LEVEL SETS OF POLYNOMIALS IN $n$ REAL VARIABLES

MORRIS MARDEN AND PETER A. MCCOY
LEVEL SETS OF POLYNOMIALS 
IN \( n \) REAL VARIABLES

MORRIS MARDEN AND PETER A. McCoy

The methods used in studying the zeros of a polynomial in a single complex variable are here adapted to investigating the level surfaces of a real polynomial in \( E^n \), with respect to their intersection and finite or asymptotic tangency with certain cones. Special attention is given to the equipotential surfaces generated by an axisymmetric harmonic polynomial in \( E^3 \).

A principal interest is the application of reasoning used by Cauchy [2, p. 123] in obtaining bounds on the zeros of polynomials in one complex variable. We thereby seek the level sets

\[
L_\alpha (H) = \{ X \in E^n \mid H(X) = \alpha \}
\]

generated from the real polynomials

\[
H(X) - \alpha = \sum_{0 \leq j_1 + \cdots + j_n \leq n} \alpha_{j_1 \cdots j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n},
\]

\(X = (x_1, x_2, \ldots, x_n)\), \( r = |X| = [x_1^2 + x_2^2 + \cdots + x_n^2]^{1/2} \).

It is convenient to introduce direction numbers \( \lambda_j = x_j r^{-1} \), \( 1 \leq j \leq n \), connected by \( \lambda_1^2 + \cdots + \lambda_n^2 = 1 \) and cones \( \Lambda_j : \lambda_j = \text{constant} \), about the \( j \)th axis. On the intersection of the cones \( \Lambda_j \), these polynomials become

\[
H(r \Lambda_j) - \alpha = \sum_{k=0}^n r^k A_k (\Lambda_j)
\]

where

\[
A_k = A_k (\Lambda_j) = \sum_{j_1 + \cdots + j_n = k} \alpha_{j_1 \cdots j_n} \lambda_1^{j_1} \cdots \lambda_n^{j_n}, \quad 0 \leq k \leq n.
\]

At the origin the level set \( L_\alpha (H) \) has \( \nu \)th order contact with \( \Lambda_\nu \) if \( A_k (\Lambda_\nu) = 0 \) for \( 0 \leq k \leq \nu - 1 \) but \( A_\nu (\Lambda_\nu) \neq 0 \) and \( A_n (\Lambda_\nu) \neq 0 \). For such sets we introduce the ratios

\[
M_\nu = M_\nu (\Lambda_\nu) = \max_{0 \leq k \leq n - 1} \left| \frac{A_k}{A_n} \right|
\]
\[ m_\nu = m_\nu (\Lambda_\nu) = \min_{\nu + 1 \leq k \leq n} \left( |A_\nu| / (|A_\nu| + |A_k|) \right) \]

\[ \mu_\nu = \mu_\nu (\Lambda_\nu) = \max_{\nu + 1 \leq k \leq n} |A_k / A_\nu|. \]

Then, by considering points common to the level set \( L_\alpha (H) \) and the cone \( \Lambda_\nu \) exterior to the unit ball \( r > 1 \) (about the origin), we deduce an inequality

\[ (2) \quad |H(r \Lambda_\nu) - \alpha| > |A_n| |r^n - \sum_{k=\nu}^{n-1} |A_k| |r^k \geq |A_n| r^n \left[ 1 - M_\nu \sum_{k=\nu+1}^{n} r^{-k} \right] = \]

\[ (2a) \quad |A_n| r^n \left[ 1 - (M_\nu (1 - r^{-n}) (r - 1))^{-1} \right] > |A_n| r^n (r - 1 - M_\nu) / (r - 1) \]

from which it is clear that, if \( r \geq 1 + M_\nu \), \( L_\alpha (H) \) does not intersect \( \Lambda_\nu \). Likewise, if we consider the reciprocal polynomial associated with (1), derived by setting \( 1/r = \zeta > 1 \), the inequalities

\[ (3) \quad |\zeta^n [H(\zeta^{-1} \Lambda_\nu) - \alpha]| = \left| \sum_{k=\nu}^{n} \zeta^{n-k} A_k \right| \]

\[ \geq \zeta^{n-\nu} |A_\nu| - \sum_{k=\nu+1}^{n} \zeta^{n-k} |A_k| \]

\[ \geq \zeta^{n-\nu} |A_\nu| \left[ 1 - \mu_\nu \sum_{k=\nu+1}^{n} \zeta^{\nu-k} \right] \]

\[ = \zeta^{n-\nu} |A_\nu| \left[ 1 - (\mu_\nu (1 - \zeta^{-n})) (\zeta - 1)^{-1} \right] \]

\[ > \zeta^{n-\nu} |A_\nu| (\zeta - 1 - \mu_\nu)/(\zeta - 1) \]

imply that \( H(\zeta^{-1} \Lambda_\nu) \neq \alpha \) for \( \zeta \geq 1 + \mu_\nu \). Thus we infer that \( H(r \Lambda_\nu) \neq \alpha \) for

\[ r \leq (1 + \mu_\nu)^{-1} = m_\nu, \]

which brings us to

THEOREM 1. If the level set \( L_\alpha (H) \) has \( \nu \)th order contact with the cone \( \Lambda_\nu \) at the origin and if it intersects the cone at any additional finite points, then it does so at a distance \( r \) from the origin where

\[ (4) \quad m_\nu (\Lambda_\nu) < r < 1 + M_\nu (\Lambda_\nu). \]

By use of inequalities (3a) and (2a), we replace inequality (4) in Theorem 1 by
where \( r_1 \) is the larger positive root of the equation
\[
1 - (1 + \mu_\nu)r + \mu_\nu r^{n-\nu+1} = 0
\]
and \( r_2 \) the larger positive root of the equation
\[
r^{n+1-\nu} - (1 + M_\nu)r^{n-\nu} + M_\nu = 0,
\]
\( r = 1 \) being a root of both equations.

A natural question arising from this theorem is that of determining the point of tangency of the level sets with the cones \( \Lambda_\nu \). Let us consider the \( k \)th term in the polynomial (1),
\[
r^k A_k (r^{-1} X) = \sum_{j_1 + \ldots + j_m = k} \alpha_{j_1} x_1^{j_1} \cdots x_n^{j_n}.
\]
As this sum is composed of homogeneous polynomials of degree \( k \), we may apply Euler's Identity [1, p. 141] to find that
\[
X \cdot \nabla [r^k A_k (r^{-1} X)] = kr^k A_k (r^{-1} X),
\]
where the left side is the scalar product of vector \( X \) and the gradient of the bracket. On account of this relation, the orthogonality condition
\[
X \cdot \nabla H(X) = 0
\]
becomes
\[
\sum_{k=\nu}^n kr^k A_k (\Lambda_\nu) = 0.
\]
Let us define
\[
m^*(\Lambda_\nu) = \min \left[ (kA_k + A_\nu)/(A_\nu) \right]_{2 \leq k \leq n}
\]
\[
M^*(\Lambda_\nu) = \max_{\nu \leq k \leq n-1} (kA_k/nA_n).
\]

Theorem 1 and equation 6 lead to

**Corollary 1.1.** If the level set \( L_\alpha (H) \) has \( \nu \)th order contact with the
cone \( \Lambda_j \) at its vertex and is tangent to the cone at a positive distance \( r \) from the origin, then

\[
m_j^*(\Lambda_j) < r < 1 + M_j^*(\Lambda_j).
\]

As equation (6) may be viewed as

\[
\frac{\partial [H(X) - \alpha]}{\partial r} = 0,
\]

we may use Rolle's Theorem to conclude

**Corollary 1.2.** If the ray \((\lambda_1, \cdots, \lambda_n) \in \bigcap_{j=1}^n \Lambda_j\), the level surface \( L_\alpha(H) \) has a finite tangential contact point between successive pairs of intersections of \( L_\alpha(H) \) with the ray.

The influence of the coefficient \( A_n = A_n(X) \) on the structure of \( L_\alpha(H) \) near infinity is found by selecting a sequence of points \( \{X_k\} \), \( r_k = |X_k| \to \infty \), such that \( H(X_k) = \alpha \). Each of these points is located on a cone \( \Lambda_j^{(k)} \). This leads to the bound \( r_k < 1 + M_n(\Lambda_j^{(k)}) \) and the limit \( A_n(\Lambda_j^{(k)}) \to 0 \) due to \( r_k \to \infty \). From the continuity of \( A_n \), the sequence \( \Lambda_j^{(k)} \) converges to the cone \( \Lambda_j \), where \( A_n(\Lambda_j) = 0 \). We conclude that \( L_\alpha(H) \) is asymptotic to a set imbedded in the null cones of \( A_n \). Level sets which are asymptotic to these cones are unbounded. Hence

**Theorem 2.** The level set \( L_\alpha(H) \) is unbounded if and only if it is asymptotic to a set imbedded in a cone \( \Lambda_j \) such that \( A_n(\Lambda_j) = 0 \).

Let us turn our attention to the influence of the algebraic sign of the coefficients of these polynomials on their level sets. It is of course clear that, if a level set \( L_\alpha(H) \) has contact with a cone \( \Lambda_j \) on \( p \) spheres, then \( L_\alpha(H) \) has contact with these same spheres on each cone \( \Lambda_j \) for which the coefficients \( A_k \) agree term wise. A more explicit conclusion is obtained thru the use of Descartes' rule of signs in

**Theorem 3.** If the number of variations in sign of the terms in the sequence of coefficients

\[
A_0(\Lambda_j), \cdots, A_n(\Lambda_j)
\]

generated from the polynomial \( H(X) - \alpha \) on the cone \( \Lambda_j \) is \( p \), then the number of intersections of surface \( L_\alpha(H) \) and cone \( \Lambda_j \) is \( p \) or is less than \( p \) by an even positive integer. If the number of permanences in sign for (9) is \( q \), then surface \( L_\alpha(H) \) and cone \( \Lambda_j^* = (-\Lambda_j) \) have at most of \( q \) intersections.
A sufficient condition for such an intersection is found in

**Corollary 3.1.** If \( \Lambda_i \) is a cone for which the signs of the coefficients \( A_0(\Lambda_i) \) and \( A_n(\Lambda_i) \) are opposite, then the level set \( L_a(H) \) has positive contact with \( \Lambda_i \).

Additional connections between these level sets and the coefficients \( A_k \) are found in the equation

\[
H(r\lambda \Lambda_i) - \alpha = \sum_{k=0}^{n} A_k(\Lambda_i) \lambda^k r^k = \sum_{k=0}^{n} A_k(\Lambda_i) r^k = H(r\Lambda_i) - \alpha
\]

which hold on cones \( \Lambda_i \) and \( \Lambda_j \) for which \( \lambda^k A_k(\Lambda_i) = A_k(\Lambda_j) \), \( A_k(\Lambda_i) \), \( 0 \leq k \leq n \), for some real constant \( \lambda \). This equation establishes a relation between the intersections of level sets with cones about \( j \)th and \( l \)th axes, as stated in

**Theorem 4.** Let the level set \( L_a(H) \) meet the cone \( \Lambda_i \) at the positive distances \( r_1, \cdots, r_p \). Then \( L_a(H) \) meets each cone \( \Lambda_i \) for which there exists a positive constant \( \lambda \) such that

\[
\lambda^k A_k(\Lambda_i) = A_k(\Lambda_i), \quad 0 \leq k \leq n
\]

at the distances \( \lambda r_1, \lambda r_2, \cdots, \lambda r_p \).

Let us now focus our attention upon equipotential surfaces generated by axisymmetric harmonic polynomials in \( E^3 \). These surfaces arise when the coefficients \( A_k(\Lambda_i) \) reduce to \( P_k(\cos \theta) \), the Legendre polynomial of degree \( k \) in \( \cos \theta = xr^{-1} \) and the polynomial \( H(X) - \alpha \) becomes the real harmonic polynomial of degree \( n \)

\[
H(r, \theta) - \alpha = \sum_{k=0}^{n} a_k r^k P_k(\cos \theta), \quad a_n \neq 0.
\]

Elementary reasoning based on the fact that on the cone \( \theta = \theta_0 \), \( H(r, \theta) - \alpha \) is a polynomial of degree \( n \) in the variable \( r \) leads us to geometrical properties of these surfaces which are summarized in

**Theorem 5.** For each axisymmetric harmonic polynomial \( H \), every finite point of \( E^3 \) belongs to some equipotential of \( H \). In particular, if the equipotential surfaces \( L_a(H) \) and \( L_\beta(H) \) have contact with a cone on the spheres \( r = r_0 \) and \( r = R_0 \), respectively, then for each choice of \( \lambda \) between \( \alpha \) and \( \beta \) the equipotential surface \( L_\lambda(H) \) has contact with this cone between these spheres.
Although equipotential surfaces generated from distinct harmonic polynomials of degree $n$ with common zeroth order contact at the origin have no more than $n - 1$ common circles of intersection on any fixed cone, near infinity these surfaces have nearly identical structure. To bring forth this asymptotic property, we apply Theorem 2 to equation (10) to conclude

**Theorem 6.** An equipotential surface generated from an axisymmetric harmonic polynomial of degree $n$ is unbounded if and only if it is asymptotic to at least one of the cones $\theta = \theta_j$ for which $P_n(\cos \theta_j) = 0$, $1 \leq j \leq n$.

Having established these properties of equipotentials, let us now estimate the growth of these surfaces in a neighborhood of infinity. To accomplish this, consider an unbounded equipotential surface generated from an $n$th degree harmonic polynomial with $\nu$th order contact at the origin.

At large distances from the origin this surface either coincides with or approaches some cone $\theta = \theta_j$, $P_n(\cos \theta_j) = 0$. In the latter case assuming the equipotential meets the cone $\theta = \theta_0$ for $\theta_0 > \theta_j$, we select $\epsilon (\epsilon > 0)$ sufficiently small so that $P'_n(\cos \theta)P_n(\cos \theta) \neq 0$ for $0 \leq \theta_j < \theta + \epsilon < \pi$. Let us now apply the Mean Value Theorem on the interval $J_\epsilon = [\theta_j, \theta]$, $\theta_j < \theta < \theta_j + \epsilon$ to find $\eta \in J_\epsilon$ so that

$$P'_n(\cos \eta) = \frac{P_n(\cos \theta) - P_n(\cos \theta_j)}{(\cos \theta - \cos \theta_j)}$$

We then use the relations

$$- \cos \theta + \cos \theta_j = 2\sin ((\theta + \theta_j)/2) \sin ((\theta - \theta_j)/2)$$

$$> (\sin \theta_j)(\theta - \theta_j) \sin (\epsilon/2)/\epsilon$$

to deduce that

$$\left| \frac{P_k(\cos \theta)}{P_n(\cos \theta)} \right| = \left| \frac{P_k(\cos \theta)}{(\cos \theta - \cos \theta_j)} \frac{(\cos \theta - \cos \theta_j)}{(P_n(\cos \theta) - P_n(\cos \theta_j))} \right| \leq \frac{K(\epsilon)}{(\theta - \theta_j)|P'_n(\cos \eta)|}$$

From this estimate we find that on the equipotential surface $L_\alpha(H)$,

$$r(\theta) \leq 1 + \max_{\nu \leq k \leq n-1} \left| \frac{a_k P_k(\cos \theta)}{a_n P_n(\cos \theta)} \right| < 1 + M_{\epsilon}/(\theta - \theta_j)$$

for $\theta_j < \theta < \theta_j + \epsilon$ and $\nu + 1 < n$ establishing
THEOREM 7. If an equipotential surface generated by an axisymmetric harmonic polynomial in $E^3$ is unbounded and in neighborhood of infinity meets the cone $\theta = \theta_0$ at a distance $r = r(\theta_0)$, then

$$r(\theta_0) = 0\{\max_{1 \leq j \leq n} (1/|\theta_0 - \theta_j|)\}$$

where $P_n(\cos \theta_j) = 0$, $1 \leq j \leq n$ for $\theta_0 \neq \theta_j$.

We now turn to some analytic results on the zeros $r_1, \ldots, r_{n-\nu}$ of the function

$$H(r, \theta_0) - \alpha = \sum_{k=\nu}^n a_k r^k P_k(\cos \theta_0), \quad r > 0,$$

from which we infer that

$$H(r, \theta_0) - \alpha = a_n P_n(\cos \theta_0) r^n (r_1 + \cdots + r_{n-\nu} + \cdots + s_{\nu+1} r + s_\nu)$$

where $s_k = a_k P_k(\cos \theta_0)/a_n P_n(\cos \theta_0)$, $\nu \leq k \leq n-1$. The coefficients $s_k$ of the equation

$$r_1 + \cdots + r_{n-\nu} = - (a_{n-1} P_{n-1}(\cos \theta_0)/a_n P_n(\cos \theta_0))$$

are symmetric functions of its roots $r_k$. Thus,

$$s_{n-1} = -(r_1 + \cdots + r_{n-\nu})$$

$$s_{\nu+1} = (-1)^{\nu-1}(r_2 r_3 \cdots r_{n-\nu} + r_1 r_3 \cdots r_{n-\nu} + \cdots + r_1 r_2 \cdots r_{n-\nu-1})$$

$$s_\nu = (-1)^{\nu-1}(r_1 \cdots r_{n-\nu}).$$

From these symmetric functions we find

$$r_1 + \cdots + r_{n-\nu} = -(a_{n-1} P_{n-1}(\cos \theta_0)/a_n P_n(\cos \theta_0))$$

(11)

$$1/r_1 + \cdots + 1/r_{n-\nu} = -(a_{\nu+1} P_{\nu+1}(\cos \theta_0)/a_\nu P_\nu(\cos \theta_0)).$$

which bring us to

THEOREM 8. Let the equipotential surface generated by the harmonic polynomial

$$H(r, \theta) - \alpha = \sum_{k=\theta}^n a_k r^k P_k(\cos \theta)$$
having \( v \)th order contact with the cone \( \theta = \theta_0 \) at the origin meet this cone in \( n - v \) additional finite circles at the distances \( r_1, \cdots, r_{n-v} \). Let \( M_a, M_g \) and \( M_h \) be respectively the arithmetic, geometric and harmonic means of \( r_1, \cdots, r_{n-v} \) and let \( b_k = |a_k/a_n| \) and \( \tau_k(\theta_0) = |P_k(\cos \theta_0)/P_n(\cos \theta_0)| \). If \( P_n(\cos \theta_0)P_{v+1}(\cos \theta_0) \neq 0 \) and \( a_n a_{v+1} \neq 0 \), then

\[
M_a = (n - v)^{-1} b_{n-1} \tau_{n-1}(\theta_0),
M_g = [b_v \tau_v(\theta_0)]^{(n-v)},
M_h = (n - v) (b_v/b_{v+1}) [\tau_v(\theta_0)/\tau_{v+1}(\theta_0)].
\]

Bounds on the circles of intersection having maximum and minimum radii are found in

**Corollary 8.1.** The maximum circle of intersection of the equipotential surface \( L_a(H) \) and the cone \( \theta = \theta_0 \) lies exterior to the sphere about the origin with a radius \( \max \{M_a, M_h\} \) and the minimum circle of intersection not on the origin lies interior to the sphere about the origin with a radius \( \min \{M_a, M_h\} \).

When contact at the origin is zeroth order, from the facts that the distances \( r_1, \cdots, r_{n-v} \) are positive, \( P_0(\cos \theta) = 1, P_1(\cos \theta) = \cos \theta \) and equations (11) we deduce

**Corollary 8.2.** For an equipotential surface \( L_a(H) \) having zeroth order contact with the cone \( \theta = \theta_0 \) at the origin to intersect this cone in \( n \) finite circles, it is necessary that \( 0 \leq \theta_0 < \pi/2 \) if \( a_1/a_0 < 0 \) and \( \pi/2 < \theta_0 < \pi \) if \( a_1/a_0 > 0 \).

**References**


Received October 3, 1974. Research of the first author partly supported by a grant of the University of Wisconsin–Milwaukee Graduate School from W.A.R.F. funds.

**California Polytechnic State University — San Luis Obispo**

**and**

**U.S. Naval Academy, Annapolis**
<table>
<thead>
<tr>
<th>Author</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gerald A. Beer</td>
<td>Tax structures whose progressivity is inflation neutral</td>
<td>305</td>
</tr>
<tr>
<td>William M. Cornette</td>
<td>A generalization of the unit interval</td>
<td>313</td>
</tr>
<tr>
<td>David E. Evans</td>
<td>Unbounded completely positive linear maps on $C^*$-algebras</td>
<td>325</td>
</tr>
<tr>
<td>Hector O. Fattorini</td>
<td>Some remarks on convolution equations for vector-valued distributions</td>
<td>347</td>
</tr>
<tr>
<td>Amassa Courtney Fauntleroy</td>
<td>Automorphism groups of unipotent groups of Chevalley type</td>
<td>373</td>
</tr>
<tr>
<td>Christian C. Fenske and Heinz-Otto Peitgen</td>
<td>On fixed points of zero index in asymptotic fixed point theory</td>
<td>391</td>
</tr>
<tr>
<td>Atsushi Inoue</td>
<td>On a class of unbounded operator algebras. II</td>
<td>411</td>
</tr>
<tr>
<td>Herbert Meyer Kamowitz</td>
<td>The spectra of endomorphisms of algebras of analytic functions</td>
<td>433</td>
</tr>
<tr>
<td>Jimmie Don Lawson</td>
<td>Embeddings of compact convex sets and locally compact cones</td>
<td>443</td>
</tr>
<tr>
<td>William Lindgren and Peter Joseph Nyikos</td>
<td>Spaces with bases satisfying certain order and intersection properties</td>
<td>455</td>
</tr>
<tr>
<td>Emily Mann Peck</td>
<td>Lattice projections on continuous function spaces</td>
<td>477</td>
</tr>
<tr>
<td>Morris Marden and Peter A. McCoy</td>
<td>Level sets of polynomials in n real variables</td>
<td>491</td>
</tr>
<tr>
<td>Francis Joseph Narcowich</td>
<td>An imbedding theorem for indeterminate Hermitian moment sequences</td>
<td>499</td>
</tr>
<tr>
<td>John Dacey O’Neill</td>
<td>Rings whose additive subgroups are subrings</td>
<td>509</td>
</tr>
<tr>
<td>Chull Park and David Lee Skoug</td>
<td>Wiener integrals over the sets bounded by sectionally continuous barriers</td>
<td>523</td>
</tr>
<tr>
<td>Vladimir Scheffer</td>
<td>Partial regularity of solutions to the Navier-Stokes equations</td>
<td>535</td>
</tr>
<tr>
<td>Eugene Spiegel and Allan Trojan</td>
<td>On semi-simple group algebras. II</td>
<td>553</td>
</tr>
<tr>
<td>Katsuo Takano</td>
<td>On Cameron and Storvick’s operator valued function space integral</td>
<td>561</td>
</tr>
</tbody>
</table>