LEVEL SETS OF POLYNOMIALS
IN n REAL VARIABLES

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The methods used in studying the zeros of a polynomial in a single complex variable are here adapted to investigating the level surfaces of a real polynomial in $E^n$, with respect to their intersection and finite or asymptotic tangency with certain cones. Special attention is given to the equipotential surfaces generated by an axisymmetric harmonic polynomial in $E^3$.

A principal interest is the application of reasoning used by Cauchy [2, p. 123] in obtaining bounds on the zeros of polynomials in one complex variable. We thereby seek the level sets

$$L_\alpha(H) = \{X \in E^n \mid H(X) = \alpha\}$$

generated from the real polynomials

$$(1) \quad H(X) - \alpha = \sum_{0 \leq j_1 + \cdots + j_n \leq n} \alpha_{j_1 \cdots j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n},$$

$$X = (x_1, x_2, \cdots, x_n), \quad r = |X| = [x_1^2 + x_2^2 + \cdots + x_n^2]^{1/2}.$$

It is convenient to introduce direction numbers $\lambda_j = x_j r^{-1}, \ 1 \leq j \leq n$, connected by $\lambda_1^2 + \cdots + \lambda_n^2 = 1$ and cones $\Lambda_j : \lambda_j = \text{constant}$, about the $j$th axis. On the intersection of the cones $\Lambda_n$, these polynomials become

$$H(r\Lambda_j) - \alpha = \sum_{k=0}^{n} r^k A_k(\Lambda_j)$$

where

$$A_k = A_k(\Lambda_j) = \sum_{j_1 + \cdots + j_n = k} \alpha_{j_1 \cdots j_n} \lambda_1^{j_1} \cdots \lambda_n^{j_n}, \quad 0 \leq k \leq n.$$

At the origin the level set $L_\alpha(H)$ has $\nu$th order contact with $\Lambda_n$ if $A_k(\Lambda_j) = 0$ for $0 \leq k \leq \nu - 1$ but $A_\nu(\Lambda_j) \neq 0$ and $A_n(\Lambda_j) \neq 0$. For such sets we introduce the ratios

$$M_\nu = M_\nu(\Lambda_j) = \max_{\nu \leq k \leq n-1} |A_k/A_n|$$
\[ m_v = \min_{v \leq k \leq n} ( |A_v| / (|A_v| + |A_k|) ) \]

\[ \mu_v = \max_{v \leq k \leq n} |A_k/A_v| = \mu_v(\Lambda_v) \]

Then, by considering points common to the level set \( L_\alpha(H) \) and the cone \( \Lambda_v \) exterior to the unit ball \( r > 1 \) (about the origin), we deduce an inequality

\[ |H(r\Lambda_v) - \alpha| > |A_n| r^n - \sum_{k=n}^{n-v} |A_k| r^k \geq |A_n| r^n \left[ 1 - M_v \sum_{k=1}^{n-v} r^{-k} \right] \]

\[ |H(r\Lambda_v) - \alpha| > |A_n| r^n \left[ 1 - M_v (1 - r^{-n}) (r - 1)^{-1} \right] \]

from which it is clear that, if \( r \geq 1 + M_v \), \( L_\alpha(H) \) does not intersect \( \Lambda_v \). Likewise, if we consider the reciprocal polynomial associated with (1), derived by setting \( 1/r = \zeta > 1 \), the inequalities

\[ |\zeta^n[H(\zeta^{-1}\Lambda_v) - \alpha]| = \left| \sum_{k=1}^{n} \zeta^{n-k} A_k \right| \]

\[ \geq |A_n| - \sum_{k=n+1}^{n} \zeta^{n-k} |A_k| \]

\[ \geq |A_n| \left[ 1 - \mu_v \sum_{k=n+1}^{n} \zeta^{n-k} \right] \]

\[ = |A_n| \left[ 1 - (\mu_v (1 - \zeta^{-n}) (\zeta - 1)^{-1}) \right] \]

\[ > |A_n| (\zeta - 1 - \mu_v) / (\zeta - 1) \]

imply that \( H(\zeta^{-1}\Lambda_v) \neq \alpha \) for \( \zeta \geq 1 + \mu_v \). Thus we infer that \( H(r\Lambda_v) \neq \alpha \) for

\[ r \leq (1 + \mu_v)^{-1} = m_v \]

which brings us to

**Theorem 1.** If the level set \( L_\alpha(H) \) has \( v \)th order contact with the cone \( \Lambda_v \) at the origin and if it intersects the cone at any additional finite points, then it does so at a distance \( r \) from the origin where

\[ m_v(\Lambda_v) < r < 1 + M_v(\Lambda_v). \]

By use of inequalities (3a) and (2a), we replace inequality (4) in Theorem 1 by
where \( r_1 \) is the larger positive root of the equation
\[
1 - (1 + \mu_v) r + \mu_v r^{n-v+1} = 0
\]
and \( r_2 \) the larger positive root of the equation
\[
r^{n+1-v} - (1 + M_v) r^{n-v} + M_v = 0,
\]
\( r = 1 \) being a root of both equations.

A natural question arising from this theorem is that of determining the point of tangency of the level sets with the cones \( \Lambda_r \). Let us consider the \( k \)th term in the polynomial (1),
\[
r^k A_k (r^X) = \sum_{j_1 + \cdots + j_n = k} \alpha_{j_1 \cdots j_n} x_1^{j_1} \cdots x_n^{j_n}.
\]
As this sum is composed of homogeneous polynomials of degree \( k \), we may apply Euler’s Identity [1, p. 141] to find that
\[
X \cdot \nabla [r^k A_k (r^X)] = kr^k A_k (r^X),
\]
where the left side is the scalar product of vector \( X \) and the gradient of the bracket. On account of this relation, the orthogonality condition
\[
X \cdot \nabla H(X) = 0
\]
becomes
\[
\sum_{k=\nu} \sum_{j_1 + \cdots + j_n = k} kr^k A_k (\Lambda_{j}) = 0.
\]
Let us define
\[
m^*(\Lambda_{j}) = \min_{2 \leq k \leq n} \left[ (kA_k + A_v)/(A_v) \right]
\]
\[
M^*(\Lambda_{j}) = \max_{1 \leq k \leq n-1} (kA_k/nA_n).
\]
Theorem 1 and equation 6 lead to

**Corollary 1.1.** *If the level set \( L_\alpha (H) \) has \( \nu \)th order contact with the*
cone \( \Lambda \), at its vertex and is tangent to the cone at a positive distance \( r \) from the origin, then

\[
m^*_i(\Lambda_j) < r < 1 + M^*_i(\Lambda_j).
\]

As equation (6) may be viewed as

\[
\frac{\partial}{\partial \alpha} \{H(X) - \alpha\} = 0,
\]

we may use Rolle’s Theorem to conclude

**Corollary 1.2.** If the ray \((\lambda_1, \cdots, \lambda_n) \in \bigcap_{\alpha=1}^n \Lambda_{\alpha}\), the level surface \(L_\alpha(H)\) has a finite tangential contact point between successive pairs of intersections of \(L_\alpha(H)\) with the ray.

The influence of the coefficient \(A_n = A_n(X)\) on the structure of \(L_\alpha(H)\) near infinity is found by selecting a sequence of points \(\{X_k\}\), \(r_k = |X_k| \to \infty\), such that \(H(X_k) = \alpha\). Each of these points is located on a cone \(\Lambda^*_k\). This leads to the bound \(r_k < 1 + M_\alpha(\Lambda^*_k)\) and the limit \(A_n(\Lambda^*_k) \to 0\) due to \(r_k \to \infty\). From the continuity of \(A_\alpha\), the sequence \(\Lambda^*_k\) converges to the cone \(\Lambda\), where \(A_\alpha(\Lambda) = 0\). We conclude that \(L_\alpha(H)\) is asymptotic to a set imbedded in the null cones of \(A_n\). Level sets which are asymptotic to these cones are unbounded. Hence

**Theorem 2.** The level set \(L_\alpha(H)\) is unbounded if and only if it is asymptotic to a set imbedded in a cone \(\Lambda\) such that \(A_\alpha(\Lambda) = 0\).

Let us turn our attention to the influence of the algebraic sign of the coefficients of these polynomials on their level sets. It is of course clear that, if a level set \(L_\alpha(H)\) has contact with a cone \(\Lambda\) on \(p\) spheres, then \(L_\alpha(H)\) has contact with these same spheres on each cone \(\Lambda\) for which the coefficients \(A_k\) agree term wise. A more explicit conclusion is obtained thru the use of Descartes’ rule of signs in

**Theorem 3.** If the number of variations in sign of the terms in the sequence of coefficients

\[
A_0(\Lambda), \cdots, A_n(\Lambda)
\]

generated from the polynomial \(H(X) - \alpha\) on the cone \(\Lambda\), is \(p\), then the number of intersections of surface \(L_\alpha(H)\) and cone \(\Lambda\) is \(p\) or is less than \(p\) by an even positive integer. If the number of permanences in sign for (9) is \(q\), then surface \(L_\alpha(H)\) and cone \(\Lambda^*_i = (-\Lambda)\) have at most of \(q\) intersections.
A sufficient condition for such an intersection is found in

**Corollary 3.1.** If \( \Lambda_i \) is a cone for which the signs of the coefficients \( A_0(\Lambda_i) \) and \( A_n(\Lambda_i) \) are opposite, then the level set \( L_\alpha(H) \) has positive contact with \( \Lambda_i \).

Additional connections between these level sets and the coefficients \( A_k \) are found in the equation

\[
H(r\lambda \Lambda_i) - \alpha = \sum_{k=0}^{n} A_k(\Lambda_i) r^k = \sum_{k=0}^{n} A_k(\Lambda_i) r^k = H(r\Lambda_i) - \alpha
\]

which hold on cones \( \Lambda_i \) and \( \Lambda_j \) for which \( \lambda^k A_k(\Lambda_i) = A_k(\Lambda_j) \), \( 0 \leq k \leq n \), for some real constant \( \lambda \). This equation establishes a relation between the intersections of level sets with cones about \( j \)th and \( l \)th axes, as stated in

**Theorem 4.** Let the level set \( L_\alpha(H) \) meet the cone \( \Lambda_i \) at the positive distances \( r_1, \ldots, r_p \). Then \( L_\alpha(H) \) meets each cone \( \Lambda_i \) for which there exists a positive constant \( \lambda \) such that

\[
\lambda^k A_k(\Lambda_i) = A_k(\Lambda_j), \quad 0 \leq k \leq n
\]

at the distances \( \lambda r_1, \lambda r_2, \ldots, \lambda r_p \).

Let us now focus our attention upon equipotential surfaces generated by axisymmetric harmonic polynomials in \( \mathbb{E}^3 \). These surfaces arise when the coefficients \( A_k(\Lambda_i) \) reduce to \( P_k(\cos \theta) \), the Legendre polynomial of degree \( k \) in \( \cos \theta = x r^{-1} \) and the polynomial \( H(X) - \alpha \) becomes the real harmonic polynomial of degree \( n \)

(10) \[
H(r, \theta) - \alpha = \sum_{k=0}^{n} a_k r^k P_k(\cos \theta), \quad a_n \neq 0.
\]

Elementary reasoning based on the fact that on the cone \( \theta = \theta_0 \), \( H(r, \theta) - \alpha \) is a polynomial of degree \( n \) in the variable \( r \) leads us to geometrical properties of these surfaces which are summarized in

**Theorem 5.** For each axisymmetric harmonic polynomial \( H \), every finite point of \( \mathbb{E}^3 \) belongs to some equipotential of \( H \). In particular, if the equipotential surfaces \( L_\alpha(H) \) and \( L_\beta(H) \) have contact with a cone on the spheres \( r = r_0 \) and \( r = R_0 \), respectively, then for each choice of \( \lambda \) between \( \alpha \) and \( \beta \) the equipotential surface \( L_\lambda(H) \) has contact with this cone between these spheres.
Although equipotential surfaces generated from distinct harmonic polynomials of degree \( n \) with common zeroth order contact at the origin have no more than \( n - 1 \) common circles of intersection on any fixed cone, near infinity these surfaces have nearly identical structure. To bring forth this asymptotic property, we apply Theorem 2 to equation (10) to conclude

**Theorem 6.** An equipotential surface generated from an axisymmetric harmonic polynomial of degree \( n \) is unbounded if and only if it is asymptotic to at least one of the cones \( \theta = \theta_j \) for which \( P_n(\cos \theta_j) = 0 \), \( 1 \leq j \leq n \).

Having established these properties of equipotentials, let us now estimate the growth of these surfaces in a neighborhood of infinity. To accomplish this, consider an unbounded equipotential surface generated from an \( n \)th degree harmonic polynomial with \( \nu \)th order contact at the origin.

At large distances from the origin this surface either coincides with or approaches some cone \( \theta = \theta_0 \), \( P_n(\cos \theta_0) = 0 \). In the latter case assuming the equipotential meets the cone \( \theta = \theta_0 \) for \( \theta_0 > \theta \), we select \( \epsilon (\epsilon > 0) \) sufficiently small so that \( P'_n(\cos \theta)P_n(\cos \theta) \neq 0 \) for \( 0 \leq \theta_j < \theta + \epsilon < \pi \). Let us now apply the Mean Value Theorem on the interval \( J_0 = [\theta_0, \theta] \), \( \theta_j < \theta < \theta_j + \epsilon \) to find \( \eta \in J_0 \) so that

\[
P'_n(\cos \eta) = \frac{[P_n(\cos \theta) - P_n(\cos \theta))] / (\cos \theta - \cos \theta_j)}{(\cos \theta_j)}.
\]

We then use the relations

\[
-cos \theta + cos \theta_j = 2sin((\theta + \theta_j)/2)sin((\theta - \theta_j)/2)
\]

\[
> (\sin \theta_j)(\theta - \theta_j)sin(\epsilon/2)/\epsilon
\]

to deduce that

\[
\left| \frac{P_k(\cos \theta)}{P_n(\cos \theta)} \right| = \left| \frac{P_k(\cos \theta)}{(\cos \theta - cos \theta_j)} \frac{(\cos \theta - cos \theta)}{(P_n(\cos \theta) - P_n(\cos \theta))] / (\cos \theta - \cos \theta_j)} \right|
\]

\[
\leq \frac{K(\epsilon)}{(\theta - \theta_j)|P'_n(\cos \eta)|}
\]

From this estimate we find that on the equipotential surface \( L_a(H) \),

\[
r(\theta) \leq 1 + \max_{\nu + 1 < n - 1} \left| \frac{a_nP_k(\cos \theta)}{a_nP_n(\cos \theta)} \right| < 1 + M_\epsilon / (\theta - \theta_j)
\]

for \( \theta_j < \theta < \theta_j + \epsilon \) and \( \nu + 1 < n \) establishing
Theorem 7. If an equipotential surface generated by an axisymmetric harmonic polynomial in \(E^3\) is unbounded and in neighborhood of infinity meets the cone \(\theta = \theta_0\) at a distance \(r = r(\theta_0)\), then

\[ r(\theta_0) = \theta_0 \left\{ \max_{1 \leq j \leq n} \left( \frac{1}{|\theta_0 - \theta_j|} \right) \right\} \]

where \(P_n(\cos \theta_j) = 0\), \(1 \leq j \leq n\) for \(\theta_0 \neq \theta_j\).

We now turn to some analytic results on the zeros \(r_1, \ldots, r_{n-v}\) of the function

\[ H(r, \theta_0) - \alpha = \sum_{k=0}^{n} a_k r^k P_k(\cos \theta_0), \quad r > 0, \]

from which we infer that

\[ H(r, \theta_0) - \alpha = a_n P_n(\cos \theta_0) r^v (r^{n-v} + s_{n-1} r^{n-v-1} + \cdots + s_{v+1} r + s_v) \]

where \(s_k = a_k P_k(\cos \theta_0)/a_n P_n(\cos \theta_0)\), \(\nu \leq k \leq n - 1\). The coefficients \(s_k\) of the equation

\[ r^{n-v} + s_{n-1} r^{n-v-1} + \cdots + s_{v+1} r + s_v = 0 \]

are symmetric functions of its roots \(r_k\). Thus,

\[ s_{n-1} = -(r_1 + \cdots + r_{n-v}) \]
\[ s_{v+1} = (-1)^{n-v-1}(r_2 r_3 \cdots r_{n-v} + r_1 r_3 \cdots r_{n-v} + \cdots + r_1 r_2 \cdots r_{n-v-1}) \]
\[ s_v = (-1)^{n-v}(r_1 \cdots r_{n-v}). \]

From these symmetric functions we find

\[ r_1 + \cdots + r_{n-v} = -(a_{n-1} P_{n-1}(\cos \theta_0)/a_n P_n(\cos \theta_0)) \]

\[ 1/r_1 + \cdots + 1/r_{n-v} = -(a_{v+1} P_{v+1}(\cos \theta_0)/a_v P_v(\cos \theta_0)). \]

which bring us to

Theorem 8. Let the equipotential surface generated by the harmonic polynomial

\[ H(r, \theta) - \alpha = \sum_{k=0}^{n} a_k r^k P_k(\cos \theta) \]
having \( v \)th order contact with the cone \( \theta = \theta_0 \) at the origin meet this cone in \( n - v \) additional finite circles at the distances \( r_1, \ldots, r_{n-v} \). Let \( M_a, M_g \) and \( M_h \) be respectively the arithmetic, geometric and harmonic means of \( r_1, \ldots, r_{n-v} \) and let \( b_k = |a_k/a_n| \) and \( \tau_k(\theta_0) = |P_k(\cos \theta_0)/P_n(\cos \theta_0)| \). If \( P_n(\cos \theta_0)P_{n+1}(\cos \theta_0) \neq 0 \) and \( a_n a_{n+1} \neq 0 \), then

\[
M_a = (n - v)^{-1}b_{n-1}\tau_{n-1}(\theta_0),
\]
\[
M_g = [b_n\tau_n(\theta_0)]^{1/(n-v)},
\]
\[
M_h = (n - v)(b_n/b_{n+1})[\tau_n(\theta_0)/\tau_{n+1}(\theta_0)].
\]

Bounds on the circles of intersection having maximum and minimum radii are found in

**Corollary 8.1.** The maximum circle of intersection of the equipotential surface \( L_a(H) \) and the cone \( \theta = \theta_0 \) lies exterior to the sphere about the origin with a radius \( \max\{M_a, M_h\} \) and the minimum circle of intersection not on the origin lies interior to the sphere about the origin with a radius \( \min\{M_a, M_h\} \).

When contact at the origin is zeroth order, from the facts that the distances \( r_1, \ldots, r_{n-v} \) are positive, \( P_0(\cos \theta) = 1 \), \( P_1(\cos \theta) = \cos \theta \) and equations (11) we deduce

**Corollary 8.2.** For an equipotential surface \( L_a(H) \) having zeroth order contact with the cone \( \theta = \theta_0 \) at the origin to intersect this cone in \( n \) finite circles, it is necessary that \( 0 \leq \theta_0 < \pi/2 \) if \( a_1/a_0 < 0 \) and \( \pi/2 < \theta_0 < \pi \) if \( a_1/a_0 > 0 \).

**References**


Received October 3, 1974. Research of the first author partly supported by a grant of the University of Wisconsin-Milwaukee Graduate School from W.A.R.F. funds.
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