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# LEVEL SETS OF POLYNOMIALS IN $n$ REAL VARIABLES

MORRIS MARDEN AND PETER A. MCCOY

**The methods used in studying the zeros of a polynomial in a single complex variable are here adapted to investigating the level surfaces of a real polynomial in $E^n$, with respect to their intersection and finite or asymptotic tangency with certain cones. Special attention is given to the equipotential surfaces generated by an axisymmetric harmonic polynomial in $E^3$.**

A principal interest is the application of reasoning used by Cauchy [**2**, p. 123] in obtaining bounds on the zeros of polynomials in one complex variable. We thereby seek the level sets

$$L_\alpha(H) = \{X \in E^n \mid H(X) = \alpha\}$$

generated from the real polynomials

$$H(X) - \alpha = \sum_{0 \leqq j_1 + \cdots + j_n \leqq n} \alpha_{j_1 \cdots j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}, \tag{1}$$

$$X = (x_1, x_2, \cdots, x_n),\ r = |X| = [x_1^2 + x_2^2 + \cdots + x_n^2]^{1/2}.$$

It is convenient to introduce direction numbers $\lambda_j = x_j r^{-1}$, $1 \leqq j \leqq n$, connected by $\lambda_1^2 + \cdots + \lambda_n^2 = 1$ and cones $\Lambda_j : \lambda_j =$ constant, about the $j$th axis. On the intersection of the cones $\Lambda_j$, these polynomials become

$$H(r\Lambda_j) - \alpha = \sum_{k=0}^{n} r^k A_k(\Lambda_j)$$

where

$$A_k = A_k(\Lambda_j) = \sum_{j_1 + \cdots j_n = k} \alpha_{j_1 \cdots j_n} \lambda_1^{j_1} \cdots \lambda_n^{j_n}, \qquad 0 \leqq k \leqq n.$$

At the origin the level set $L_\alpha(H)$ has $\nu$th order contact with $\Lambda_j$, if $A_k(\Lambda_j) = 0$ for $0 \leqq k \leqq \nu - 1$ but $A_\nu(\Lambda_j) \neq 0$ and $A_n(\Lambda_j) \neq 0$. For such sets we introduce the ratios

$$M_\nu = M_\nu(\Lambda_j) = \max_{\nu \leqq k \leqq n-1} |A_k / A_n|$$

$$m_\nu = m_\nu(\Lambda_j) = \min_{\nu+1 \leqq k \leqq n} (|A_\nu|/(|A_\nu| + |A_k|))$$

$$\mu_\nu = \mu_\nu(\Lambda_j) = \max_{\nu+1 \leqq k \leqq n} |A_k/A_\nu|.$$

Then, by considering points common to the level set $L_\alpha(H)$ and the cone $\Lambda_j$ exterior to the unit ball $r > 1$ (about the origin), we deduce an inequality

$$\text{(2)} \quad |H(r\Lambda_j) - \alpha| > |A_n| r^n - \sum_{k=\nu}^{n-1} |A_k| r^k \geqq |A_n| r^n \left[1 - M_\nu \sum_{k=1}^{n-\nu} r^{-k}\right] =$$

$$\text{(2a)} \quad |A_n| r^n [1 - (M_\nu(1 - r^{\nu-n})(r-1)^{-1}] > |A_n| r^n (r - 1 - M_\nu)/(r-1)$$

from which it is clear that, if $r \geqq 1 + M_\nu$, $L_\alpha(H)$ does not intersect $\Lambda_j$. Likewise, if we consider the reciprocal polynomial associated with (1), derived by setting $1/r = \zeta > 1$, the inequalities

$$\text{(3)} \quad |\zeta^n[H(\zeta^{-1}\Lambda_j) - \alpha]| = \left|\sum_{k=\nu}^{n} \zeta^{n-k} A_k\right|$$

$$\geqq \zeta^{n-\nu}|A_\nu| - \sum_{k=\nu+1}^{n} \zeta^{n-k}|A_k|$$

$$\geqq \zeta^{n-\nu}|A_\nu| \left[1 - \mu_\nu \sum_{k=\nu+1}^{n} \zeta^{\nu-k}\right]$$

$$\text{(3a)} \quad = \zeta^{n-\nu}|A_\nu|[1 - (\mu_\nu(1 - \zeta^{\nu-n}))(\zeta - 1)^{-1}]$$

$$> \zeta^{n-\nu}|A_\nu|(\zeta - 1 - \mu_\nu)/(\zeta - 1)$$

imply that $H(\zeta^{-1}\Lambda_j) \neq \alpha$ for $\zeta \geqq 1 + \mu_\nu$. Thus we infer that $H(r\Lambda_j) \neq \alpha$ for

$$r \leqq (1 + \mu_\nu)^{-1} = m_\nu,$$

which brings us to

THEOREM 1. *If the level set $L_\alpha(H)$ has $\nu$th order contact with the cone $\Lambda_j$ at the origin and if it intersects the cone at any additional finite points, then it does so at a distance $r$ from the origin where*

$$\text{(4)} \quad m_\nu(\Lambda_j) < r < 1 + M_\nu(\Lambda_j).$$

By use of inequalities (3a) and (2a), we replace inequality (4) in Theorem 1 by

(4)′
$$r_1 \leqq r \leqq r_2$$

where $r_1$ is the larger positive root of the equation

$$1-(1+\mu_\nu)r+\mu_\nu r^{n-\nu+1}=0$$

and $r_2$ the larger positive root of the equation

$$r^{n+1-\nu}-(1+M_\nu)r^{n-\nu}+M_\nu=0,$$

$r=1$ being a root of both equations.

A natural question arising from this theorem is that of determining the point of tangency of the level sets with the cones $\Lambda_j$. Let us consider the $k$th term in the polynomial (1),

$$r^kA_k(r^{-1}X)=\sum_{j_1+\cdots+j_n=k}\alpha_{j_1\cdots j_n}x_1^{j_1}\cdots x_n^{j_n}.$$

As this sum is composed of homogeneous polynomials of degree $k$, we may apply Euler's Identity [**1**, p. 141] to find that

(5)
$$X\cdot\nabla[r^kA_k(r^{-1}X)]=kr^kA_k(r^{-1}X),$$

where the left side is the scalar product of vector $X$ and the gradient of the bracket. On account of this relation, the orthogonality condition

$$X\cdot\nabla H(X)=0$$

becomes

(6)
$$\sum_{k=\nu}^{n}kr^kA_k(\Lambda_j)=0.$$

Let us define

$$m^*(\Lambda_j)=\min_{2\leqq k\leqq n}[(kA_k+A_\nu)/(A_\nu)]$$

$$M^*(\Lambda_j)=\max_{\nu\leqq k\leqq n-1}(kA_k/nA_n).$$

Theorem 1 and equation 6 lead to

COROLLARY 1.1. *If the level set $L_\alpha(H)$ has $\nu$th order contact with the* *cone $\Lambda_j$ at its vertex and is tangent to the cone at a positive distance r from the origin, then*

$$m_\nu^*(\Lambda_j) < r < 1 + M_\nu^*(\Lambda_j). \tag{7}$$

As equation (6) may be viewed as

$$\partial[H(X) - \alpha]/\partial r = 0, \tag{8}$$

we may use Rolle's Theorem to conclude

COROLLARY 1.2. *If the ray $(\lambda_1, \cdots, \lambda_n) \in \bigcap_{j=1}^n \Lambda_j$, the level surface $L_\alpha(H)$ has a finite tangental contact point between successive pairs of intersections of $L_\alpha(H)$ with the ray.*

The influence of the coefficient $A_n = A_n(X)$ on the structure of $L_\alpha(H)$ near infinity is found by selecting a sequence of points $\{X_k\}$, $r_k = |X_k| \to \infty$, such that $H(X_k) = \alpha$. Each of these points is located on a cone $\Lambda_j^{(k)}$. This leads to the bound $r_k < 1 + M_\nu(\Lambda_j^{(k)})$ and the limit $A_n(\Lambda_j^{(k)}) \to 0$ due to $r_k \to \infty$. From the continuity of $A_n$, the sequence $\Lambda_j^{(k)}$ converges to the cone $\Lambda_j$, where $A_n(\Lambda_j) = 0$. We conclude that $L_\alpha(H)$ is asymptotic to a set imbedded in the null cones of $A_n$. Level sets which are asymptotic to these cones are unbounded. Hence

THEOREM 2. *The level set $L_\alpha(H)$ is unbounded if and only if it is asymptotic to a set imbedded in a cone $\Lambda_j$ such that $A_n(\Lambda_j) = 0$.*

Let us turn our attention to the influence of the algebraic sign of the coefficients of these polynomials on their level sets. It is of course clear that, if a level set $L_\alpha(H)$ has contact with a cone $\Lambda_j$ on $p$ spheres, then $L_\alpha(H)$ has contact with these same spheres on each cone $\Lambda_l$ for which the coefficients $A_k$ agree term wise. A more explicit conclusion is obtained thru the use of Descartes' rule of signs in

THEOREM 3. *If the number of variations in sign of the terms in the sequence of coefficients*

$$A_0(\Lambda_j), \cdots, A_n(\Lambda_j) \tag{9}$$

*generated from the polynomial $H(X) - \alpha$ on the cone $\Lambda_j$ is p, then the number of intersections of surface $L_\alpha(H)$ and cone $\Lambda_j$ is p or is less than p by an even positive integer. If the number of permanences in sign for* (9) *is q, then surface $L_\alpha(H)$ and cone $\Lambda_j^* = (-\Lambda_j)$ have at most of q intersections.*

A sufficient condition for such an intersection is found in

COROLLARY 3.1. *If $\Lambda_j$ is a cone for which the signs of the coefficients $A_0(\Lambda_j)$ and $A_n(\Lambda_j)$ are opposite, then the level set $L_\alpha(H)$ has positive contact with $\Lambda_j$.*

Additional connections between these level sets and the coefficients $A_k$ are found in the equation

$$H(r\lambda\Lambda_l)-\alpha=\sum_{k=0}^{n}A_k(\Lambda_l)\lambda^k r^k=\sum_{k=0}^{n}A_k(\Lambda_j)r^k=H(r\Lambda_j)-\alpha$$

which hold on cones $\Lambda_l$ and $\Lambda_j$ for which $\lambda^k A_k(\Lambda_l)=A_k(\Lambda_j)$, $A_k(\Lambda_j)$, $0\leqq k\leqq n$, for some real constant $\lambda$. This equation establishes a relation between the intersections of level sets with cones about $j$th and $l$th axes, as stated in

THEOREM 4. *Let the level set $L_\alpha(H)$ meet the cone $\Lambda_j$ at the positive distances $r_1,\cdots,r_p$. Then $L_\alpha(H)$ meets each cone $\Lambda_l$ for which there exists a positive constant $\lambda$ such that*

$$\lambda^k A_k(\Lambda_l)=A_k(\Lambda_j),\qquad 0\leqq k\leqq n$$

*at the distances $\lambda r_1,\lambda r_2,\cdots,\lambda r_p$.*

Let us now focus our attention upon equipotential surfaces generated by axisymmetric harmonic polynomials in $E^3$. These surfaces arise when the coefficients $A_k(\Lambda_j)$ reduce to $P_k(\cos\theta)$, the Legendre polynomial of degree $k$ in $\cos\theta=xr^{-1}$ and the polynomial $H(X)-\alpha$ becomes the real harmonic polynomial of degree $n$

$$H(r,\theta)-\alpha=\sum_{k=0}^{n}a_k r^k P_k(\cos\theta),\qquad a_n\neq 0. \tag{10}$$

Elementary reasoning based on the fact that on the cone $\theta=\theta_0$, $H(r,\theta)-\alpha$ is a polynomial of degree $n$ in the variable $r$ leads us to geometrical properties of these surfaces which are summarized in

THEOREM 5. *For each axisymmetric harmonic polynomial $H$, every finite point of $E^3$ belongs to some equipotential of $H$. In particular, if the equipotential surfaces $L_\alpha(H)$ and $L_\beta(H)$ have contact with a cone on the spheres $r=r_0$ and $r=R_0$, respectively, then for each choice of $\lambda$ between $\alpha$ and $\beta$ the equipotential surface $L_\lambda(H)$ has contact with this cone between these spheres.*

Although equipotential surfaces generated from distinct harmonic polynomials of degree $n$ with common zeroth order contact at the origin have no more than $n-1$ common circles of intersection on any fixed cone, near infinity these surfaces have nearly identical structure. To bring forth this asymptotic property, we apply Theorem 2 to equation (10) to conclude

THEOREM 6. *An equipotential surface generated from an axisymmetric harmonic polynomial of degree n is unbounded if and only if it is asymptotic to at least one of the cones $\theta = \theta_j$ for which $P_n(\cos\theta_j) = 0$, $1 \leqq j \leqq n$.*

Having established these properties of equipotentials, let us now estimate the growth of these surfaces in a neighborhood of infinity. To accomplish this, consider an unbounded equipotential surface generated from an $n$th degree harmonic polynomial with $\nu$th order contact at the origin.

At large distances from the origin this surface either coincides with or approaches some cone $\theta = \theta_j$, $P_n(\cos\theta_j) = 0$. In the latter case assuming the equipotential meets the cone $\theta = \theta_0$ for $\theta_0 > \theta_j$ we select $\epsilon(\epsilon > 0)$ sufficiently small so that $P_n'(\cos\theta)P_n(\cos\theta) \neq 0$ for $0 \leqq \theta_j < \theta + \epsilon < \pi$. Let us now apply the Mean Value Theorem on the interval $J_\theta = [\theta_j, \theta]$, $\theta_j < \theta < \theta_j + \epsilon$ to find $\eta \in J_\theta$ so that

$$P_n'(\cos\eta) = [P_n(\cos\theta) - P_n(\cos\theta_j)]/(\cos\theta - \cos\theta_j).$$

We then use the relations

$$\begin{aligned} -\cos\theta + \cos\theta_j &= 2\sin((\theta+\theta_j)/2)\sin((\theta-\theta_j)/2) \\ &> (\sin\theta_j)(\theta-\theta_j)\sin(\epsilon/2)/\epsilon \end{aligned}$$

to deduce that

$$\begin{aligned} \left|\frac{P_k(\cos\theta)}{P_n(\cos\theta)}\right| &= \left|\frac{P_k(\cos\theta)}{(\cos\theta-\cos\theta_j)}\,\frac{(\cos\theta-\cos\theta_j)}{(P_n(\cos\theta)-P_n(\cos\theta_j))}\right| \\ &\leqq \frac{K(\epsilon)}{(\theta-\theta_j)|P_n'(\cos\eta)|} \end{aligned}$$

From this estimate we find that on the equipotential surface $L_\alpha(H)$,

$$r(\theta) \leqq 1 + \max_{\nu \leqq k \leqq n-1}\left|\frac{a_k P_k(\cos\theta)}{a_n P_n(\cos\theta)}\right| < 1 + M_\epsilon/(\theta-\theta_j)$$

for $\theta_j < \theta < \theta_j + \epsilon$ and $\nu + 1 < n$ establishing

THEOREM 7. *If an equipotential surface generated by an axisymmetric harmonic polynomial in $E^3$ is unbounded and in neighborhood of infinity meets the cone $\theta = \theta_0$ at a distance $r = r(\theta_0)$, then*

$$r(\theta_0) = \mathbf{0}\{\max_{1 \leqq j \leqq n} (1/|\theta_0 - \theta_j|)\}$$

*where $P_n(\cos \theta_j) = 0$, $1 \leqq j \leqq n$ for $\theta_0 \neq \theta_j$.*

We now turn to some analytic results on the zeros $r_1, \cdots, r_{n-\nu}$ of the function

$$H(r, \theta_0) - \alpha = \sum_{k=\nu}^{n} a_k r^k P_k(\cos \theta_0), \qquad r > 0,$$

from which we infer that

$$H(r, \theta_0) - \alpha = a_n P_n(\cos \theta_0) r^\nu (r^{n-\nu} + s_{n-1} r^{n-\nu-1} + \cdots + s_{\nu+1} r + s_\nu)$$

where $s_k = a_k P_k(\cos \theta_0)/a_n P_n(\cos \theta_0)$, $\nu \leqq k \leqq n-1$. The coefficients $s_k$ of the equation

$$r^{n-\nu} + s_{n-1} r^{n-\nu-1} + \cdots + s_{\nu+1} r + s_\nu = 0$$

are symmetric functions of its roots $r_k$. Thus,

$$\begin{aligned} s_{n-1} &= -(r_1 + \cdots + r_{n-\nu}) \\ s_{\nu+1} &= (-1)^{n-\nu-1}(r_2 r_3 \cdots r_{n-\nu} + r_1 r_3 \cdots r_{n-\nu} + \cdots + r_1 r_2 \cdots r_{n-\nu-1}) \\ s_\nu &= (-1)^{n-\nu}(r_1 \cdots r_{n-\nu}). \end{aligned}$$

From these symmetric functions we find

$$\begin{aligned} r_1 + \cdots + r_{n-\nu} &= -(a_{n-1} P_{n-1}(\cos \theta_0)/a_n P_n(\cos \theta_0)) \\ 1/r_1 + \cdots + 1/r_{n-\nu} &= -(a_{\nu+1} P_{\nu+1}(\cos \theta_0)/a_\nu P_\nu(\cos \theta_0)). \end{aligned} \tag{11}$$

which bring us to

THEOREM 8. *Let the equipotential surface generated by the harmonic polynomial*

$$H(r, \theta) - \alpha = \sum_{k=\theta}^{n} a_k r^k P_k(\cos \theta)$$

*having $\nu$th order contact with the cone $\theta = \theta_0$ at the origin meet this cone in $n - \nu$ additional finite circles at the distances $r_1, \cdots, r_{n-\nu}$. Let $M_a$, $M_g$ and $M_h$ be respectively the arithmetic, geometric and harmonic means of $r_1 \cdots r_{n-\nu}$ and let $b_k = |a_k/a_n|$ and $\tau_k(\theta_0) = |P_k(\cos\theta_0)/P_n(\cos\theta_0)|$. If $P_n(\cos\theta_0)P_{\nu+1}(\cos\theta_0) \neq 0$ and $a_n a_{\nu+1} \neq 0$, then*

$$M_a = (n - \nu)^{-1} b_{n-1}\tau_{n-1}(\theta_0),$$
$$M_g = [b_\nu \tau_\nu(\theta_0)]^{1/(n-\nu)},$$
$$M_h = (n - \nu)(b_\nu / b_{\nu+1})[\tau_\nu(\theta_0)/\tau_{\nu+1}(\theta_0)].$$

Bounds on the circles of intersection having maximum and minimum radii are found in

COROLLARY 8.1. *The maximum circle of intersection of the equipotential surface $L_\alpha(H)$ and the cone $\theta = \theta_0$ lies exterior to the sphere about the origin with a radius max $\{M_a, M_h\}$ and the minimum circle of intersection not on the origin lies interior to the sphere about the origin with a radius* $\min\{M_a, M_h\}$.

When contact at the origin is zeroth order, from the facts that the distances $r_1, \cdots, r_{n-\nu}$ are positive, $P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$ and equations (11) we deduce

COROLLARY 8.2. *For an equipotential surface $L_\alpha(H)$ having zeroth order contact with the cone $\theta = \theta_0$ at the origin to intersect this cone in $n$ finite circles, it is necessary that $0 \leqq \theta_0 < \pi/2$ if $a_1/a_0 < 0$ and $\pi/2 < \theta_0 < \pi$ if $a_1/a_0 > 0$.*

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