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CHULL PARK AND DAVID LEE SKOUG

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C. Park and D. L. Skoug

Let $C_W \equiv C[0, T]$ denote the Wiener space on $[0, T]$. The Wiener integrals of various functionals $F[x]$ over the space C_W are well-known. In this paper we establish formulas for the Wiener integrals of $F[x]$ over the subsets of C_W bounded by sectionally continuous functions.

1. Introduction. Let $C_W \equiv C[0, T]$ be the Wiener space on $[0, T]$, i.e., the space of all real-valued continuous functions on $[0, T]$ vanishing at the origin. The standard Wiener process $\{X(t) \equiv X(t, \cdot): 0 \leqq t \leqq T\}$ and C_W are related by $X(t, x) = x(t)$ for each x in C_W. Evaluation formulas for the Wiener integral

$$\int_{C_W} F[x] d_W x \equiv E\{F[x]\}$$

of various functionals $F[x]$ are of course well-known (for example see [7] for some of these formulas). Now, consider sets of the type

$$\Gamma_f \equiv \left\{ \sup_{0 \leqq t \leqq T} X(t) - f(t) < 0 \right\}$$

$$= \left\{ x \in C_W: \sup_{0 \leqq t \leqq T} x(t) - f(t) < 0 \right\}$$

where $f(t)$ is sectionally continuous on $[0, T]$ and $f(0) \geqq 0$.

It is well-known that for $b \geqq 0$

$$P[\Gamma_b] = 2\Phi(bT^{-1/2}) - 1$$

and

$$P[\Gamma_{at+b}] = \Phi[(aT + b)T^{-1/2}] - e^{-2ab}\Phi[(aT - b)T^{-1/2}]$$

where Φ is the standard normal distribution function. In [3], [5], and [6] more general functions $f(t)$ are considered and formulas given for the probabilities of the sets Γ_f.

The main purpose of this paper is to derive formulas for Wiener integrals over the sets Γ_f. In §2 we state and prove the main results, while in §3 we discuss some applications and examples.

2. Integration formulas. Our first theorem is preliminary; it plays a key role in the proof of Theorem 2.

THEOREM 1. *Let* $0 = t_0 < t_1 < \cdots < t_n = T$ *be a partition of* $[0, T]$. *For* $j = 1, 2, \cdots, n$ *let* $f_j(t)$ *be continuous on* $[t_{j-1}, t_j]$. *Then the conditional probability*

$$P\left[\sup_{t_{j-1} \leq t \leq t_j} X(t) - f_j(t) < 0, 1 \leq j \leq n \,\middle|\, X(t_j) = u_j, 1 \leq j \leq n \right]$$

$$= \prod_{j=1}^{n} P\left[\sup_{0 \leq t \leq \Delta t_j} X(t) - \{f_j(t + t_{j-1}) - u_{j-1}\} < 0 \,\middle|\, X(\Delta t_j) = \Delta u_j \right]$$

$$= \prod_{j=1}^{n} P\left(\sup_{0 \leq t < \infty} X(t) - \left\{ \frac{t + t\Delta t_j}{\Delta t_j} \left[f_j\left(\frac{\Delta t_j}{1 + t\Delta t_j} + t_{j-1} \right) - u_{j-1} \right] - \frac{\Delta u_j}{\Delta t_j} \right\} < 0 \right)$$

where $\Delta t_j = t_j - t_{j-1}$ *and* $\Delta u_j = u_j - u_{j-1}$ *with* $u_0 = 0$.

Proof. First we note that

$$P\left[\sup_{t_{j-1} \leq t \leq t_j} X(t) - f_j(t) < 0, 1 \leq j \leq n \,\middle|\, X(t_j) = u_j, 1 \leq j \leq n \right]$$

$$= P\left\{ \sup_{t_{j-1} \leq t \leq t_j} X(t) - X(t_{j-1}) - [f_j(t) - u_{j-1}] < 0, \right.$$

$$\left. 1 \leq j \leq n \,\middle|\, X(t_j) - X(t_{j-1}) = \Delta u_j, 1 \leq j \leq n \right\}.$$

Now since the Wiener process has independent increments, the above expression equals

$$\prod_{j=1}^{n} P\left\{ \sup_{t_{j-1} \leq t \leq t_j} X(t) - X(t_{j-1}) - [f_j(t) - u_{j-1}] < 0 \,\middle|\, X(t_j) - X(t_{j-1}) = \Delta u_j \right\}.$$

Hence the first equality in the theorem follows from the fact that stationarity implies that $X(t) - X(t_{j-1})$ is the same process as $X(t - t_{j-1})$. To prove the second equality in the theorem, we note that $X(t)$ and $tX(1/t)$ are identical Wiener processes for $t > 0$ by checking the covariance function. Thus

$$P\left\{ \sup_{0 \leq t \leq \Delta t_j} X(t) - [f_j(t + t_{j-1}) - u_{j-1}] < 0 \,\middle|\, X(\Delta t_j) = \Delta u_j \right\}$$

$$= P\left\{ \sup_{0 < t \leq \Delta t_j} X\left(\frac{1}{t}\right) - \frac{1}{t} [f_j(t + t_{j-1}) - u_{j-1}] < 0 \,\middle|\, X\left(\frac{1}{\Delta t_j}\right) = \frac{\Delta u_j}{\Delta t_j} \right\}$$

$$= P\left\{ \sup_{0 < t \leq \Delta t_j} X\left(\frac{1}{t}\right) - X\left(\frac{1}{\Delta t_j}\right) - \frac{1}{t} [f_j(t + t_{j-1}) - u_{j-1}] + \frac{\Delta u_j}{\Delta t_j} < 0 \right\}.$$

The result now follows by the transformation

$$t^{-1} - (\Delta t_j)^{-1} \to t.$$

THEOREM 2. *Let* $0 = t_0 < t_1 < \cdots < t_n = T$ *be a partition of* $[0, T]$. *Let* $g(u_1, \cdots, u_n)$ *be a Lebesgue measurable function on* R^n *and for* $x \in C_w$ *let* $G[x] = g(x(t_1), \cdots, x(t_n))$. *For* $j = 1, 2, \cdots, n$ *let* $f_j(t)$ *be a continuous function on* $[t_{j-1}, t_j]$. *Then the Wiener integral of* $G[x]$ *over the set*

$$\Gamma_f \equiv \left\{ x \in C_w : \sup_{t_{j-1} \leqq t \leqq t_j} x(t) - f_j(t) < 0, j = 1, 2, \cdots, n \right\}$$

is given by

$$\int_{\Gamma_f} G[x] d_w x = \int_{-\infty}^{\lambda_1} \cdot (n) \cdot \int_{-\infty}^{\lambda_n} g(u_1, \cdots, u_n) H(u_1, \cdots, u_n) du_n \cdots du_1$$

in the sense the existence of either side implies that of their equality, where

$$\lambda_j = \begin{cases} \min\{f_j(t_j), f_{j+1}(t_j)\}, & j = 1, 2, \cdots, n-1 \\ \\ f_n(T), & j = n \end{cases}$$

$$f(t) = \begin{cases} f_j(t), & t_{j-1} < t < t_j, & j = 1, 2, \cdots, n \\ f_1(0), & t = 0 \\ \lambda_j, & t = t_j, & j = 1, 2, \cdots, n \end{cases},$$

and

$$H(u_1, \cdots, u_n) = \prod_{j=1}^{n} (2\pi \Delta t_j)^{-1/2} \exp\{-(\Delta u_j)^2/(2\Delta t_j)\}$$

$$P\left[\sup_{0 \leqq t \leqq \Delta t_j} X(t) - \{f_j(t + t_{j-1}) - u_{j-1}\} < 0 \,|\, X(\Delta t_j) = \Delta u_j \right].$$

Proof. First consider the case where $G[x]$ is the characteristic function of a Wiener interval I. That is to say I has the form

$$I = \{x \in C_w \,|\, [x(t_1), \cdots, x(t_n)] \in E\}$$

for some Lebesgue measurable set E in R^n. Then $G[x] = \chi_I(x) = \chi_E[x(t_1), \cdots, x(t_n)]$ and so in this case

$$\int_{\Gamma_f} G[x]d_w x = \int_{\Gamma_f} \chi_I(x)d_w x = P[\Gamma_f \cap I]$$

$$= \int_{-\infty}^{\lambda_1} \cdot (n) \cdot \int_{-\infty}^{\lambda_n} P\left\{ \sup_{t_{j-1} \le t \le t_j} X(t) - f_j(t) < 0, \; 1 \le j \le n, \right.$$

$$\left. [x(t_1), \cdots, x(t_n)] \in E \,|\, x(t_j) = u_j, \; 1 \le j \le n \right\} K(\vec{t}, \vec{u})du_n \cdots du_1$$

where

$$K(\vec{t}, \vec{u}) \equiv \prod_{j=1}^{n} (2\pi \Delta t_j)^{-1/2} \exp\{-(\Delta u_j)^2/(2\Delta t_j)\}.$$

Next we observe that

$$P\left\{ \sup_{t_{j-1} \le t \le t_j} X(t) - f_j(t) < 0, \; 1 \le j \le n, [x(t_1), \cdots, x(t_n)] \in E \,| \right.$$

$$\left. x(t_j) = u_j, \; 1 \le j \le n \right\} = \chi_E(u_1, \cdots, u_n) P\left[\sup_{t_{j-1} \le t \le t_j} X(t) - f_j(t) < 0, \right.$$

$$\left. 1 \le j \le n \,|\, x(t_j) = u_j, \; 1 \le j \le n \right].$$

Next, applying Theorem 1 to the last conditional probability above gives the desired result for this case.   The general case follows by the usual arguments in integration theory.

THEOREM 3.   *Let* $f(t)$ *be sectionally continuous on* $[0, T]$ *with* $f(0) \ge 0$.   *Let* $g(u_1, \cdots, u_n)$ *be a Lebesgue measurable function on* R^n, *and let* $\alpha(t) \in BV[0, T]$.   *Then*

$$\int_{\Gamma_f} g[x(t_1), \cdots, x(t_n)]e^{\int_0^T \alpha(t)dx(t)} d_w x$$

$$= e^{1/2\int_0^T \alpha^2(t)dt} \int_{\Gamma_{f(t)-\int_0^t \alpha(s)ds}} g\left[x(t_1) + \int_0^{t_1} \alpha(s)ds, \cdots, x(t_n) \right.$$

$$\left. + \int_0^{t_n} \alpha(s)ds \right] d_w x$$

in the sense the existence of either side implies that of the other and their equality.

COROLLARY.   *If* f *and* α *satisfy the conditions in Theorem 3, then*

$$\int_{\Gamma_f} e^{\int_0^T \alpha(t)dx(t)} d_w x = e^{1/2 \int_0^T \alpha^2(t)dt} P\left\{ \sup_{0 \le t \le T} X(t) - \left[f(t) - \int_0^t \alpha(s)ds \right] < 0 \right\}.$$

Proof (of Theorem 3).   Using the Cameron–Martin translation theorem (see **[1]** or **[7]**) with the translation

$$x(t) \rightarrow x(t) + \int_0^t \alpha(u)\,du$$

we obtain

$$\int_{\Gamma_f} g[x(t_1), \cdots, x(t_n)]e^{\int_0^T \alpha(t)dx(t)}\,d_w x$$

$$= e^{-1/2 \int_0^T \alpha^2(t)dt} \int_{\Gamma_{f(t)} - \int_0^t \alpha(s)ds} g\left[x(t_1) + \int_0^{t_1} \alpha(u)du, \cdots, x(t_n) + \int_0^{t_n} \alpha(u)du\right]$$

$$\cdot e^{\int_0^T \alpha(t)d[x(t) + \int_0^t \alpha(u)du]} e^{-\int_0^T \alpha(t)dx(t)}\,d_w x.$$

The result now follows by simplifying the last expression.

THEOREM 4.   *Assume that $g(z) = \sum_0^\infty a_n z^n$ is an entire function such that for some M, N and $\gamma \in (0,2)$, $|g(z)| \leq M \exp(N|z|^\gamma)$ for all complex numbers z. For some $r \in (2,\infty]$ and $b > 0$ assume that $\theta(t,u) \in L_{1r}([0,T] \times (-\infty, b])$, i.e., $\int_{-\infty}^b |\theta(t,u)|\,du \in L_r[0,T]$.   Then for any $\psi(u) \in L_1(-\infty, b] \cup L_\infty(-\infty, b]$,*

(1) $$\int_{\Gamma_b} g\left[\int_0^T \theta(t, x(t))dt\right] \psi(x(T))d_w x = \sum_0^\infty a_n J_n(T)n!$$

where for $n = 0, 1, 2, \cdots$

$$J_n(T) \equiv \int_0^{t_{n+1}} \cdot (n) \cdot \int_0^{t_2} \int_{-\infty}^b \cdot (n+1) \cdot \int_{-\infty}^b \psi(u_{n+1}) \prod_{j=1}^n \theta(t_j, u_j)$$

(2) $$\prod_{j=1}^{n+1} (2\pi \Delta t_j)^{-1/2} \exp\{-(\Delta u_j)^2/(2\Delta u_j)\}$$

$$\cdot [1 - \exp\{-2(b - u_{j-1})(b - u_j)/\Delta t_j\}]$$

$$\cdot du_{n+1} \cdots du_1 dt_1 \cdots dt_n$$

and where $u_0 \equiv 0$, and $0 = t_0 < t_1 < \cdots < t_{n+1} = T$.

Proof.   Proceeding formally we obtain

$$\int_{\Gamma_b} g\left[\int_0^T \theta(t, x(t))dt\right] \psi(x(T))d_w x$$

$$= \sum_0^\infty a_n \int_{\Gamma_b} \left[\int_0^T \theta(t, x(t))dt\right]^n \psi(x(T))d_w x.$$

But for $n = 1, 2, \cdots$

$$\int_{\Gamma_b} \left[\int_0^T \theta(t, x(t)) dt \right]^n \psi(x(T)) d_w x$$

$$= \int_{\Gamma_b} \left[\int_0^T \cdot (n) \cdot \int_0^T \prod_{j=1}^n \theta(t_j, x(t_j)) dt_1 \cdots dt_n \right] \psi(x(T)) d_w x$$

$$= n! \int_0^{t_{n+1}} \cdot (n) \cdot \int_0^{t_2} \left[\int_{\Gamma_b} \psi(x(t_{n+1})) \prod_{j=1}^n \theta(t_j, x(t_j)) d_w x \right] dt_1 \cdots dt_n$$

$$= n! \int_0^{t_{n+1}} \cdot (n) \cdot \int_0^{t_2} \int_{-\infty}^b \cdot (n+1) \cdot \int_{-\infty}^b \psi(u_{n+1}) \prod_{j=1}^n \theta(t_j, u_j) \prod_{j=1}^{n+1} \{ (2\pi \Delta t_j)^{-1/2}$$

$$\cdot \exp \left[-(\Delta u_j)^2/(2\Delta t_j) \right] [1 - \exp(-2(b - u_{j-1})(b - u_j)/\Delta t_j)] \}$$

$$\cdot du_{n+1} \cdots du_1 dt_1 \cdots dt_n$$

where the last equality above is obtained using Theorem 2, Theorem 1, and the fact that

$$P \left(\sup_{0 \leq t < \infty} X(t) - \{ (b_j - u_{j-1}) t + (b_j - u_j)/\Delta t_j \} < 0 \right)$$

$$= 1 - \exp[-2(b - u_{j-1})(b - u_j)/\Delta t_j].$$

Thus proceeding formally we have obtained equation (1). The Theorem follows readily once the absolute convergence of the series

$$\sum_0^\infty n! \, a_n J_n(T)$$

is established.

Recall that $r \in (2, \infty]$. We will establish the absolute convergence when $2 < r < \infty$ and $\psi \in L_1(-\infty, b]$; then other cases are similar, but easier. Let p satisfy $1/r + 1/p = 1$. Then $1 < p < 2$ and by Hölder's inequality we obtain

$$|J_n(T)| \leq \| \psi \|_1 \int_0^{t_{n+1}} \cdot (n) \cdot \int_0^{t_2} \prod_{j=1}^n \| \theta(t_j, \cdot) \|_1 \prod_{j=1}^{n+1} (2\pi \Delta t_j)^{-1/2} dt_1 \cdots dt_n$$

$$\leq \| \psi \|_1 \left\{ \int_0^{t_{n+1}} \cdot (n) \cdot \int_0^{t_2} [t_1(t_2 - t_1) \cdots (t_{n+1} - t_n)]^{-p/2} dt_1 \cdots dt_n \right\}^{1/p}$$

$$\cdot \left\{ \int_0^{t_{n+1}} \cdot (n) \cdot \int_0^{t_2} \prod_{j=1}^n \| \theta(t_j, \cdot) \|_1^r dt_1 \cdots dt_n \right\}^{1/r} (2\pi)^{-(n+1)/2}.$$

But

$$\left\{\int_0^{t_{n+1}} \cdot (n) \cdot \int_0^{t_2} \prod_{j=1}^n \| \theta(t_j, \cdot) \|_1^r \, dt_1 \cdots dt_n \right\}^{1/r}$$

$$= \left\{ \frac{1}{n!} \int_0^T \cdot (n) \cdot \int_0^T \prod_{j=1}^n \| \theta(t_j, \cdot) \|_1^r \, dt_1 \cdots dt_n \right\}^{1/r}$$

$$= \left(\frac{1}{n!} \right)^{1/r} \left(\int_0^T \| \theta(t, \cdot) \|_1^r \, dt \right)^{n/r}$$

$$= \left(\frac{1}{n!} \right)^{1/r} \| \theta \|_{1r}^n.$$

In addition

$$\int_0^{t_{n+1}} \cdot (n) \cdot \int_0^{t_2} [t_1(t_2 - t_1) \cdots (t_{n+1} - t_n)]^{-p/2} \, dt_1 \cdots dt_n$$

$$= \frac{T^{n(2-p)/2} \{\Gamma(1 - p/2)\}^{n+1}}{T^{p/2} \Gamma[(n+1)(1 - p/2)]}$$

where $\Gamma(z)$ denotes the Gamma function.

Thus the series $\Sigma_0^\infty |a_n J_n(T)| n!$ is dominated by the series

(3)
$$\sum_0^\infty \frac{n! |a_n| (n!)^{-1/r} \| \theta \|_{1r}^n T^{n(2-p)/2p} \{\Gamma(1 - p/2)\}^{(n+1)/p}}{T^{1/2} \{\Gamma[(n+1)(1 - p/2)]\}^{1/p}}$$

$$= \sum_0^\infty \frac{|a_n| \| \theta \|_{1r}^n T^{n(2-p)/2p} \{\Gamma(1 - p/2)\}^{(n+1)/p}}{T^{1/2}} \left\{ \frac{n!}{\Gamma[(n+1)(1 - p/2)]} \right\}^{1/p}.$$

But since $g(z)$ is an entire function of order at most γ we know that

$$\limsup_{n \to \infty} \left(\frac{n \ln n}{- \ln |a_n|} \right) \leqq \gamma < \frac{\gamma + 2}{2}$$

and so for n sufficiently large we obtain that

$$|a_n| < n^{-2n/(\gamma+2)}.$$

But $\Gamma(z) = z^{z - 1/2} e^{-z} (2\pi)^{1/2} (1 + 0(1))$ and hence for positive z sufficiently large

$$\frac{1}{\Gamma(z)} < \frac{2e^z z^{1/2}}{(2\pi)^{1/2} z^z}.$$

Also by Stirling's formula

$$n! \leq (n/e)^n (2\pi n)^{1/2} \exp\left(\frac{1}{12n}\right).$$

Thus for n sufficiently large we obtain

$$|a_n| \left\{\frac{n!}{\Gamma[(n+1)(1-p/2)]}\right\}^{1/p}$$

(4)
$$\leq 2^{1/p} \exp\left(\frac{12n+1}{12np}\right) n^{1/2} e^{-(n+1)/2} \left(\frac{2}{2-p}\right)$$
$$\times [(n+1)(2-p)-1]/2p_n^{-n(2-\gamma)/2(\gamma+2)}.$$

Now using inequality (4) the convergence of the series (3) follows by the root test.

COROLLARY 1.   *Let $\theta(t, u)$ be as in Theorem 4.   Then*

$$\int_{\Gamma_b} \exp\left[\int_0^T \theta(s, x(s))ds\right] d_W x = \sum_0^\infty J_n(T)$$

where $J_n(T)$ is given by (2) with $\psi \equiv 1$.

COROLLARY 2.   *Let $\alpha(t)$ be of bounded variations on $[0, T]$.   Then for any $b > 0$,*

$$\int_{\Gamma_b} e^{\int_0^T \alpha(t)dx(t)} d_W x = \sum_0^\infty (-1)^n K_n(T)$$

where

$$K_n(T) \equiv \int_0^{t_{n+1}} \cdot (n) \cdot \int_0^{t_2} \int_{-\infty}^b \cdot (n+1) \cdot \int_{-\infty}^b e^{\alpha(T)u_{n+1}} \prod_{j=1}^n u_j \prod_{j=1}^{n+1} (2\pi\Delta t_j)^{-1/2}$$

(5)
$$\cdot \exp\left[-(\Delta u_j)^2/(2\Delta t_j)\right]\{1 - \exp\left[-2(b - u_{j-1})(b - u_j)/\Delta t_j\right]\}$$
$$\cdot du_{n+1} \cdots du_1 d\alpha(t_1) \cdots d\alpha(t_n).$$

Proof (of Corollary 2).   The Corollary follows quite readily once the absolute convergence of the series $\sum_0^\infty (-1)^n K_n(T)$ is established.   Now proceeding formally we see that

$$\int_{\Gamma_b} e^{\int_0^T \alpha(t)dx(t)} d_W x = \int_{\Gamma_b} e^{\alpha(T)x(T) - \int_0^T x(t)d\alpha(t)} d_W x$$

$$= \int_{\Gamma_b} \sum_{n=0}^\infty \frac{(-1)^n}{n!} e^{\alpha(T)x(T)} \left[\int_0^T x(t)d\alpha(t)\right]^n d_W x$$

and so

$$|K_n(T)| = \left| \frac{1}{n!} \int_{\Gamma_b} e^{\alpha(T)x(T)} \left[\int_0^T x(t)d\alpha(t) \right]^n d_W x \right|$$

$$\leqq \frac{1}{n!} \int_{C_W} e^{\alpha(T)x(T)} \left| \int_0^T x(t)d\alpha(t) \right|^n d_W x.$$

Thus

$$\sum_0^\infty |K_n(T)| \leqq \sum_0^\infty \frac{1}{n!} \int_{C_W} e^{\alpha(T)x(T)} \left| \int_0^T x(t)d\alpha(t) \right|^n d_W x$$

$$= \int_{C_W} e^{\alpha(T)x(T)} e^{|\int_0^T x(t)d\alpha(t)|} d_W x$$

$$\leqq \int_{C_W} e^{2|\alpha(T)x(T)|} e^{|\int_0^T \alpha(t)dx(t)|} d_W x$$

$$\leqq \left[\int_{C_W} e^{4|\alpha(T)x(T)|} d_W x \right]^{1/2} \left[\int_{C_W} e^{2|\int_0^T \alpha(t)dx(t)|} d_W x \right]^{1/2}$$

$$< \infty \quad \text{since} \quad \alpha(t) \in L^2[0, T].$$

3. Applications and examples.

A. *Application* 1. For our first application we obtain a formula for the probability that a Wiener path always stays below the broken line segments $f_j(t) = a_j t + b_j$, $t_{j-1} \leqq t \leqq t_j$, $1 \leqq j \leqq n$, where $b_1 > 0$. Using Theorems 1 and 2 we obtain

$$P\left[\sup_{t_{j-1} \leqq t \leqq t_j} X(t) - (a_j t + b_j) < 0, 1 \leqq j \leqq n \right]$$

$$= \int_{-\infty}^{\lambda_1} \cdot (n) \cdot \int_{-\infty}^{\lambda_n} \prod_{j=1}^n P\left(\sup_{0 \leqq t < \infty} X(t) - \left\{ \frac{1 + t\Delta t_j}{\Delta t_j} \left[a_j \left(\frac{\Delta t_j}{1 + t\Delta t_j} + t_{j-1} \right) \right. \right. \right.$$

$$\left. \left. \left. + b_j - u_{j-1} \right] - \frac{\Delta u_j}{\Delta t_j} \right\} < 0 \right) K(\vec{t}, \vec{u}) du_n \cdots du_1$$

where

$$\lambda_j = \begin{cases} \min\{a_j t_j + b_j, a_{j+1} t_j + b_{j+1}\}, & 1 \leqq j < n \\ \\ a_n T + b_n, & j = n \end{cases}$$

and

$$K(\vec{t}, \vec{u}) = \prod_{j=1}^{n} (2\pi\Delta t_j)^{-1/2} \exp\{-(\Delta u_j)^2/(2\Delta t_j)\}.$$

But the probability in the integrand simplifies into

$$P\left(\sup_{0 \le t < \infty} X(t) - \{(a_j t_{j-1} + b_j - u_{j-1})t + (a_j t_j + b_j - u_j)/\Delta t_j\} < 0\right)$$

which, using Doob [2, p. 397], equals the expression

$$[1 - \exp\{-2(a_j t_{j-1} + b_j - u_{j-1})(a_j t_j + b_j - u_j)/\Delta t_j\}].$$

Thus, we finally obtain the formula

$$P\left[\sup_{t_{j-1} \le t \le t_j} X(t) - (a_j t + b_j) < 0, 1 \le j \le n\right]$$

$$= \int_{-\infty}^{\lambda_1} \cdot (n) \cdot \int_{-\infty}^{\lambda_n} \prod_{j=1}^{n} [1 - \exp\{-2(a_j t_{j-1} + b_j - u_{j-1})(a_j t_j + b_j - u_j)/\Delta t_j\}]$$

$$\cdot [2\pi\Delta t_j]^{-1/2} \exp\{-(\Delta u_j)^2/(2\Delta t_j)\} du_n \cdots du_1.$$

   B.   Example 1.   For $j = 1, 2, \cdots, n$ let the a_j and b_j be as above.   Let

$$E \equiv \left\{x \in C_W \;\middle|\; \sup_{t_{j-1} \le t \le t_j} X(t) - (a_j t + b_j) < 0, 1 \le j \le n\right\}.$$

Let $g(u_1, \cdots, u_n)$ be Lebesgue measurable on R^n.   Then

$$\int_E g(x(t_1), \cdots, x(t_n)) d_W x$$

$$= \int_{-\infty}^{\lambda_1} \cdot (n) \cdot \int_{-\infty}^{\lambda_n} g(u_1, \cdots, u_n) \prod_{j=1}^{n} [2\pi\Delta t_j]^{-1/2} \exp\{-(\Delta u_j)^2/(2\Delta t_j)\}$$

$$\cdot \prod_{j=1}^{n} [1 - \exp\{-2(a_j t_{j-1} + b_j - u_{j-1})(a_j t_j + b_j - u_j)/\Delta t_j\}] du_n \cdots du_1$$

in the sense the existence of either side implies that of the other and their equality.

   C.   Application 2.   Assume $\alpha(t)$ is of bounded variation on $[0, T]$ and let $f(t) \equiv \int_0^t \alpha(s) ds$.   For $b > 0$ we want to find the probability of the set

$$\Gamma_{f(\cdot)+b} = \left\{ \sup_{0 \leq t \leq T} X(t) - f(t) < b \right\}.$$

First, using the Corollary to Theorem 3, we have that

$$P\left[\Gamma_{f(\cdot)+b}\right] = e^{-1/2 \int_0^T \alpha^2(t)dt} \int_{\Gamma_b} e^{-\int_0^T \alpha(t)dx(t)} d_W x.$$

Next, using Corollary 2 of Theorem 4, we obtain

$$P\left[\Gamma_{f(\cdot)+b}\right] = e^{-1/2 \int_0^T \alpha^2(t)dt} \sum_{n=0}^{\infty} K_n(T)$$

where $K_n(T)$ is given by equation (5). This expression is an entirely different series expansion of the probability than the ones given by Park and Paranjape in [5].

   D. *Application* 3. The Corollary to Theorem 3 is also useful to evaluate integrals of the type

$$\int_{\Gamma_f} e^{\int_0^T \alpha(t)dx(t)} d_W x$$

numerically for given $\alpha(t)$, $f(t)$, and T. By the Corollary, the above integral is equal to

$$e^{1/2 \int_0^T \alpha^2(t)dt} P\left\{ \sup_{0 \leq t \leq T} X(t) - \left[f(t) - \int_0^t \alpha(s)ds\right] < 0 \right\},$$

and the last probability can be evaluated numerically using the Park–Schuurmann method [6].

   The following table was computed by an IBM/168 with the unit interval divided into 2^9 equal subintervals.

Estimates of $\int_{\Gamma_f} e^{\int_0^1 \alpha(t)dx(t)} d_W x$

$\alpha(t)$	$\sin t$	e^t	t	$\sqrt{t}$
$f(t)$	$t+1$	t^2+t+1	$\cos t$	$ln(t+1)+1$
The integral	.976414	3.729278	.467819	.939285

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