); Let $T \in \mathfrak{A}_b^+$. Since $\overline{\mathfrak{A}_b} \cap L_2^\omega(\varphi_0) = \overline{\pi_0((\mathscr{D}_0)_b)}$,

$$\bar{\varphi}(\bar{T}) = \varphi(T) = \begin{cases} (x \mid x), & \text{if } \overline{T^{1/2}} = \overline{\pi_0(x)},\ x \in L_2^\omega(\mathscr{D}_0); \\ \infty, & \text{if otherwise} \end{cases}$$

$$= \begin{cases} (x \mid x), & \text{if } \overline{T^{1/2}} = \overline{\pi_0(x)},\ x \in (\mathscr{D}_0)_b; \\ \infty, & \text{if otherwise} \end{cases}$$

$$= \varphi_0(\bar{T}).$$

(5); Let $T \in \mathfrak{M}_\varphi^+$. By (2) there exists an element x of $L_2^\omega(\mathscr{D}_0)$ such that $\overline{T^{1/2}} = \overline{\pi_0(x)}$. Then we have $\varphi(T) = (x \mid x) = \mu(\bar{T})$, and so $\dot{\varphi}(T) = \mu(\bar{T})$, $T \in \mathfrak{M}_\varphi$.

(6); Let π be the left regular representation of $\mathscr{E}$. We can easily show that

$$T\pi(\xi) = \pi(T\xi), \quad T \in \mathfrak{A}, \quad \xi \in (\mathscr{D}_0)_b \subset \mathscr{E}.$$

Therefore $\pi(T\xi) = T\pi(\xi) \in \mathfrak{A}$ and $\overline{\pi(T\xi)} = \overline{\pi_0(T\xi)}$, $T\xi \in \mathscr{E} \subset L_2^\omega(\mathscr{D}_0)$, and so $T\pi(\xi) \in \mathfrak{N}_\varphi$.

(7); Let $T \in \mathfrak{A}$ and let $T = U|T|$ be the polar decomposition of T. Let $\overline{|T|} = \int_0^\infty \lambda d\overline{E_T(\lambda)}$ be the spectral resolution of $\overline{|T|}$. Since $\overline{|T|}$ is a φ_0-restrictedly measurable operator, $\overline{E_T(\lambda_0)^{\perp}} \in \overline{(\mathfrak{M}_{\varphi})_b}^+$ for some $\lambda_0 > 0$. By (6) $\mathfrak{A}(\mathfrak{M}_{\varphi})_b \subset \mathfrak{M}_{\varphi}$, and so putting

$$T_1 = TE_T(\lambda_0)^{\perp} = U\,|\,T\,|\,E_T(\lambda_0)^{\perp} \text{ and } T_0 = TE_T(\lambda_0),$$

$T_0 \in \mathfrak{A}_b$, $T_1 \in \mathfrak{M}_{\varphi}$ and $T = T_0 + T_1$.

(8); Let $T \in \mathfrak{A}$. By (7) we have

$$\begin{aligned} \bar{T} &= \overline{T_0} + \overline{T_1}, \quad T_0 \in \mathfrak{A}_b, \quad T_1 \in \mathfrak{M}_{\varphi} \\ &= \overline{T_0} + \overline{\pi_0(x)}, \quad x \in L^{\omega}(\mathscr{D}_0) \\ &= \overline{(T_0/\mathscr{D}_0)} + \overline{\pi_0(x)} = \overline{T_0/\mathscr{D}_0 + \pi_0(x)} = \overline{T/\mathscr{D}_0}. \end{aligned}$$

(1); We shall show that φ is a trace on $\mathfrak{A}^+$, i.e.,

(a) $\varphi(S+T) = \varphi(S) + \varphi(T)$, $S, T \in \mathfrak{A}^+$;

(b) $\varphi(\lambda S) = \lambda\varphi(S)$, $\lambda \geqq 0$, $S \in \mathfrak{A}^+$;

(c) $\varphi(S^{\#}S) = \varphi(SS^{\#})$, $S \in \mathfrak{A}$.

(a); Let $S, T \in \mathfrak{A}^+$. Suppose $\varphi(S+T) < \infty$. Since $\bar{S}$ (or $\bar{T}$) $\leqq \bar{S} + \bar{T}$ and $\bar{S} + \bar{T} \in \overline{\mathfrak{M}_{\varphi}}^+$, $\bar{S}$ and $\bar{T}$ in $\overline{\mathfrak{M}_{\varphi}}^+$, and so $\varphi(S) = \mu(\bar{S}) < \infty$ and $\varphi(T) = \mu(\bar{T}) < \infty$ by (5). Suppose $\varphi(S) < \infty$ and $\varphi(T) < \infty$. Since $\bar{S}$ and $\bar{T}$ in $L^1(\varphi_0)^+$, by Theorem 3.5. we have $\bar{S} + \bar{T} \in L^1(\varphi_0)^+$ and

$$\varphi(S) + \varphi(T) = \mu(\bar{S}) + \mu(\bar{T}) = \mu(\bar{S} + \bar{T}) = \mu(\overline{S+T}) = \varphi(S+T).$$

(b); clear.

(c); Let $S \in \mathfrak{A}$. Suppose $\varphi(S^{\#}S) < \infty$. Let $S = U|S|$ be the polar decomposition of S. Then $\overline{|S|} = \pi_0(x)$, $x \in L_2^{\omega}(\mathscr{D}_0)$ and $\overline{|S^{\#}|} = |S^*| = \pi_0(x^*)$, and so we get

$$\varphi(S^{\#}S) = (x \mid x) = (x^* \mid x^*) = \varphi(SS^{\#}).$$

Consequently φ is a trace on $\mathfrak{A}^+$. Since $\bar{\varphi} = \varphi_0$ by (4), $\bar{\varphi}$ is a faithful normal semifinite trace on $\overline{\mathfrak{A}_b}^+$. We can easily show that φ is faithful. We shall show that φ is normal. Let $T_{\alpha} \uparrow T$, $T_{\alpha}, T \in \mathfrak{A}^+$. Suppose $\varphi(T) < \infty$. Then there exist $\{x_{\alpha}\} \subset L_2^{\omega}(\mathscr{D}_0)$ and $x \in L_2^{\omega}(\mathscr{D}_0)$ such that $\overline{T_{\alpha}^{1/2}} = \pi_0(x_{\alpha})$ and $\overline{T^{1/2}} = \pi_0(x)$. We can easily show that $\varphi(T_{\alpha}) = \|x_{\alpha}\|^2 \uparrow \varphi(T) = \|x\|^2$. Suppose $\varphi(T) = \infty$ and $\sup_{\alpha}\varphi(T_{\alpha}) < \infty$. There exists a net $\{x_{\alpha}\}$ of $L_2^{\omega}(\mathscr{D}_0)$ such that $\overline{T_{\alpha}^{1/2}} = \pi_0(x_{\alpha})$. Let $\bar{T} = \int_0^{\infty} \lambda d\overline{E_T(\lambda)}$ be the spectral resolution of $\bar{T}$. Since $\bar{T}$ is φ_0-restrictedly measurable, $\overline{E_T(\lambda_0)^{\perp}} \in \overline{(\mathfrak{M}_{\varphi})_b}^+$ for some $\lambda_0 > 0$, and so by (5) we get

$$TE_T(\lambda_0)^\perp \in \mathfrak{M}_\varphi^+ \quad \text{and} \quad \bar{T} = \int_0^{\lambda_0} \lambda d\overline{E_T(\lambda)} + \overline{TE_T(\lambda_0)^\perp}.$$

From $\varphi(T) = \infty$, we have $\bar{\varphi}\left(\int_0^{\lambda_0} \lambda d\overline{E_T(\lambda)}\right) = \infty$. Since $T_\alpha \uparrow T$, we get $E_T(\lambda_0)T_\alpha E_T(\lambda_0) \in \mathfrak{A}_b$ and

$$E_T(\lambda_0)T_\alpha E_T(\lambda_0) \uparrow E_T(\lambda_0)TE_T(\lambda_0) = \int_0^{\lambda_0} \lambda dE_T(\lambda).$$

Then we can show that

$$\overline{E_T(\lambda_0)T_\alpha E_T(\lambda_0)} \uparrow \int_0^{\lambda_0} \lambda d\overline{E_T(\lambda)},$$

and so by the normality of $\bar{\varphi}$

$$\bar{\varphi}\,(\overline{E_T(\lambda_0)T_\alpha E_T(\lambda_0)}) \uparrow \bar{\varphi}\left(\int_0^{\lambda_0} \lambda d\overline{E_T(\lambda)}\right) = \infty.$$

On the other hand we have

$$\begin{aligned}
\bar{\varphi}\left(\int_0^{\lambda_0} \lambda d\overline{E_T(\lambda)}\right) &= \sup_\alpha \bar{\varphi}\,(\overline{E_T(\lambda_0)T_\alpha E_T(\lambda_0)}) \\
&= \sup_\alpha \bar{\varphi}\,(\overline{E_T(\lambda_0)} \cdot \overline{\pi_0(x_\alpha)}^2 \cdot \overline{E_T(\lambda_0)}) \\
&= \sup_\alpha \bar{\varphi}\,(\overline{\pi_0(\overline{E_T(\lambda_0)}x_\alpha) \cdot \pi_0(\overline{E_T(\lambda_0)}x_\alpha^*)^*}) \\
&= \sup_\alpha (\overline{E_T(\lambda_0)}x_\alpha \mid \overline{E_T(\lambda_0)}x_\alpha^*) \\
&\leqq \sup_\alpha \|x_\alpha\|^2 = \sup_\alpha \varphi(T_\alpha) < \infty.
\end{aligned}$$

This contradicts $\bar{\varphi}\left(\int_0^{\lambda_0} \lambda d\overline{E_T(\lambda)}\right) = \infty$. Consequently φ is normal. Finally we shall show that φ is semifinite. Since $\bar{\varphi}$ is semifinite, there exists a net $\{T_\alpha\}$ of $(\mathfrak{M}_\varphi)_b^+$ such that $\bar{T}_\alpha \uparrow \bar{I}$. Let $T \in \mathfrak{A}^+$. By (6) we have

$$T^{\frac{1}{2}}T_\alpha T^{\frac{1}{2}} \in \mathfrak{M}_\varphi^+ \quad \textit{and} \quad T^{\frac{1}{2}}T_\alpha T^{\frac{1}{2}} \uparrow T,$$

and so φ is semifinite.

DEFINITION 4.3. The trace φ of Theorem 4.2. is called the natural trace on $\mathfrak{A}^+$.

COROLLARY 4.4. *For every $A \in \mathfrak{A}$ and $x \in L_2^\omega(\mathscr{D}_0)$ we have*

$$\bar{\mathfrak{A}} L_2^\omega(\mathscr{D}_0) \subset L_2^\omega(\mathscr{D}_0) \quad \textit{and} \quad \bar{A} \cdot \overline{\pi_0(x)} = \overline{\pi_0(\bar{A}x)}.$$

In particular, we have

$$\mathfrak{A}\mathfrak{N}_\varphi \subset \mathfrak{N}_\varphi \quad \textit{and} \quad \mathfrak{A}\mathfrak{M}_\varphi \subset \mathfrak{M}_\varphi.$$

Proof. By Theorem 4.2.(7) we get $A = A_0 + A_1$, $A_0 \in \mathfrak{A}_b$, $A_1 \in \mathfrak{M}_\varphi$, and so $\bar{A} = \overline{A_0} + \overline{\pi_0(y)}$, $y \in L^\omega(\mathscr{D}_0)$. Hence $\mathfrak{D}(\bar{A}) = \mathfrak{D}(\overline{\pi_0(y)}) \supset L_2^\omega(\mathscr{D}_0)$ and we have

$$\begin{aligned} \bar{A} L_2^\omega(\mathscr{D}_0) &= \overline{A_0} L_2^\omega(\mathscr{D}_0) + \overline{A_1} L_2^\omega(\mathscr{D}_0) \\ &\subset L_2^\omega(\mathscr{D}_0), \end{aligned}$$

and

$$\begin{aligned} \bar{A} \cdot \overline{\pi_0(x)} &= (\overline{A_0} + \overline{\pi_0(y)}) \cdot \overline{\pi_0(x)} \\ &= \overline{\overline{A_0}\, \overline{\pi_0(x)} + \overline{\pi_0(y)} \cdot \overline{\pi_0(x)}} \\ &= \overline{\pi_0(\overline{A_0}x)} + \overline{\pi_0(\overline{\pi_0(y)}x)} \\ &= \overline{\pi_0(\overline{A_0}x + \overline{A_1}x)} \\ &= \overline{\pi_0(\bar{A}x)}. \end{aligned}$$

Moreover, since $\overline{\mathfrak{N}_\varphi} = \bar{\mathfrak{A}} \cap L_2^\omega(\varphi_0)$ and $\overline{\mathfrak{M}_\varphi} = \bar{\mathfrak{A}} \cap L^\omega(\varphi_0)$, we have $\mathfrak{A}\mathfrak{N}_\varphi \subset \mathfrak{N}_\varphi$ and $\mathfrak{A}\mathfrak{M}_\varphi \subset \mathfrak{M}_\varphi$.

For every $A \in \mathfrak{A}$ putting

$$\tilde{A}x = \bar{A}x, \qquad x \in L_2^\omega(\mathscr{D}_0),$$

$\tilde{A}$ is a linear operator on $L_2^\omega(\mathscr{D}_0)$ by Corollary 4.4.. Let $\tilde{\mathfrak{A}} = \{\tilde{A}\,;\, A \in \mathfrak{A}\}$. Then we have

$$\tilde{A}\tilde{B} = \widetilde{AB}, \quad \lambda\tilde{A} = \widetilde{\lambda A} \quad \text{and} \quad \tilde{A}^\# = A^*/L_2^\omega(\mathscr{D}_0) = \widetilde{A^\#}$$

for every $A, B \in \mathfrak{A}$ and $\lambda \in \mathfrak{C}$. We can easily show that $\tilde{\mathfrak{A}}$ equals the left $EW^\#$-algebra $\mathscr{U}(\mathfrak{N}(\mathscr{D}_0))$ of a pure unbounded Hilbert algebra $\mathfrak{N}(\mathscr{D}_0)$. So, we obtain the following theorem.

THEOREM 4.5. *Let $\mathscr{D}$ be a pure unbounded Hilbert algebra over $\mathscr{D}_0$*

and let $\mathscr{E}$ be a pure unbounded Hilbert algebra over $\underline{(\mathscr{D}_0)_b}$ containing $\mathscr{D}$. Let $\mathfrak{A}$ be a measurable $EW^{\#}$-algebra on $\mathscr{E}$ such that $\overline{\mathfrak{A}_b} = \mathscr{U}_0(\mathscr{D}_0)$ and $\bar{\mathfrak{A}} \supset \pi(\mathscr{D})$. Then $\mathfrak{A}$ is regarded as the left $EW^{\#}$-algebra $\mathscr{U}(\mathfrak{N}(\mathscr{D}_0))$ of a pure unbounded Hilbert algebra $\mathfrak{N}(\mathscr{D}_0)$ over $(\mathscr{D}_0)_b$ containing $\mathscr{D}$.

Finally we shall show that an $EW^{\#}$-algebra with a faithful normal semifinite trace is isomorphic to a left $EW^{\#}$-algebra of a pure unbounded Hilbert algebra (Theorem 4.11). Let $\mathfrak{A}$ be an $EW^{\#}$-algebra on $\mathfrak{D}$ and let φ be a faithful trace on $\mathfrak{A}^+$. For each $S, T \in \mathfrak{N}_\varphi$ putting

$$(\lambda(S) \mid \lambda(T)) = \dot{\varphi}(T^{\#}S),$$

$(\ \mid\ )$ is an inner product on $\lambda(\mathfrak{N}_\varphi)$ and by, for each $S, T \in \mathfrak{N}_\varphi$ and $\alpha \in \mathfrak{C}$,

$$\lambda(S) + \lambda(T) = \lambda(S+T), \quad \alpha\lambda(S) = \lambda(\alpha S),$$

$\lambda(\mathfrak{N}_\varphi)$ is a pre-Hilbert space. Let $\mathfrak{H}_\varphi$ be the completion of $\lambda(\mathfrak{N}_\varphi)$. Let $\mathfrak{A}$ be a φ-measurable $EW^{\#}$-algebra on $\mathfrak{D}$ and let φ be a faithful normal semifinite trace on $\mathfrak{A}^+$ satisfying $\mathfrak{A}(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi$.

LEMMA 4.6. *The property "$\mathfrak{A}(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi$" leads the property "$\mathfrak{A}\mathfrak{N}_\varphi \subset \mathfrak{N}_\varphi$".*

Proof. Let $A \in \mathfrak{A}$ and $S \in \mathfrak{N}_\varphi$. Let $S = U|S|$ be the polar decomposition of S and let $\overline{|S|} = \int_0^\infty \lambda d\overline{E_S(\lambda)}$ be the spectral resolution of $\overline{|S|}$. Since $\overline{|S|}$ is a $\bar{\varphi}$-restrictedly measurable operator, $\overline{E_S(\lambda_0)^\perp} \in \overline{(\mathfrak{M}_\varphi)_b}^+$ for some $\lambda_0 > 0$, and so we have

$$\begin{aligned} AS = AU|S| &= AU\Big(\int_0^{\lambda_0} \lambda dE_S(\lambda) + |S|E_S(\lambda_0)^\perp\Big) \\ &= AU\int_0^{\lambda_0} \lambda dE_S(\lambda) + ASE_S(\lambda_0)^\perp \\ &\in \mathfrak{A}(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi. \end{aligned}$$

LEMMA 4.7. *Let $A \in \mathfrak{A}$. Then there exist $A_0 \in \mathfrak{A}_b$ and $A_1 \in \mathfrak{M}_\varphi$ such that*

$$A = A_0 + A_1.$$

Proof. Let $A = U|A|$ be the polar decomposition of A and let $\overline{|A|} = \int_0^\infty \lambda d\overline{E_A(\lambda)}$ be the spectral resolution. Since $\overline{|A|}$ is $\bar{\varphi}$-restrictedly measurable, $\overline{E_A(\lambda_0)^\perp} \in \overline{(\mathfrak{M}_\varphi)_b}^+$ for some $\lambda_0 > 0$. Putting

$$A_0 = U\left(\int_0^{\lambda_0} \lambda dE_A(\lambda)\right) \quad \text{and} \quad A_1 = AE_A(\lambda_0)^{\perp},$$

$A_0 \in \mathfrak{A}_b$, $A_1 \in \mathfrak{A}(\mathfrak{M}_\varphi)_b \subset \mathfrak{M}_\varphi$ and $A = A_0 + A_1$.

LEMMA 4.8. *The pre-Hilbert space $\lambda(\mathfrak{N}_\varphi)$ is a pure unbounded Hilbert algebra over $\lambda((\mathfrak{N}_\varphi)_b)$.*

Proof. We shall show that $\lambda((\mathfrak{N}_\varphi)_b)$ is dense in $\lambda(\mathfrak{N}_\varphi)$. For each $T \in \mathfrak{N}_\varphi$ let $T = U|T|$ be the polar decomposition of T. Then $|T| = U^* T \in \mathfrak{N}_\varphi^+$. Let $\overline{|T|} = \int_0^\infty \lambda d\overline{E_T(\lambda)}$ be the spectral resolution of $\overline{|T|}$. Putting

$$\overline{S_n} = \int_0^n \lambda d\overline{E_T(\lambda)},$$

$S_n \in (\mathfrak{N}_\varphi)_b^+$ and $\{S_n\}$ converges σ-strongly to $|T|$, and so $S_n^2 \uparrow |T|^2$ and since φ is normal, we get

$$\|\lambda(S_n)\|^2 = \varphi(S_n^2) \uparrow \varphi(|T|^2) = \|\lambda(|T|)\|^2$$

and

$$\begin{aligned}(\lambda(|T|) \mid \lambda(S_n)) &= \dot{\varphi}(|T| S_n) \\ &= \varphi(|T|^{\frac{1}{2}} S_n |T|^{\frac{1}{2}}) \uparrow \varphi(|T|^2) = \|\lambda(|T|)\|^2,\end{aligned}$$

and hence

$$\lim_{n\to\infty} \|\lambda(US_n) - \lambda(T)\| \leqq \lim_{n\to\infty} \|\lambda(S_n) - \lambda(|T|)\| = 0.$$

Therefore $\lambda((\mathfrak{N}_\varphi)_b)$ is dense in $\lambda(\mathfrak{N}_\varphi)$. Since $\bar{\varphi}$ is a faithful normal semifinite trace on $\overline{\mathfrak{A}_b}^+$, $\lambda(\overline{(\mathfrak{N}_\varphi)_b}) = \lambda(\mathfrak{N}_{\bar{\varphi}})$ is a maximal Hilbert algebra, and so we can easily show that $\lambda((\mathfrak{N}_\varphi)_b)$ is a maximal Hilbert algebra. For every $S, T \in \mathfrak{N}_\varphi$ we define the operations on $\lambda(\mathfrak{N}_\varphi)$ as follows;

$$\lambda(S)\lambda(T) = \lambda(ST), \qquad \alpha\lambda(S) = \lambda(\alpha S),$$
$$\lambda(S)^* = \lambda(S^*), \qquad (\lambda(S) \mid \lambda(T)) = \dot{\varphi}(T^* S).$$

Then it is not difficult to show that $\lambda(\mathfrak{N}_\varphi)$ is an unbounded Hilbert algebra over $\lambda((\mathfrak{N}_\varphi)_b)$. Finally we shall show that $\lambda(\mathfrak{N}_\varphi)$ is pure. By

Lemma 4.7. every element A of $\mathfrak{A}$ is represented by $A = A_0 + A_1$, $A_0 \in \mathfrak{A}_b$, $A_1 \in \mathfrak{M}_\varphi$. If $A \in \mathfrak{A} - \mathfrak{A}_b$, then $A_1 \in \mathfrak{M}_\varphi - (\mathfrak{M}_\varphi)_b$, and so $\lambda((\mathfrak{N}_\varphi)_b) \neq \lambda(\mathfrak{N}_\varphi)$ and $\lambda((\mathfrak{N}_\varphi)_b)$ is a maximal Hilbert algebra. Therefore $\lambda(\mathfrak{N}_\varphi)$ is pure.

LEMMA 4.9. *For every $A \in \mathfrak{A}$ putting*

$$\Psi(A)\lambda(T) = \lambda(AT), \quad T \in \mathfrak{N}_\varphi,$$

$\Psi(A)$ is a linear operator on $\lambda(\mathfrak{N}_\varphi)$. $\Psi(\mathfrak{A})$ is a measurable $EW^\#$-algebra on $\lambda(\mathfrak{N}_\varphi)$ such that $\overline{\Psi(\mathfrak{A})_b} = \overline{\Psi(\mathfrak{A}_b)} = \mathscr{U}_0(\lambda((\mathfrak{N}_\varphi)_b))$ and $\overline{\Psi(\mathfrak{A})} \supset \overline{\pi(\lambda(\mathfrak{N}_\varphi))}$ and Ψ is an isomorphism of $\mathfrak{A}$ onto $\Psi(\mathfrak{A})$.

Proof. By Lemma 4.6. $\mathfrak{A}\mathfrak{N}_\varphi \subset \mathfrak{N}_\varphi$, and so $\Psi(A)$ is a linear operator on $\lambda(\mathfrak{N}_\varphi)$. For every $S \in \mathfrak{N}_\varphi$ we have $\Psi(S) = \pi(\lambda(S))$, where π is the left regular representation of the pure unbounded Hilbert algebra $\lambda(\mathfrak{N}_\varphi)$. We shall show $\overline{\Psi(\mathfrak{A})_b} = \overline{\Psi(\mathfrak{A}_b)}$. Clearly we have $\Psi(\mathfrak{A}_b) \subset \Psi(\mathfrak{A})_b$. Conversely let $\Psi(A) \in \Psi(\mathfrak{A})_b$. By Lemma 4.7. $A = A_0 + A_1$, $A_0 \in \mathfrak{A}_b$, $A_1 \in \mathfrak{M}_\varphi$, and so $\Psi(A_1) = \pi(\lambda(A_1)) \in \Psi(\mathfrak{M}_\varphi)_b$. Since $\lambda((\mathfrak{N}_\varphi)_b)$ is a maximal Hilbert algebra, $\lambda(A_1) \in \lambda((\mathfrak{N}_\varphi)_b)$, i.e., $A_1 \in (\mathfrak{N}_\varphi)_b$. Therefore $A = A_0 + A_1 \in \mathfrak{A}_b$, and so $\Psi(A) \in \Psi(\mathfrak{A}_b)$. By the theory of von Neumann algebras, $\overline{\Psi(\mathfrak{A}_b)} = \mathscr{U}_0(\lambda((\mathfrak{N}_\varphi)_b))$. Moreover it is easy to show that $\Psi(\mathfrak{A}) \supset \Psi(\mathfrak{N}_\varphi) = \pi(\lambda(\mathfrak{N}_\varphi))$ and Ψ is an isomorphism of $\mathfrak{A}$ onto $\Psi(\mathfrak{A})$. Since $\mathfrak{A}$ is φ-measurable, we can easily show that $\Psi(\mathfrak{A})$ is measurable.

LEMMA 4.10. *Let ψ be the natural trace on $\Psi(\mathfrak{A})^+$. Then we have*

$$\varphi(A) = \psi(\Psi(A)), \quad A \in \mathfrak{A}^+.$$

Proof. By the definition of the natural trace ψ we get

$$\mathfrak{M}_\psi^+ = \pi(\lambda(\mathfrak{M}_\varphi^+)) = \Psi(\mathfrak{M}_\varphi^+)$$

and moreover for every $A \in \mathfrak{M}_\varphi^+$

$$\varphi(A) = \|\lambda(A^{\frac{1}{2}})\|^2 = \psi(\pi(\lambda(A))) = \psi(\Psi(A)).$$

By Lemma 4.6. ~ 4.10. and Theorem 4.5. we obtain the following theorem.

THEOREM 4.11. *Let $\mathfrak{A}$ be an $EW^\#$-algebra and let φ be a faithful normal semifinite trace on $\mathfrak{A}^+$. Suppose that $\mathfrak{A}$ is a φ-measurable*

$EW^{\#}$-algebra and $\mathfrak{A}(\mathfrak{N}_\varphi)_b \subset \mathfrak{N}_\varphi$. Then $\lambda(\mathfrak{N}_\varphi)$ is a pure unbounded Hilbert algebra over $\lambda((\mathfrak{N}_\varphi)_b)$ and putting

$$\Psi(A)\lambda(S) = \lambda(AS), \quad S \in \mathfrak{N}_\varphi$$

for every $A \in \mathfrak{A}$, $\Psi(A)$ is a linear operator on $\lambda(\mathfrak{N}_\varphi)$. The isomorphism Ψ is extended to an isomorphism Φ of $\mathfrak{A}$ onto the left $EW^{\#}$-algebra $\mathcal{U}(\lambda(\mathfrak{N}_\varphi))$ of $\lambda(\mathfrak{N}_\varphi)$. Let ψ be the natural trace on $\Phi(\mathfrak{A})^+$. Then $\varphi = \psi \circ \Phi$.

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