ON CAMERON AND STORVICK’S OPERATOR VALUED FUNCTION SPACE INTEGRAL

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In this paper "the probability density of path space" is introduced by the formula

\[ p_\alpha^*(t, u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left( -iu\eta - \frac{t}{\lambda} |\eta|^\alpha \right) d\eta, \quad (\alpha > 0). \]

If \( \alpha = 2 \) and \( \lambda > 0 \), \( p_2^*(t, u) \) is the normal probability density. But if \( \alpha > 2 \) this density cannot be considered as a probability density. By this generalization, one can generalize the operator valued function space integral based on the Wiener integral.

**Introduction.** Cameron and Storvick introduced the operator valued function space integral in [1]. Johnson and Skoug [9] developed Cameron and Storvick's theory and improved the results obtained in [1]. To make the arguments in the following sections comprehensible, we will quote the operators \( I_1^\sigma(F) \) and \( I_\lambda^\sigma(F) \) from [1], which played an important role in [1], [9]. Let \( B[a, b] \) denote the space of real valued functions on an interval \([a, b]\) which are continuous except for a finite number of finite jump discontinuities. Let \( F(x) \) be a functional on \( B[a, b] \) and \( \psi \in L_2(-\infty, \infty) \), \( \xi \in (-\infty, \infty) \). Then for \( \text{Re} \lambda > 0 \) and any partition \( \sigma: a = t_0 < t_1 < \cdots < t_n = b \), the operator \( I_1^\sigma(F) \) is defined by the formula

\[
(I_1^\sigma(F)\psi)(\xi) = \lambda^{n/2}[(2\pi)^n(t_1 - a) \cdots (t_n - t_{n-1})]^{-1/2} \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} \psi(v_n) \cdot f_\sigma(\xi, v_1, \cdots, v_n) \exp \left( - \sum_{j=1}^{n} \frac{\lambda(v_j - v_{j-1})^2}{2(t_j - t_{j-1})} \right) dv_1 \cdots dv_n
\]

(0.1)

where \( v_0 = \xi, f_\sigma(\xi, v_1, \cdots, v_n) = F[z(\sigma, \xi, v_1, \cdots, v_n, \cdot)] \),

\[
z(\sigma, \xi, v_1, \cdots, v_n, t) = \begin{cases} v_j & \text{if} \quad t_j \leq t < t_{j+1}, \quad j = 0, 1, \cdots, n - 1, \\ v_n & \text{if} \quad t = b, \end{cases}
\]

and where if \( n \) is odd we always choose \( \lambda^{n/2} \) with nonnegative real part. Here \( \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} \psi(v_n) \) means the \( n \)-fold integral. If \( \lambda > 0 \), by using the Wiener integral, this can be written as
\[(I^\tau_\sigma (F)\psi)(\xi) = \int_{C_{[a,b]}} F(\lambda^{-1/2}x\sigma + \xi)\psi(\lambda^{-1/2}x(b) + \xi)dx\]

where

\[x_\sigma(t) = \begin{cases} x(t_{j-1}) & \text{if } t_{j-1} \leq t < t_j, \\ x(b) & \text{if } t = b. \end{cases}\]

As an example important in quantum theory, the functional

\[F(x) = \exp \left\{ \int_a^b \theta(s, x(s))ds \right\} \]

is discussed in [1], [9]. In this case \(f_\sigma(\xi, v_1, \cdots, v_n)\) is given by

\[\exp \left\{ \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \theta(s, v_{j-1})ds \right\}.

Let us denote \(\max \{t_j - t_{j-1}\}\) by norm \(\sigma\) or \(\|\sigma\|\). For \(\text{Re}\lambda > 0\) the operator \(I^{\text{sec}}_\sigma(F)\) is defined by

\[(0.2) \quad I^{\text{sec}}_\sigma(F) = \lim_{\|\sigma\| \to 0} I^\sigma_\lambda(F),\]

where \(\text{w lim}\) refers to the limit with respect to the weak operator topology. [1] proved that for a class of functionals \(F\), \(I^{\text{sec}}_\sigma(F)\) exists by using the Wiener integral, and furthermore that \(I^{\text{sec}}_\sigma(F)\) converges in the weak operator topology as \(\lambda = p - iq \to +0 - iq\) for almost all \(q \neq 0\). [9] proved that for a class of functionals \(F\), \(I^{\text{sec}}_\sigma(F)\) exists as the strong operator limit, and furthermore that \(I^{\text{sec}}_\sigma(F)\) converges in the strong operator topology as \(\lambda = p - iq \to +0 - iq\) for all \(q \neq 0\) by using the analytic continuation of the Wiener integral. In this paper we introduce the following operator from \(L^2\) to \(L^2\) in §2 corresponding to (0.1),

\[(\mathcal{F}_\lambda^{\sigma\sigma}(F)\psi)(\xi) = \left| \int_{-\infty}^{\infty} p^{\sigma}_\lambda(t_1 - a, v_1 - \xi)dv_1 \int_{-\infty}^{\infty} p^{\sigma}_\lambda(t_2 - t_1, v_2 - v_1)dv_2 \right| \]

\[\cdots \int_{-\infty}^{\infty} p^{\sigma}_\lambda(t_n - t_{n-1}, v_n - v_{n-1})\psi(v_n)f_\sigma(v_1, v_2, \cdots, v_n)dv_n,\]

for \(\text{Re}\lambda \geq 0, \lambda \neq 0\). We note that if \(\text{Re}\lambda > 0\), \(p^{\lambda}_\sigma(t, u)\) and \(p^{\lambda\sigma}_\sigma(t, u)\) are especially given by
respectively.
In general \( p^*_\lambda(t, u) \) is given by the formula

\[
(0.4) \quad p^*_\lambda(t, u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left( -iu\eta - \frac{t}{\lambda} |\eta|^\alpha \right) d\eta
\]

for \( \alpha > 0, \ t > 0 \) and \( \text{Re}\ \lambda > 0 \). [1]–[3] and [6]–[10] used only (0.3). Even by this generalization we can also show that for a class of functionals \( F, \mathcal{F}_{\lambda}^{\alpha}(F) \) converges in the strong operator topology as norm \( \sigma \rightarrow 0 \). Furthermore we can show that the same integral equation as in [1] [9] holds.

1. **Stable density and semigroup.** The stable density of exponent \( \alpha \) is

\[
(1.1) \quad p^*_\alpha(t, u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left( -iu\eta - t |\eta|^\alpha \right) d\eta
\]

where \( 0 < \alpha \leq 2, \ 0 < t < \infty \). cf. [12].

It is well-known that (1.1) satisfies the Chapman and Kolmogorov equation, that is, when \( \lambda = 1 \)

\[
(1.2) \quad p^*_\lambda(t+s, u) = \int_{-\infty}^{\infty} p^*_\lambda(s, u-y)p^*_\lambda(t, y)dy.
\]

If we consider \( p^*_\lambda(t, u) \) in the operator’s sense in the Hilbert space \( L_2 \), (1.2) holds for \( \text{Re}\ \lambda > 0 \). We shall denote the Fourier transform of \( f \in L_2 \) by \( Uf \) and the inverse Fourier transform by \( U^*f \), that is,

\[
\begin{align*}
(a) \quad (Uf)(y) &= (2\pi)^{-1/2(\alpha)} \int_{-\infty}^{\infty} \exp(-iyx)f(x)dx, \\
(b) \quad (U^*f)(y) &= (2\pi)^{-1/2(\alpha)} \int_{-\infty}^{\infty} \exp(iyx)f(x)dx,
\end{align*}
\]

where \( (y) \) denotes the so-called limit in the mean.

In what follows, let us assume that \( \alpha > 0 \).
Let

$$\begin{equation}
(P^*_\lambda(t)f)(y) = U^* \left( \exp \left( -\frac{t}{\lambda} |\xi|^a \right) (Uf)(\xi) \right)(y),
\end{equation}$$

where $f \in L_2$ and $\lambda \in D = \{ \lambda : \Re \lambda \geq 0, \lambda \neq 0 \}$.

**Theorem 1.1.** $\{P^*_\lambda(t); t \geq 0\}$ is a strongly continuous semigroup of contraction operators on $L_2$. Furthermore $P^*_\lambda(t)$ is also strongly continuous with respect to $\lambda$ on $D$.

This follows from the fact that the Fourier transformation is a unitary operator on $L_2$.

**Lemma 1.1.** Let $\Re \lambda > 0$ and $t > 0$. Then

$$p^*_\lambda(t, \cdot)$$
is $L_2$-integrable,

$$|p^*_\lambda(t, u)| \leq M(\alpha, t, \lambda) < \infty,$$

and $p^*_\lambda(t, u)$ is continuous in $t$, $\lambda$ and $u$.

**Proof.** Let $\Re \lambda > 0$. Then

$$|p^*_\lambda(t, u)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \exp \left( -\frac{t}{\lambda} |\eta|^a \right) \right| d\eta.$$

Let $t/\lambda = \delta - i\gamma$, ($\delta > 0$). We obtain

$$|p^*_\lambda(t, u)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\delta |\eta|^a) d\eta = \frac{1}{\pi} \int_{0}^{\infty} \exp(-\delta \eta^a) d\eta$$

$$= \frac{1}{\pi} \beta \delta^{-\beta} \Gamma(\beta) < \infty,$$

where $\beta = 1/\alpha$ and $\Gamma(\beta)$ is the gamma function. On using the Dominated Convergence Theorem, we can show that $p^*_\lambda(t, u)$ is continuous in $t$, $\lambda$ and $u$. From the above proof, clearly it holds that

$$\begin{equation}
(2\pi)^{-1/2} \exp \left( -\frac{t}{\alpha} |\eta|^a \right) \in L_p, \quad (p = 1, 2, \cdots).
\end{equation}$$

We may write (0.4) as

$$\begin{equation}
p^*_\lambda(t, u) = U \left[ (2\pi)^{-1/2} \exp \left( -\frac{t}{\alpha} |\cdot|^a \right) \right](u).
\end{equation}$$
Since the Fourier transformation is a unitary operator on $L_2$, $p^*_\lambda(t, u)$ is $L_2$-integrable in $u$.

**Lemma 1.2.** Let $\text{Re } \lambda > 0$ and $t > 0$. Then $p^*_\lambda(t, \cdot)$ is $L_1$-integrable.

**Proof.** Let

$$f(\xi) = (2\pi)^{-1/2} \exp \left( -\frac{t}{\lambda} |\xi|^\alpha \right).$$

The derivative of $f(\xi)$ is

$$f'(\xi) = (2\pi)^{-1/2} \exp \left( -\frac{t}{\lambda} |\xi|^\alpha \right) \left( -\frac{t}{\lambda} |\xi|^{\alpha-1} \right) \alpha \text{ sgn } \xi,$$

where

$$\text{sgn } \xi = \begin{cases} -1 & \text{if } \xi < 0, \\ 1 & \text{if } \xi > 0. \end{cases}$$

Then since $f$ is absolutely continuous on each bounded interval and $f'$ is absolutely integrable, it holds that $(Uf')(\xi) = i\xi(Uf)(\xi)$. Therefore in order to show that $Uf$ is $L_1$-integrable, by [13. 12.42. p. 382], it is sufficient to show that $Uf' \in L_q(1, +\infty)$ and $(Uf')(-\cdot) \in L_q(1, +\infty)$ since $1/\xi \in L_p(1, +\infty)$, where $1/p + 1/q = 1$, $p, q > 1$. With respect to $f'$, it holds that

$$\int_{-\infty}^{\infty} |f'(\xi)|^p d\xi = 2(2\pi)^{-p/2} \alpha^{p-1} \left| \frac{t}{\lambda} \right|^{p-q} \int_0^\infty r^{(\beta-1)(1-p)} \exp(-\delta r) d\tau,$$

where $1/\alpha = \beta$ and $\text{Re } (t/\lambda) = \delta$. Hence $f'$ is $L_p$-integrable if $(\beta - 1)(1 - p) < -1$ holds. For a fixed $\alpha$, there always exists a number $p$ in the interval $(1, 2]$ such that $(\beta - 1)(1 - p) > -1$. Therefore we can consider that $f'$ is $L_p$-integrable for some $p(>1)$. When $f'$ is $L_p$-integrable, by using [14. Theorem (3.2) p. 254], we obtain that

$$F(\xi, a) = (2\pi)^{-1/2} \int_{-a}^{a} \exp(-i\xi x) f'(x) dx$$

converges in mean with exponent $q$ as $a \to \infty$. Since $f'$ is $L_1$-integrable, $F(\xi, a)$ also converges for all $\xi$ as $a \to \infty$. Then it follows from [13. 12.5 12.51] that the pointwise limit is equal to the limit in mean with exponent...
for almost all $\xi$. Therefore $Uf'$ is $L_q$-integrable, hence $(Uf')(\xi)$ and $(Uf')(\xi)$ are in $L_q(1, + \infty)$.

**Lemma 1.3.** Let $\Re A > 0$ and $t > 0$. Then for $f \in L_2$

$$\int_{-\infty}^{\infty} p^*_\lambda(t, u - y)f(u)du$$

is $L_2$-integrable and continuous in $y$.

**Proof.** Let us put

$$g(y) = \int_{-\infty}^{\infty} p^*_\lambda(t, u - y)f(u)du.$$  

By Lemma 1.1 and the Schwartz inequality, we have

$$|g(y) - g(y + h)| \leq \|f\| \left[ \int_{-\infty}^{\infty} |p^*_\lambda(t, u - y) - p^*_\lambda(t, u - y - h)|^2 du \right]^{1/2}.$$  

Hence from [13, 19, p. 397], it follows that $g(y)$ is continuous in $y$. It holds that

$$\|g\|^2 \leq \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |p^*_\lambda(t, x - y)||f(x)|dx \right) \left( \int_{-\infty}^{\infty} |p^*_\lambda(t, u - y)||f(u)|du \right) dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} |p^*_\lambda(t, x - y)||p^*_\lambda(t, u - y)||f(x)||f(u)|du \right] dxdy.$$  

Let us make a change of variables, $x = x, x - y = z, u - y = v$. Then we have

$$\frac{\partial}{\partial (x, v, z)} = 1.$$  

Hence we obtain by Lemma 1.2 that

$$\|g\|^2 \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |p^*_\lambda(t, z)||p^*_\lambda(t, v)||f(x)||f(v + x - z)|dxdvdz$$

(1.6)

$$\leq \|f\|\|p^*_\lambda(t, \cdot)\|_1,$$

where $\| \cdot \|_1$ denotes $L_1$-norm.

**Theorem 1.2.** Let $\Re \lambda > 0$ and $t > 0$. Then for $f \in L_2$
(1.7) \( (P_x^a(t)f)(y) = \int_{-\infty}^{\infty} p_x^a(t, u - y)f(u)du. \)

Proof. Case I. Let \( f \in L_1 \cap L_2 \). Then \( Uf \in L_2 \). By this and (1.4), we can change the order of integration of (1.3) by the Fubini Theorem, therefore it follows from this fact that

\[
(P_e^a(t)f)(y) = \int_{-\infty}^{\infty} p_e^a(t, u - y)f(u)du.
\]

Case II. Let \( f \in L_2 \). Then we put

\[
f_n(x) = \begin{cases} f(x) & \text{if } |x| \leq n, \\ 0 & \text{if } |x| > n. \end{cases}
\]

Since \( f_n \in L_1 \cap L_2 \), it holds that

\[
(P_e^a(t)f_n)(y) = \int_{-\infty}^{\infty} p_e^a(t, u - y)f_n(u)du,
\]

and

\[
\| P_e^a(t)f - P_e^a(t)f_n \| \leq \| f_n - f \| \to 0 \quad \text{as} \quad n \to \infty.
\]

Let us put

\[
g(y) = \int_{-\infty}^{\infty} p_e^a(t, u - y)f(u)du
\]

and

\[
g_n(y) = \int_{-\infty}^{\infty} p_e^a(t, u - y)f_n(u)du.
\]

We obtain the following by using the inequality (1.6),

\[
\| g - g_n \| \leq d \| f - f_n \| \to 0 \quad \text{as} \quad n \to \infty.
\]

Therefore (1.7) holds.

Corollary 1.1. Let \( q \) be a nonzero real number and let

\[
p_{\lambda a}(t, x) = \left( \frac{iq}{4\pi t} \right)^{1/2} \exp \left( -\frac{iq}{4t} x^2 \right).
\]
Then for $f \in L_2$
\[
(P^2_{i\eta}(t)f)(y) = \int_{-\infty}^{\infty} p^2_{i\eta}(t, u - y)f(u)du.
\]

Proof. Let $\Re \lambda > 0$ and $f \in L_2$. We see that
\[
\left\| \int_{-\infty}^{\infty} p^2_{i\eta}(t, u - \cdot)f(u)du - (P^2_{i\eta}(t)f)(\cdot) \right\|
\]
\[
\leq \left\| \int_{-\infty}^{\infty} p^2_{i\eta}(t, u - \cdot)f(u)du - \int_{-\infty}^{\infty} p^2_{i\eta}(t, u - \cdot)f(u)du \right\|
\]
\[
+ \|(P^2_{i\eta}(t)f)(\cdot) - (P^2_{i\eta}(t)f)(\cdot)\| \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow i\eta,
\]
by Theorem 1.1, Theorem 1.2 and the proof of Theorem in [6, p. 778].

NOTE 1.1. From the assertion of Theorem 1.2, for convenience, we write $(P_{i\eta}^{\lambda}(t)f)(y)$ as
\[
(1.8) \quad \int_{-\infty}^{\infty} p_{i\eta}^{\lambda}(t, u - y)f(u)du
\]
even if $\lambda = i\eta$, where $\eta$ is a nonzero real number. In what follows, we always use the notation (1.8) instead of $P_{i\eta}^{\lambda}(t)f$ for $\lambda \in D$.

LEMMA 1.4. Let $t > 0$. Then $\|p_{i\eta}^{\lambda}(t, \cdot)\|$ and $\|p_{i\eta}^{\lambda}(t, \cdot)\|_1$ are continuous functions of $\lambda$ on $C^+ = \{\lambda : \Re \lambda > 0\}$.

Proof. By (1.5) we have
\[
\int_{-\infty}^{\infty} |p_{i\eta}^{\lambda}(t, \xi)|^2 d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \exp \left( -\frac{t}{\lambda} |\eta|^2 \right) \right|^2 d\eta.
\]
Therefore $\|p_{i\eta}^{\lambda}(t, \cdot)\|$ is continuous in $\lambda$. From (1.5) and the proof of Lemma 1.2, and using $f$ as defined there we have
\[
\int_{-\infty}^{\infty} |p_{i\eta}^{\lambda}(t, \xi)| d\xi = \int_{-1}^{1} |(Uf)(\xi)| d\xi + \int_{1}^{\infty} |(Uf^\prime)(\xi)| d\xi
\]
\[
+ \int_{-1}^{1} \frac{1}{\xi} |(Uf^\prime)(-\xi)| d\xi.
\]
The first term is continuous in $\lambda$. By [14. Theorem (3.2), p. 254] it holds that
where let $p$ satisfy the inequalities $(\beta - 1)(1 - p) > -1$ and $1 < p \leq 2$, and $K$ is constant. By replacing $f$ in the above by

$$h(\xi, \lambda, \Delta \lambda) = (2\pi)^{-1/2} \exp\left(-\frac{t}{\lambda + \Delta \lambda} |\xi|^\alpha\right) - \exp\left(-\frac{t}{\lambda} |\xi|^\alpha\right),$$

we can show that the second term of (1.9) is continuous in $\lambda$. The above argument can be applicable to the third term of (1.9). Therefore the left side of (1.9) is continuous in $\lambda$.

2. Definition of operator valued function space integral. Let $C[a, b]$ denote the space of real valued right continuous functions defined on the interval $[a, b]$ and $C_0[a, b]$ denote those $x \in C[a, b]$ such that $x(a) = 0$. Let $\sigma$ be any partition of $[a, b]$, $\sigma: a = t_0 < t_1 < \cdots < t_n = b$. For any $x \in C[a, b]$, let $x_\sigma(t) = x(t)$ if $t_{i-1} < t \leq t_i$ and $x_\sigma(a) = x(a)$. Let $F(x)$ be a bounded functional defined on $B[a, b]$. We suppose that $F(x_\sigma)$ has the form

$$(2.1) \quad F(x_\sigma) = f_\sigma(x(t_1), x(t_2), \cdots, x(t_n)), \quad (x \in C[a, b])$$

where $f_\sigma(v_1, v_2, \cdots, v_n)$ is a bounded Borel function on $R^n$, and that for each $\lambda \in D$ the operator $F_\lambda^{x, \sigma}(F)$ on $L_2$ can be defined by

$$(2.2) \quad (F_\lambda^{x, \sigma}(F)\psi)(\xi) = \int_{-\infty}^\infty p_\lambda^x(t_1 - a, v_1 - \xi)dv_1 \int_{-\infty}^\infty p_\lambda^x(t_2 - t_1, v_2 - v_1)dv_2 \cdots \int_{-\infty}^\infty p_\lambda^x(b - t_{n-1}, v_n - v_{n-1})f_\sigma(v_1, v_2, \cdots, v_n)\psi(v_n)dv_n,$$

where $\psi \in L_2$.

DEFINITION 2.1. If $F_\lambda^{x, \sigma}(F)$ converges in the weak operator topology as norm $\sigma$ tends to 0 for $\lambda \in D$, we denote its limit by $F_\lambda^x(F)$ and for the moment we call $F_\lambda^x(F)$ the operator valued function space integral of $F$.

We should note from (0.3) that if $Re \lambda > 0$, (2.2) corresponds to (0.2) except the fact that $f_\sigma$ in (0.2) has the variable $v_0$. To compare the operator valued function space integral with the Wiener integral, it is convenient to use the following notations;
(\mathcal{F}_\lambda^{\alpha, \sigma}(F)\psi)(\xi) = \int_{C[a, b]} F(x_\alpha + \xi)\psi(x(b) + \xi)dp_\alpha^*(x) \\
\text{for (2.2) and} \\
(\mathcal{F}_\alpha^*(F)\psi)(\xi) = \int_{C[a, b]} F(x + \xi)\psi(x(b) + \xi)dp_\alpha^*(x). \\

\text{LEMMA 2.1. Let } F(x) \text{ be a factorable functional given by} \\
F(x) = f_1(x(s_1))f_2(x(s_2)) \cdots f_m(x(s_m)), \quad a < s_1 < \cdots < s_m = b, \\
\text{where } f_i(v) \text{ are bounded Borel functions. Then for } \lambda \in D, \mathcal{F}_\lambda^{\alpha, \sigma}(F)\psi \text{ converges to} \\
(\mathcal{F}_\lambda^{\alpha, \sigma}(F)\psi)(\xi) = \int_{-\infty}^{\infty} p_\lambda^{\alpha}(s_i - a, v_1 - \xi)f_1(v_1)dv_1 \cdots \\
\cdot \int_{-\infty}^{\infty} p_\lambda^{\alpha}(b - s_m, v_m - \xi)f_m(v_m)\psi(v_m)dv_m \\
\begin{equation}
\tag{2.3}
\end{equation}
\text{in the norm topology as norm } \sigma \to 0, \text{ where } \psi \in L_2. \\

\text{Proof. Let } \sigma \text{ be any partition of } [a, b], \sigma: a = t_0 < t_1 < \cdots < t_n = b. \text{ Clearly } F(x_\sigma) \text{ can be expressed as} \\
F(x_\sigma) = f_1(x(t_{i(1)}))f_2(x(t_{i(2)})) \cdots f_m(x(b)) \\
\text{where } t_{i(j)} < s_j \leq t_{i(j)}. \text{ Then} \\
(\mathcal{F}_\lambda^{\alpha, \sigma}(F)\psi)(\xi) = \int_{-\infty}^{\infty} p_\lambda^{\alpha}(t_1 - a, v_1 - \xi)f_1(v_1)dv_1 \int_{-\infty}^{\infty} p_\lambda^{\alpha}(t_2 - t_1, v_2 - v_1)f_2(v_2)dv_2 \\
\cdots \int_{-\infty}^{\infty} p_\lambda^{\alpha}(b - t_{n-1}, v_n - v_{n-1})f_1(v_{i(1)})f_2(v_{i(2)}) \\
\cdots f_m(v_n)\psi(v_m)dv_m. \\
\begin{equation}
\tag{2.4}
\end{equation}
\text{Here } \mathcal{F}_\lambda^{\alpha, \sigma}(F) \text{ is a well-defined operator on } L_2 \text{ for each } \lambda \in D. \text{ By the} \\
\text{semigroup property of Theorem 1.1 and Note 1.1, we obtain when norm } \sigma < \text{Min } \{s_i - s_{i-1}\} \\
(\mathcal{F}_\lambda^{\alpha, \sigma}(F)\psi)(\xi) = \int_{-\infty}^{\infty} p_\lambda^{\alpha}(t_{i(1)} - a, v_1 - \xi)f_1(v_1)dv_1 \cdots \\
\cdot \int_{-\infty}^{\infty} p_\lambda^{\alpha}(b - t_{i(m-1)}, v_m - v_{m-1})f_m(v_m)\psi(v_m)dv_m.
Since $t_{(j)}$ converge to $s_j$ ($j = 1, 2, \cdots, m$) as norm $\sigma \to 0$, by using the strong continuity of semigroup of Theorem 1.1 and the boundedness of $f_n$, it follows that $\mathcal{F}^\sigma\lambda(F)\psi \to \mathcal{F}^\sigma\lambda(F)\psi$ in the $L_2$-norm topology as norm $\sigma \to 0$ for $\lambda \in D$.

**Lemma 2.2.** If $F$ is a functional to which $\mathcal{F}^\sigma\lambda(F)$ applies, then $\mathcal{F}^\sigma\lambda(F)$ is a linear operator defined on a linear manifold $\mathcal{M}$ of functions $\psi$. Moreover the operator valued integral $\mathcal{F}^\sigma\lambda$ is linear in the sense that if the linear operators $\mathcal{F}^\sigma\lambda(F)$ and $\mathcal{F}^\sigma\lambda(G)$ are defined on the same manifold $\mathcal{M}$, then $\mathcal{F}^\sigma\lambda(c_1F + c_2G)$ is defined on $\mathcal{M}$ for each pair of complex numbers $c_1$, $c_2$, and

$$
(2.5) \quad \mathcal{F}^\sigma\lambda(c_1F + c_2G) = c_1\mathcal{F}^\sigma\lambda(F) + c_2\mathcal{F}^\sigma\lambda(G).
$$

In particular, if $F$ and $G$ satisfy the hypotheses of Lemma 2.1, $\mathcal{F}^\sigma\lambda(c_1F + c_2G)$ maps $L_2$ into $L_2$ and satisfies $(2.5)$.

**Proof.** For each fixed $\sigma$, we note from (2.1) that $f_\sigma$ depends linearly on $F$, and hence it follows from (2.2) that $(\mathcal{F}^\sigma\lambda(F)\psi)(\xi)$ depends linearly on $F$ as well as on $\psi$. Hence the lemma follows from Definition 2.1 and the linearity of the weak limit and Lemma 2.1.

**Note 2.1.** If $\text{Re} \lambda > 0$, by Lemma 1.1, Lemma 1.2 and the Fubini Theorem, we can write (2.3), (2.4) for all $\xi \in R$ as follows

$$
(\mathcal{F}^\sigma\lambda(F)\psi)(\xi) = \int_{-\infty}^{\infty} (m) \int_{-\infty}^{\infty} f_1(v_1) \cdots f_m(v_m) \psi(v_m)p^\sigma\lambda(s_1 - a, v_1 - \xi) \\
\cdots p^\sigma\lambda(b-s_{m-1}, v_m - v_{m-1})dv_1 \cdots dv_m,
$$

$$
(\mathcal{F}^\sigma\lambda(F)\psi)(\xi) = \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} f_1(v_{(1)}) \cdots f_m(v_n) \psi(v_n) \\
\cdots p^\sigma\lambda(t_1 - a, v_1 - \xi) \cdots p^\sigma\lambda(b-t_{n-1}, v_n - v_{n-1})dv_1 \cdots dv_n.
$$

### 3. The operator valued function space integral of a product integral.

Let us consider the following functional,

$$
(3.1) \quad F(x) = \prod_{j=1}^{m} \int_{a}^{b} \theta_j(s, x(s))ds,
$$

where

$$
(3.2) \quad \theta_j(s, u) \text{ are Borel measurable functions on } [a, b] \times R, \quad \text{and}
$$

$$
(3.3) \quad |\theta_j(s, u)| \leq M_j < \infty, \quad (j = 1, 2, \cdots, m).
$$
**Lemma 3.1.** Let $F(x)$ be a functional given by (3.1), (3.2) and (3.3). Then for each $\psi$ in $L_2$, $\mathcal{F}^\alpha_{\lambda,\sigma}(F)\psi$ is a strongly continuous function of $\lambda$ on $D$ and analytic in $C^\ast$.

**Proof.** For a partition $\sigma: t_0 = a < t_1 < \cdots < t_n = b$ and any $x \in C[a, b]$, we have

$$F(x_\sigma) = \int_a^b (m) \int_a^b \theta_1(s_1, x_\sigma(s_1)) \cdots \theta_m(s_m, x_\sigma(s_m)) ds_1 \cdots ds_m.$$  

Since $x_\sigma(s) = x(t_i)$ if $t_{i-1} < s \leq t_i$ and $x_\sigma(a) = x(a)$, it holds that

$$\int_a^b \theta_1(s, x_\sigma(s)) ds = \int_a^{t_1} \theta_1(s, x(t_1)) ds + \cdots + \int_{t_{n-1}}^{t_n} \theta_1(s, x(t_n)) ds.$$  

Let us denote

$$\int_{t_i}^{t_{i+1}} \theta_j(s, v) ds$$

by $\phi^i_j(v)$. Then we have

$$F(x_\sigma) = \sum_{i(1)=1}^{n} \cdots \sum_{i(m)=1}^{n} \phi^1_{i(1)}(x(t_{i(1)})) \cdots \phi^m_{i(m)}(x(t_{i(m)})).$$

Since $\phi^i_j(v)$ are bounded Borel measurable functions on $R$, we can define the following operator on $L_2$ for each $\lambda \in D$,

$$(K^\alpha_{\lambda}(F)\psi)(\xi) = \int_{-\infty}^{\xi} p^\alpha_{\lambda}(t_1 - a, v_1 - \xi) dv_1 \int_{-\infty}^{v_1} p^\alpha_{\lambda}(t_2 - t_1, v_2 - v_1) dv_2 \cdots$$

$$\cdot \int_{-v_{n-1}}^{\xi} p^\alpha_{\lambda}(t_n - t_{n-1}, v_n - v_{n-1}) \phi^1_{i(1)}(v_{i(1)}) \cdots \phi^m_{i(m)}(v_{i(m)}) \psi(v_n) dv_n,$$

where $\psi \in L_2$. As we have stated in the proof of Lemma 2.2, it holds that

$$\mathcal{F}^\alpha_{\lambda,\sigma}(F)\psi(\xi) = \sum_{i(1)=1}^{n} \cdots \sum_{i(m)=1}^{n} (K^\alpha_{\lambda}(F)\psi)(\xi),$$

and by the boundedness of $\theta_\mu$ it holds that

$$\|K^\alpha_{\lambda}(F)\psi\| \leq M_1(t_{i(1)} - t_{i(1)-1}) \cdots M_m(t_{i(m)} - t_{i(m)-1}) \|\psi\|.$$
By using the boundedness of \( \phi_i(v) \) and Theorem 1.1, we can show that \( K^\sigma_\lambda(F)\psi \) is strongly continuous in \( \lambda \) on \( D \). Therefore \( F_\lambda^{\sigma_\lambda}(F)\psi \) is a continuous function of \( \lambda \) on \( D \). Next we wish to show that \( K^\sigma_\lambda(F)\psi \) is an analytic function of \( \lambda \) in \( C^+ \) for a fixed \( \psi \in L_2 \). Let \( g(\lambda) = (K^\sigma_\lambda(F)\psi, \varphi) \), \( \varphi \in L_2 \). As we have shown above, since \( g(\lambda) \) is a continuous function of \( \lambda \) on \( D \), we show that

\[
\int g(\lambda) \, d\lambda = 0
\]

for triangular path \( \Gamma \) in \( C^+ \). Then by Morera's Theorem, \( g(\lambda) \) is the analytic function on \( C^+ \). We can consider the ordered integration of \( K^\sigma_\lambda(F)\psi \) as an \( n \)-fold Lebesgue integral by Lemma 1.1 and Lemma 1.2 since \( \lambda \) is in \( C^+ \). It holds that

\[
\int g(\lambda) \, d\lambda = \int_{-\infty}^{\infty} \varphi(\xi) \left[ \int_{\Gamma} (K^\sigma_\lambda(F)\psi)(\xi) \, d\lambda \right] \, d\xi,
\]

since \( \varphi(\xi) \cdot (K^\sigma_\lambda(F)\psi)(\xi) \) is integrable with respect to \( \lambda, \xi \) over \( \Gamma \times R \) by the Schwartz inequality and (3.6). Moreover by Lemma 1.1, Lemma 1.2 and Lemma 1.4

\begin{equation}
(3.7) \quad |\psi(v_n)| \left| p^\sigma_\lambda(t_1 - a, v_1 - \xi) \cdots p^\sigma_\lambda(t_n - t_{n-1}, v_n - v_{n-1}) \right|
\end{equation}

is integrable with respect to \( v_1, \ldots, v_n, \lambda \) over \( R^n \times \Gamma \), therefore by using the Fubini Theorem,

\[
\int_{\Gamma} (K^\sigma_\lambda(F)\psi) \, d\lambda = \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} \left[ \int_{\Gamma} p^\sigma_\lambda(t_1 - a, v_1 - \xi) \cdots p^\sigma_\lambda(t_n - t_{n-1}, v_n - v_{n-1}) \, d\lambda \right] \\
\cdot \phi^i_\lambda(v_i(1)) \cdots \phi^m_\lambda(v_i(m)) \psi(v_n) \, dv_1 \cdots dv_n.
\]

By (0.4) it holds that

\[
\int_{\Gamma} \prod_{j=1}^{n} p^\sigma_\lambda(t_j - t_{j-1}, v_j - v_{j-1}) \, d\lambda = 0.
\]

Therefore we obtain

\[
\int_{\Gamma} g(\lambda) \, d\lambda = 0.
\]
Let us put
\[
S(\tau) = \{(s_u, s_m) \in (\alpha, b)^m : a < s_{\tau(1)} < \cdots < s_{\tau(m)} < b\}.
\]
Here \(\tau\) means a permutation of \(\{1, 2, \cdots, m\}\) and \((s_1, \cdots, s_m) \in S(\tau)\) means that \((s_1, \cdots, s_m)\) satisfies the order relation \(a < s_{\tau(1)} < \cdots < s_{\tau(m)} < b\).

**Theorem 3.1.** Let \(F(x)\) be a functional given by (3.1), (3.2) and (3.3). Then for \(\lambda \in D\), \(\mathcal{F}_\lambda^\alpha(\psi)(\cdot)\) converges to
\[
(\mathcal{F}_\lambda^\alpha(F)\psi)(\xi) = \sum_{\tau} (B) \int_{S(\tau)} (m) \int [\int_{-\infty}^{\infty} p_\lambda^\alpha(s_{\tau(1)} - a, v_1 - \xi) \theta_{\tau(1)}(s_{\tau(1)}, v_1) dv_1
\]
\[
\cdots \int_{-\infty}^{\infty} p_\lambda^\alpha(b - s_{\tau(m)}, v_{m+1} - v_m) \psi(v_{m+1}) dv_{m+1} ] ds_1 \cdots ds_m
\]
in the norm topology as norm \(\sigma \to 0\), where \(\psi \in L_2\) and the sum is taken over all \(m!\) permutations \(\tau\) of \(\{1, 2, \cdots, m\}\) and \((B)\) denotes the Bochner integral with respect to Lebesgue measure on \(S(\tau)\). Furthermore
\[
(3.9) \quad \|\mathcal{F}_\lambda^\alpha(F)\| \leq (b - a)^m M_1 M_2 \cdots M_m.
\]

**Proof.** Let \(\lambda\) be in \(C^+\). For all \(\xi \in R\), (3.7) is integrable with respect to \(v_1, \cdots, v_n\) over \(R^n\). By this fact, we can change the order of integration of \(K_\lambda^\alpha(F)\psi\) by the Fubini Theorem, hence we obtain
\[
(K_\lambda^\alpha(F)\psi)(\xi)
\]
\[
(3.10) = \int_{t_{(l)}(1)}^{t_{(l)}(1)} (m) \int_{t_{(l)}(m-1)}^{t_{(l)}(m)} (H_\lambda^\alpha(s_1, \cdots, s_m)\psi)(\xi) ds_1 \cdots ds_m,
\]
where
\[
(H_\lambda^\alpha(s_1, \cdots, s_m)\psi)(\xi) = \int_{-\infty}^{\infty} p_\lambda^\alpha(t_1 - a, v_1 - \xi) dv_1 \int_{-\infty}^{\infty} p_\lambda^\alpha(t_2 - t_1, v_2 - v_1) dv_2
\]
\[
\cdots \int_{-\infty}^{\infty} p_\lambda^\alpha(t_n - t_{n-1}, v_n - v_{n-1}) \theta_1(s_1, v_{(1)}) \cdots \theta_m(s_m, v_{(m)}) \psi(v_n) dv_n.
\]
From (3.5) and (3.10) we see that

\[(\mathcal{F}_{\lambda}^{\alpha,\sigma}(F)\psi)(\xi)\]

\[= \int_{a}^{b} (m) \int_{a}^{b} ds_1 \cdots ds_m \left[ \int_{-\infty}^{\infty} p_{s_{r(1)}}^{\alpha}(t_1 - a, \nu_1 - \xi) d\nu_1 \right]
\]

\[\cdot \int_{-\infty}^{\infty} p_{s_{r(2)}}^{\alpha}(t_2 - t_1, \nu_2 - \nu_1) d\nu_2 \]

\[\cdots \int_{-\infty}^{\infty} p_{s_{r(m)}}^{\alpha}(t_m - t_{m-1}, \nu_m - \nu_{m-1}) \prod_{j=1}^{m} \theta_j(s_j, V_\sigma(s_j)) \psi(\nu_n) d\nu_n \]

\[= \sum_{r} \int_{S(\tau)}^{\infty} (m) \int_{-\infty}^{\infty} p_{s_{r(1)}}^{\alpha}(t_1 - a, \nu_1 - \xi) d\nu_1 \int_{-\infty}^{\infty} p_{s_{r(2)}}^{\alpha}(t_2 - t_1, \nu_2 - \nu_1) d\nu_2 \]

\[\cdots \int_{-\infty}^{\infty} p_{s_{r(m)}}^{\alpha}(t_m - t_{m-1}, \nu_m - \nu_{m-1}) \prod_{j=1}^{m} \theta_j(s_j, V_\sigma(s_j)) \psi(\nu_n) d\nu_n \]

\[ds_1 \cdots ds_m,\]

where \(V_\sigma(s) = v_i\) if \(t_i < s \leq t_i\).

By rearranging the product \(\theta_1(s_1, V_\sigma(s_1)) \cdots \theta_m(s_m, V_\sigma(s_m))\) of the last member of (3.11) as

\[(\mathcal{F}_{\lambda}^{\alpha,\sigma}(F)\psi)(\xi)\]

\[= \sum_{r} \int_{S(\tau)}^{\infty} (m) \int (H_{s_{r(1)}}^{\alpha,\sigma}(s_1, \cdots, s_m) \psi)(\xi) ds_1 \cdots ds_m\]

where

\[(H_{s_{r(1)}}^{\alpha,\sigma}(s_1, \cdots, s_m) \psi)(\xi)\]

\[= \int_{-\infty}^{\infty} p_{s_{r(1)}}^{\alpha}(t_1 - a, \nu_1 - \xi) d\nu_1 \int_{-\infty}^{\infty} p_{s_{r(2)}}^{\alpha}(t_2 - t_1, \nu_2 - \nu_1) d\nu_2 \cdots \]

\[\cdots \int_{-\infty}^{\infty} p_{s_{r(m)}}^{\alpha}(t_m - t_{m-1}, \nu_m - \nu_{m-1}) \theta_{r(1)}(s_{r(1)}, V_\sigma(s_{r(1)})) \cdots \theta_{r(m)}(s_{r(m)}, V_\sigma(s_{r(m)})) \psi(\nu_n) d\nu_n \]

\[\in S(\tau).\]

Now we prove that (3.13) is Bochner integrable over \(S(\tau)\). In order to show this, it suffices to show that
(1) \( H^\sigma_{\lambda, r}(s_1, \cdots, s_m)\psi \) is strongly measurable in \((s_1, \cdots, s_m)\) on \(S(\tau)\),

(2) \[ \int_{S(\tau)} (m) \int \| H^\sigma_{\lambda, r}(s_1, \cdots, s_m)\psi \| \, ds_1 \cdots ds_m < \infty. \]

Clearly it follows from Theorem 1.1 and (3.3) that

(a) \[ \| H^\sigma_{\lambda, r}(s_1, \cdots, s_m)\psi \| \leq M_1 \cdots M_m \| \psi \|, \]

for almost all \((s_1, \cdots, s_m) \in S(\tau)\).

In order to prove (1), it suffices to prove by [5. Corollary 2. p. 73] that \(H^\sigma_{\lambda, r}(s_1, \cdots, s_m)\psi\) is weakly measurable on \(S(\tau)\), that is \((H^\sigma_{\lambda, r}(s_1, \cdots, s_m)\psi, \varphi)\) is measurable in \((s_1, \cdots, s_m)\) for \(\varphi \in L_2\). If \(\varphi \in L_1 \cap L_2\), from Lemma 1.1, Lemma 1.2 and the boundedness of \(\theta_j\),

\[
\varphi(\xi) \left[ \prod_{j=1}^m \theta_{s(j), V_\sigma(s(j))} \right] \psi(v_n) p_\lambda^\sigma(t_1 - a, v_1 - \xi) \\
\cdots p_\lambda^\sigma(t_n - t_{n-1}, v_n - v_{n-1})
\]

is integrable with respect to the variables \(s_1, \cdots, s_m, \xi, v_1, \cdots, v_n\) over \(S(\tau) \times R^{n+1}\). Therefore it follows from the Fubini Theorem that \((H^\sigma_{\lambda, r}(s_1, \cdots, s_m)\psi, \varphi)\) is measurable in \((s_1, \cdots, s_m)\) on \(S(\tau)\). If \(\varphi \in L_2\), let us put

\[
\varphi_N(x) = \begin{cases} \varphi(x) & \text{if } |x| \leq N, \\ 0 & \text{if } |x| > N. \end{cases}
\]

Then since \(\|\varphi_N - \varphi\| \to 0\) as \(N \to \infty\), it holds by (a) that for almost all \((s_1, \cdots, s_m) \in S(\tau)\),

\[
(H^\sigma_{\lambda, r}(s_1, \cdots, s_m)\psi, \varphi_N) \to (H^\sigma_{\lambda, r}(s_1, \cdots, s_m)\psi, \varphi) \quad \text{as} \quad N \to \infty.
\]

By the fact that \((H^\sigma_{\lambda, r}(s_1, \cdots, s_m)\psi, \varphi_N)\) is measurable in \((s_1, \cdots, s_m)\), \((H^\sigma_{\lambda, r}(s_1, \cdots, s_m)\psi, \varphi)\) is also measurable with respect to the variables \(s_1, \cdots, s_m\) on \(S(\tau)\). Furthermore \(\| H^\sigma_{\lambda, r}(s_1, \cdots, s_m)\psi \|\) is measurable on \(S(\tau)\) by [5. Theorem 3.5.2] since \(L_2\) is separable. Therefore by (a), (2) holds. Hence we see that

(b) \( H^\sigma_{\lambda, r}(s_1, \cdots, s_m)\psi \) is Bochner integrable over \(S(\tau)\) if \(\lambda \in C^+\).

Next we wish to prove that \( H^\sigma_{\lambda, r}(s_1, \cdots, s_m)\psi \) is also Bochner integrable over \(S(\tau)\) even if \(\lambda = -iq\) \((q \text{ is a nonzero real number)}\). Here we should note that for \(\lambda = -iq\), (3.13) is well-defined for almost all \((s_1, \cdots, s_m) \in S(\tau)\) by the boundedness of \(\theta_j\). In order to
prove the Bochner integrability, by (a), (b) and [5. Theorem 3.7.9] it suffices to prove that for almost all \((s_1, \cdots, s_m)\in S(\tau),\)

\[(3.15) \quad \|H^\sigma_{\lambda, \tau}(s_1, \cdots, s_m)\psi - H^\sigma_{\lambda, \tau}(s_1, \cdots, s_m)\psi\| \to 0\]
as \(\lambda \to -iq\). By using the boundedness of \(\theta_i\) and Theorem 1.1, we can see that (3.15) holds for almost all \((s_1, \cdots, s_m)\in S(\tau)\). Hence we have

(c) \(H^\sigma_{\lambda, \tau}(s_1, \cdots, s_m)\psi\) is Bochner integrable over \(S(\tau)\),

\[(3.16) \quad \|H^\sigma_{\lambda, \tau}(s_1, \cdots, s_m)\psi\| \leq M_1 \cdots M_m \|\psi\|, \text{ for almost all } (s_1, \cdots, s_m) \in S(\tau).\]

Furthermore, by using the Dominated Convergence Theorem [5. Theorem 3.7.9] we obtain

\[
\lim_{\lambda \to -iq} (B) \int_{S(\tau)} (m) \int H^\sigma_{\lambda, \tau}(s_1, \cdots, s_m)\psi ds_1 \cdots ds_m \\
= (B) \int_{S(\tau)} (m) \int H^\sigma_{\lambda, \tau}(s_1, \cdots, s_m)\psi ds_1 \cdots ds_m.
\]

Let \(\phi \in L_2\) and \(\lambda \in C^+\). Then \(\overline{\phi(\xi)}(H^\sigma_{\lambda, \tau}(s_1, \cdots, s_m)\psi)(\xi)\) is integrable over \(S(\tau) \times R\) by (a). Hence by using the Fubini Theorem and [5. Theorem 3.7.12 and the remark following], we have

\[
\int_{S(\tau)} (m) \int (H^\sigma_{\lambda, \tau}(s_1, \cdots, s_m)\psi)(\xi) ds_1 \cdots ds_m \\
= \int_{-\infty}^{\infty} \overline{\phi(\xi)} \left[ \int_{S(\tau)} (m) \int (H^\sigma_{\lambda, \tau}(s_1, \cdots, s_m)\psi)(\xi) ds_1 \cdots ds_m \right] d\xi.
\]

From this fact, it follows that for \(\lambda \in C^+

\[
\int_{S(\tau)} (m) \int H^\sigma_{\lambda, \tau}(s_1, \cdots, s_m)\psi ds_1 \cdots ds_m \\
= (B) \int_{S(\tau)} (m) \int H^\sigma_{\lambda, \tau}(s_1, \cdots, s_m)\psi ds_1 \cdots ds_m.
\]

From (3.12) and (3.17) we have

\[
\mathcal{F}_{\lambda, \sigma, \tau}(F)\psi = \sum_{\tau} (B) \int_{S(\tau)} (m) \int H^\sigma_{\lambda, \tau}(s_1, \cdots, s_m)\psi ds_1 \cdots ds_m.
\]
for \( \lambda \in C^+ \). From this equality, by using (3.16) and Lemma 3.1, we obtain

\[
\mathcal{F}_\lambda^{\alpha, \sigma}(F) \psi = \sum_\tau (B) \int_{S(\tau)} (m) \int H^{\alpha, \tau}_{\lambda, r}(s_1, \ldots, s_m) \psi ds_1 \cdots ds_m
\]

for every \( \lambda \in D \).

Now we show that (3.18) converges to (3.8) in the \( L_2 \)-norm topology as norm \( \sigma \to 0 \). For almost all \((s_1, \ldots, s_m) \in S(\tau)\), it follows from (3.13) that

\[
(3.18) \quad \mathcal{F}_\lambda^{\alpha, \sigma}(F) \psi = \sum_\tau (B) \int_{S(\tau)} (m) \int H^{\alpha, \tau}_{\lambda, r}(s_1, \ldots, s_m) \psi ds_1 \cdots ds_m
\]

where let \( t_{r(\cdot)} - 1 < s_{r(\cdot)} \equiv t_{r(\cdot)} \). By the same argument as in the proof of Lemma 2.1, for almost all \((s_1, \ldots, s_m) \in S(\tau)\), (3.19) converges to

\[
(3.19) \quad I_\tau(s_1, \ldots, s_m) \psi(\xi) \equiv \int_{-\infty}^{\infty} p^\alpha_\lambda(t_1 - a, v_1 - \xi) \theta_{r(1)}(s_{r(1)}, v_{r(1)}) \cdots
\]

\[
\cdot \theta_{r(m)}(s_{r(m)}, v_{r(m)}) \psi(v_m) dv_m
\]

in the norm topology as norm \( \sigma \to 0 \). From this fact and from (a), (b), (c), (d) and from [5, Theorem 3.7.9], it follows that

\[
(e) \quad \| I_\tau(s_1, \ldots, s_m) \psi \| \leq M_1 \cdots M_m \| \psi \|
\]

for almost all \((s_1, \ldots, s_m) \in S(\tau)\),

\[
(f) \quad I_\tau(s_1, \ldots, s_m) \psi \text{ is also Bochner integrable over } S(\tau), \text{ furthermore}
\]

\[
\mathcal{F}_\lambda^{\alpha, \sigma}(F) \psi \to \sum_\tau (B) \int_{S(\tau)} (m) \int I_\tau(s_1, \ldots, s_m) \psi ds_1 \cdots ds_m
\]

in the norm topology as norm \( \sigma \to 0 \) for each \( \lambda \in D \).
Now we prove (3.9). From (e) and [5. Theorem 3.7.6], it follows that

$$
\| \mathcal{F}_\lambda^*(F) \psi \| \leq \sum_{m} \int_{S_{m+1}} (m) \int I_1(s_1, \ldots, s_m) \psi \| ds_1 \cdots ds_m
$$

$$
= \sum \frac{(b - a)^m}{m!} M_1 \cdots M_n \| \psi \| = (b - a)^m M_1 \cdots M_n \| \psi \|.
$$

Therefore we have

$$
\| \mathcal{F}_\lambda^*(F) \| \leq (b - a)^m M_1 \cdots M_n.
$$

**Corollary 3.1.** Let $\theta(t,u) = \theta_1(t,u) = \cdots = \theta_m(t,u)$ in Theorem 3.1. Then for $\lambda \in D$

$$
(\mathcal{F}_\lambda^*(F) \psi)(\xi)
$$

$$
= m! (B) \int_{S_{m}(a,b)} (m) \left[ \int_{-\infty}^{\infty} p_\lambda^\ast (s_1 - a, v_1 - \xi) \theta(s_1, v_1) dv_1 \right.
$$

$$
\cdots \left. \int_{-\infty}^{\infty} p_\lambda^\ast (b - s_m, v_m - v_{m-1}) \theta(v_m, v_m) dv_m \right] ds_1 \cdots ds_m,
$$

(3.21)

where

$$
S_m(a,b) = \{(s_1, \cdots, s_m) \in (a, b)^m : a < s_1 < s_2 < \cdots s_m < b\}
$$

and $\psi \in L_2$.

**Note 3.1.** Let us denote the right side of (3.21) by $m! \mathcal{J}_\lambda^*(F) \psi$. For convenience, we use a notation $\mathcal{J}_\lambda^*(F)$ in place of $\mathcal{F}_\lambda^*(F)$ when $F(x) = 1$.

4. **The operator valued function space integral of $F(x) = \Sigma_{n=1}^m F_n(x)$.**

Let us denote the set of all the functionals satisfying (3.1), (3.2) and (3.3) by $A_0$ and let us introduce the quantity

$$
N_0(F) = (b - a)^m M_1 \cdots M_n,
$$

(4.1)

where let $F$ satisfy (3.1), (3.2) and (3.3).
Let

\[ A = \left\{ F(x) = \sum_{m=1}^{\infty} F_m(x) : F_m \in A_0 \text{ and } \sum_{m=1}^{\infty} N_0(F_m) < \infty \right\}. \]

**Theorem 4.1.** Let \( F \) be in \( A \) and let \( F(x) = \sum_{m=1}^{\infty} F_m(x) \). Then for \( \lambda \in D \), \( \mathcal{F}_{\lambda}^{\alpha}(F)\psi \) converges to \( \mathcal{F}^{\alpha}(F)\psi = \sum_{m=1}^{\infty} \mathcal{F}^{\alpha}(F_m)\psi \) in the norm topology as norm \( \sigma \to 0 \), where \( \psi \in L_2 \).

Furthermore

\[ (4.2) \quad ||\mathcal{F}^{\alpha}(F)\psi|| \leq \sum_{m=1}^{\infty} N_0(F_m) ||\psi||. \]

**Note 4.1.** We do not assume that \( F \) determines \( F_1, F_2, \ldots \) uniquely or that \( F_m \) determines \( N_0(F_m) \) uniquely; but we assert that (4.2) holds for any choice of \( F_1, F_2, \ldots \) whose sum is \( F \), where each \( F_m \) satisfies (3.1), (3.2), (3.3), (4.1) with the \( m \) in those equations corresponding to the subscript of \( F_m \).

**Proof.** Let

\[ F_m(x) = \prod_{j=1}^{\infty} \int_{a}^{b} \theta_j^n(s, x(s))ds, \]

where each \( \theta_j^n \) is bounded and Borel measurable in \([a, b] \times R\); let \( \sigma \) be any partition \( a = t_0 < t_1 < \cdots < t_n = b \); and let

\[ \phi_{\sigma}^{\alpha}(v) = \int_{t_{i-1}}^{t_i} \theta_{\sigma}^{n\prime}(s, v)ds. \]

Then we have formally that

\[ (\mathcal{F}_{\lambda}^{\alpha}(F)\psi)(\xi) = \int_{-\infty}^{\infty} p_\xi\alpha(t_1 - a, v_1 - \xi)dv_1 \cdots \int_{-\infty}^{\infty} p_\xi\alpha(t_n - t_{n-1}, v_n - v_{n-1}) \]

\[ \cdot \psi(v_n) \sum_{m=1}^{\infty} F_m(V_{\sigma})dv_n. \]

where \( \psi \in L_2 \) and by (3.4)

\[ F_m(V_{\sigma}) = \sum_{i(1)=1}^{\infty} \cdots \sum_{i(m)=1}^{\infty} \phi_{i(1)}^{m\prime}(v_{i(1)}) \cdots \phi_{i(m)}^{m\prime}(v_{i(m)}). \]

If necessary, by multiplying by additional functions which are identically 1, we can write \( F_m(V_{\sigma}) \) as
\[ F_m(V_\sigma) \]

\[ = \sum_{i_1=1}^{\hat{n}} \cdots \sum_{i_m=1}^{n} \varphi_1^m(i_1, \ldots, i_m: v_1) \cdots \varphi_n^m(i_1, \ldots, i_m: v_n), \]

where notation \( i_j \) is made use of instead of \( i(j) \).

From the fact that \( F_m(x) \) is in \( A_0 \), it is clear that for almost all \( v_1, \ldots, v_n \)

\[ \sum_{i_1=1}^{\hat{n}} \cdots \sum_{i_m=1}^{n} | \varphi_1^m(i_1, \ldots, i_m: v_1) \cdots \varphi_n^m(i_1, \ldots, i_m: v_n) | \]

\[ \leq N_0(F_m), \quad (m = 1, 2, \cdots). \]

From (4.4), we have for almost all \( v_1, \ldots, v_n \),

\[ \sum_{m=1}^{\hat{n}} \left( \sum_{i_1=1}^{\hat{n}} \cdots \sum_{i_m=1}^{n} | \varphi_1^m(i_1, \ldots, i_m: v_1) \cdots \varphi_n^m(i_1, \ldots, i_m: v_n) | \right) \]

\[ \leq \sum_{m=1}^{\infty} N_0(F_m) < \infty. \]

Now we will prove that

\[ \int_{-\infty}^{\infty} p_\eta^*(b - t_{n-1}, v_n - v_{n-1}) \psi(v_n) \sum_{m=1}^{\infty} F_n(V_\sigma) dv_n \]

is \( L_2 \)-integrable with respect to \( v_{n-1} \) for almost all \( v_1, \cdots, v_{n-2} \). By Note 1.1 and (4.3), (4.6) is

\[ U^{*}_{\psi_{\sigma}} \left\{ \exp \left( -\frac{b - t_{n-1}}{\lambda} |v_n|^\sigma \right) \right\} \left\{ \psi(\eta) \sum_{m=1}^{\infty} \sum_{i_1=1}^{\hat{n}} \cdots \sum_{i_m=1}^{n} \varphi_1^m(i_1, \ldots, i_m: v_1) \cdots \varphi_n^m(i_1, \ldots, i_m: v_n) \right\} (v_{n-1}), \]

where we use the notations \((U_{\psi}f)(y)\) and \((U^*_{\psi}f)(y)\) in place of (a), (b) in §1. We wish to prove that

\[ U_{\eta} \left[ \psi(\eta) \sum_{m=1}^{\infty} \sum_{i_1=1}^{\hat{n}} \cdots \sum_{i_m=1}^{n} \varphi_1^m(i_1, \ldots, i_m: v_1) \cdots \varphi_n^m(i_1, \ldots, i_m: \eta) \right] (v_n) \]

is \( L_2 \)-integrable with respect to \( v_n \) for almost every \( v_1, \cdots, v_{n-1} \), that is, (4.8) is well-defined.
Let
\[
\psi_N(x) = \begin{cases} 
\psi(x) & \text{if } |x| \leq N, \\
0 & \text{if } |x| > N.
\end{cases}
\]

Since
\[
\psi_N(\eta) \sum_{m=1}^N \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n \varphi_{n-1}^m(i_1, \ldots, i_m : v_{i_1}) \cdots \varphi_{n-1}^m(i_1, \ldots, i_m : v_{i_m})
\]
\[
\cdot \varphi_{n}^m(i_1, \ldots, i_m : \eta)
\]
is $L_1$-integrable with respect to $\eta$ by (4.5) for almost all $v_1, \ldots, v_{n-1}$, we have for each $v_n$ and for almost every $v_1, \ldots, v_{n-1}$
\[
Q_N(v_n) = U_\eta \left[ \psi_N(\eta) \sum_{m=1}^N \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n \varphi_{n-1}^m(i_1, \ldots, i_m : v_{i_1}) \cdots \varphi_{n-1}^m(i_1, \ldots, i_m : v_{i_m}) \right](v_n)
\]
\[
\cdot \varphi_{n}^m(i_1, \ldots, i_m : v_{i_1}) \cdots \varphi_{n}^m(i_1, \ldots, i_m : v_{i_m}) \cdot \eta
\]
(4.9)
\[
= \sum_{m=1}^N \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n \varphi_{n-1}^m(i_1, \ldots, i_m : v_{i_1}) \cdots \varphi_{n-1}^m(i_1, \ldots, i_m : v_{i_m}) U_\eta [\psi_N(\eta) \varphi_{n}^m(i_1, \ldots, i_m : \eta)](v_n).
\]
From (4.4), it follows that for almost all $v_1, \ldots, v_{n-1}$
\[
\sum_{m=1}^N \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n \left| \varphi_{n-1}^m(i_1, \ldots, i_m : v_{i_1}) \cdots \varphi_{n-1}^m(i_1, \ldots, i_m : v_{i_m}) \right|
\]
\[
\cdot U_\eta [\psi_N(\eta) \varphi_{n}^m(i_1, \ldots, i_m : \eta)](\cdot)
\]
(4.10)
\[
\| \varphi_{n}^m(i_1, \ldots, i_m : \cdot) \|_\infty \cdot \| \psi_N \| \leq \| \psi_N \| \sum_{m=1}^N \| F_m \|,
\]
where $\| \cdot \|_\infty$ means the essential supremum norm.

By using (4.10) we can prove that the last member of (4.9) also converges in the $L_2$-norm topology with respect to $v_n$ for almost every $v_1, \ldots, v_{n-1}$. By [13. 12.53. Example (iii)], both limits with the infinite sum of the last member of (4.9) are equal to each other except on a null set. Therefore we see that $Q_N(v_n)$ is $L_2$-integrable with respect to $v_n$ for almost every $v_1, \ldots, v_{n-1}$. Next we wish to prove that $Q_N(v_n)$ converges to a $L_2$-integrable function $Q(v_n)$ in the $L_2$-norm topology with respect to $v_n$ as $N \to \infty$ for almost all $v_1, \ldots, v_{n-1}$. From (4.4), it follows that
\[
\left\| \sum_{m=1}^N \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n \varphi_{n-1}^m(i_1, \ldots, i_m : v_{i_1}) \cdots \varphi_{n-1}^m(i_1, \ldots, i_m : v_{i_m})
\]
\[
\cdot U_\eta [\psi_N(\eta) \varphi_{n}^m(i_1, \ldots, i_m : \eta)](\cdot)
\]
(4.11)
\[
\leq \| \psi_N - \psi \| \sum_{m=1}^N \| F_m \| \to 0 \quad \text{as} \quad N \to \infty
\]
for almost all $v_1, \cdots, v_{n-1}$, where the infinite sum is taken in the $L_2$-norm topology. Hence we obtain from (4.11) that

\[
\lim_{N \to \infty} Q_N(v_n) = \lim_{N \to \infty} \sum_{m=1}^{\infty} \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \varphi_n^m(i_1, \cdots, i_m; v_1) \cdots \varphi_{n-1}^m(i_1, \cdots, i_m; v_n) U_n[\psi_N(\eta)\varphi_n^m(i_1, \cdots, i_m; \eta)](v_n)
\]

\[
= \sum_{m=1}^{\infty} \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \varphi_n^m(i_1, \cdots, i_m; v_1) \cdots \varphi_{n-1}^m(i_1, \cdots, i_m; v_n) U_n[\psi(\eta)\varphi_n^m(i_1, \cdots, i_m; \eta)](v_n)
\]

\[
= Q(v_n)
\]

for almost every $v_1, \cdots, v_{n-1}$. Here the outer sum in the third member converges in the $L_2$-norm topology. Therefore we obtain the facts that (4.8) is $L_2$-integrable with respect to $v_n$ for almost every $v_1, \cdots, v_{n-1}$ and in the sense of $L_2$-equivalence

\[
Q(\cdot) = U_\eta \left[ \psi(\eta) \sum_{m=1}^{\infty} \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \varphi_n^m(i_1, \cdots, i_m; v_1) \cdots \varphi_{n-1}^m(i_1, \cdots, i_m; v_n) \right](\cdot)
\]

for almost every $v_1, \cdots, v_{n-1}$.

Next we wish to prove that (4.7) is $L_2$-integrable with respect to $v_{n-1}$ for almost all $v_1, \cdots, v_{n-2}$.

Let

\[
X_n(x) = \begin{cases} 
1 & \text{if } |x| \leq N, \\
0 & \text{if } |x| > N.
\end{cases}
\]

From (4.12) and from the fact that $Q$ is in $L_2$, it follows that

\[
X_n(v_n)Q(v_n) \exp\left(-\frac{b-t_{n-1}}{\lambda} |v_n|^\alpha \right)
\]

\[
= X_n(v_n) \exp\left(-\frac{b-t_{n-1}}{\lambda} |v_n|^\alpha \right) \sum_{m=1}^{\infty} \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \varphi_n^m(i_1, \cdots, i_m; v_1) \cdots \varphi_{n-1}^m(i_1, \cdots, i_m; v_n) U_n[\psi(\eta)\varphi_n^m(i_1, \cdots, i_m; \eta)](v_n)
\]

is $L_1$-integrable in $v_n$ for almost all $v_1, \cdots, v_{n-1}$. Since the infinite sum of the third member of (4.12) converges to $Q(v_n)$ in the $L_2$-norm topology, we have for almost all $v_n$ and for almost every $v_1, \cdots, v_{n-1}$,
\[ X_N(v_n)Q(v_n) \exp \left( -\frac{b - t_{n-1}}{\lambda} \mid v_n \right)^\sigma \]

\[ (4.14) \quad = \sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \phi^m(i_1, \ldots, i_m; v_1) \cdots \phi^m_{n-1}(i_1, \ldots, i_m; v_{n-1}) \]

\[ \cdot X_N(v_n) \exp \left( -\frac{b - t_{n-1}}{\lambda} \mid v_n \right)^\sigma \] \[ U_n[\psi(\eta) \phi^m(i_1, \ldots, i_m; \eta)](v_n) \]

where the infinite sum of the right side is taken in the \( L_2 \)-norm topology. With the right side of (4.14), from the Schwartz inequality and (4.4), it follows that

\[ \sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \left| \phi^m(i_1, \ldots, i_m; v_1) \cdots \phi^m_{n-1}(i_1, \ldots, i_m; v_{n-1}) \right| \]

\[ \| \phi^m_{n-1}(i_1, \ldots, i_m; \cdot) \|_\infty \]

\[ \cdot \left\| X_N \exp \left( -\frac{b - t_{n-1}}{\lambda} \mid \cdot \right)^\sigma \right\|_1 \]

\[ \cdot U_n[\psi(\eta) \phi^m(i_1, \ldots, i_m; \eta)](\cdot) \]

\[ \leq (2N)^{1/2} \| \psi \| \sum_{m=1}^{\infty} N_0(F_m) < \infty. \]

From this fact and from [13. 10.83], it follows that for almost every \( v_1, \ldots, v_{n-1} \), the infinite sum of the right side of (4.14) converges pointwisely for almost all \( v_n \). Hence by [13. 12.53. Example (iii)], both limits with the infinite sum of the right side of (4.14) are equal to each other except on a null set. By the Dominated Convergence Theorem, we have

\[ (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(iw_{n-1}v_n)X_N(v_n)Q(v_n) \exp \left( -\frac{b - t_{n-1}}{\lambda} \mid v_n \right)^\sigma \] \[ dv_n \]

\[ = \lim_{k \to \infty} (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(iw_{n-1}v_n) \sum_{m=1}^{k} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \phi^m(i_1, \ldots, i_m; v_1) \]

\[ \cdots \phi^m_{n-1}(i_1, \ldots, i_m; v_{n-1})X_N(v_n) \exp \left( -\frac{b - t_{n-1}}{\lambda} \mid v_n \right)^\sigma \]

\[ \cdot U_n[\psi(\eta) \phi^m(i_1, \ldots, i_m; \eta)](v_n) dv_n \]

(4.15)

\[ = \lim_{k \to \infty} \sum_{m=1}^{k} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \phi^m(i_1, \ldots, i_m; v_1) \]

\[ \cdots \phi^m_{n-1}(i_1, \ldots, i_m; v_{n-1})U^* \{ X_N(v_n) \exp \left( -\frac{b - t_{n-1}}{\lambda} \mid v_n \right)^\sigma \}

\[ \cdot U_n[\psi(\eta) \phi^m(i_1, \ldots, i_m; \eta)](v_n) \} (v_{n-1}) \]
\[
= \sum_{m=1}^{\infty} \sum_{i_1=1}^{n_m} \cdots \sum_{i_m=1}^{n_m} \phi_m^{(n)}(i_1, \ldots, i_m; \nu_1) \cdots \phi_m^{(n)}(i_1, \ldots, i_m; \nu_{n-1}) \\
\cdot U^*_{\psi_n} \left\{ X_n(\nu_n) \exp \left( -\frac{b - t_{n-1}}{\lambda} \right)^\alpha \right\} U_n\left[ \psi(\eta) \phi_m^{(n)}(i_1, \ldots, i_m; \eta) \right](\nu_n) \\
\times (\nu_{n-1})
\]

in the sense of the pointwise convergence with respect to \( \nu_{n-1} \) for almost every \( \nu_1, \ldots, \nu_{n-2} \). From (4.4), it follows that for almost all \( \nu_1, \ldots, \nu_{n-2} \)

\[
= \sum_{m=1}^{\infty} \sum_{i_1=1}^{n_m} \cdots \sum_{i_m=1}^{n_m} \left| \phi_m^{(n)}(i_1, \ldots, i_m; \nu_1) \right| \cdots \left| \phi_m^{(n)}(i_1, \ldots, i_m; \nu_{n-2}) \right|
\]

\[\| \phi_m^{(n)}(i_1, \ldots, i_m; \cdot, \nu_n) \|_{L^\infty} \| U^*_{\psi_n} \left\{ X_n(\nu_n) \exp \left( -\frac{b - t_{n-1}}{\lambda} \right)^\alpha \right\} \cdot U_n\left[ \psi(\eta) \phi_m^{(n)}(i_1, \ldots, i_m; \eta) \right](\nu_n) \] (\cdot) \| \leq \sum_{m=1}^{\infty} N_0(F_m) \| \psi \|

By this fact, the last member of (4.15) also converges in the \( L_2 \)-norm topology with respect to \( \nu_{n-1} \) for almost all \( \nu_1, \ldots, \nu_{n-2} \). From the fact that both limits of the last member of (4.15) are equal to each other except on a null set, by an argument similar to that used in obtaining (4.12), we can see that for almost every \( \nu_1, \ldots, \nu_{n-2} \), (4.15) converges to

\[
\sum_{m=1}^{\infty} \sum_{i_1=1}^{n_m} \cdots \sum_{i_m=1}^{n_m} \phi_m^{(n)}(i_1, \ldots, i_m; \nu_1) \cdots \phi_m^{(n)}(i_1, \ldots, i_m; \nu_{n-1}) \\
\cdot U^*_{\psi_n} \left\{ X_n(\nu_n) \exp \left( -\frac{b - t_{n-1}}{\lambda} \right)^\alpha \right\} U_n\left[ \psi(\eta) \phi_m^{(n)}(i_1, \ldots, i_m; \eta) \right](\nu_{n-1})
\]

in the \( L_2 \)-norm with respect to \( \nu_{n-1} \), where the infinite sum of (4.16) is taken in the \( L_2 \)-norm topology with respect to \( \nu_{n-1} \). From the above argument, it follows that (4.6) is \( L_2 \)-integrable in \( \nu_{n-1} \) for almost every \( \nu_1, \ldots, \nu_{n-2} \) and

\[
(4.6) = \sum_{m=1}^{\infty} \sum_{i_1=1}^{n_m} \cdots \sum_{i_m=1}^{n_m} \phi_m^{(n)}(i_1, \ldots, i_m; \nu_1) \cdots \phi_m^{(n)}(i_1, \ldots, i_m; \nu_{n-1}) \\
\cdot \int_{-\infty}^{\infty} p_\nu(a)(b - t_{n-1}, \nu_n - \nu_{n-1}) \psi(\nu_n) \phi_m^{(n)}(i_1, \ldots, i_m; \nu_n) d\nu_n
\]

in the sense of the \( L_2 \)-equivalence. By repeating the above argument, we can prove that

\[\left( \mathcal{F}_\lambda^{n^\alpha}(F)\psi \right)(\xi) = \sum_{m=1}^{\infty} \left( \mathcal{F}_\lambda^{n^\alpha}(F_m)\psi \right)(\xi),\]
where the sum is taken in the $L_2$-norm topology with respect to $\xi$. By using the fact that

\begin{equation}
\sum_{m=1}^{\infty} \| \mathcal{F}^{a,\sigma}_\lambda(F_m) \| \leq \sum_{m=1}^{\infty} N_0(F_m) < \infty,
\end{equation}

we have that

\[
\lim_{|\sigma| \to 0} \lim_{N \to \infty} \sum_{m=1}^{N} \mathcal{F}^{a,\sigma}_\lambda(F_m) \psi = \lim_{N \to \infty} \sum_{m=1}^{N} \mathcal{F}^{a}_\lambda(F_m) \psi = \sum_{m=1}^{\infty} \mathcal{F}^{a}_\lambda(F_m) \psi
\]

in the $L_2$-norm. From (4.17), (4.2) follows.

**Corollary 4.1.** Let

\[ F(x) = \exp \left\{ \int_{a}^{b} \theta(s, x(s)) ds \right\}, \]

where $\theta(s, u)$ is bounded by $M$ and measurable in $[a, b] \times \mathbb{R}$. Then for $\lambda \in D$

\[ \mathcal{F}^{a}_\lambda(F) \psi = \sum_{m=0}^{\infty} \mathcal{F}^{a}_\lambda(F_m) \psi \]

where $\psi \in L_2$ and

\[ F_m(x) = \left( \int_{a}^{b} \theta(s, x(s)) ds \right)^m, \]

and where the sum is taken in the norm topology. Furthermore

\[ \| \mathcal{F}^{a}_\lambda(F) \| \leq e^{(b-a)M}. \]

**Proof.** By expanding $F(x)$ into a series of the functionals in $A_0$ and by applying Theorem 4.1 and Corollary 3.1, we obtain Corollary.

**5. Integral equation.** Let $\theta(t, u)$ be a Borel function on $[0, t_0] \times \mathbb{R}$ and bounded by $M$. For each $t \in (0, t_0]$, let $\theta_t(s, u)$ be defined on $[0, t] \times \mathbb{R}$ by $\theta_t(s, u) = \theta(t - s, u)$. In what follows we shall consider the functional
For convenience, let us put

\[ F'(x) = \exp \left\{ \int_0^t \theta_i(s, x(s))ds \right\}. \]

where

\[ F_n^a(x) = \left\{ \int_0^t \theta_i(s, x(s))ds \right\}^m, \quad m = 0, 1, \ldots \]

**Theorem 5.1.** \( G(t, \xi, \lambda) \) satisfies the integral equation

\[
G(t, \xi, \lambda) = \int_{-\infty}^{\infty} p_n^a(t, u - \xi)\psi(u)du \\
+ (B) \int_0^t \left[ \int_{-\infty}^{\infty} p_n^a(t - s, u - \xi)\theta(s, u)G(s, u, \lambda)du \right] ds
\]

where \( t_0 \equiv t > 0, \lambda \in D \) and \( \psi \in L_2 \).

**Proof.** Let \( \lambda \in D \) and \( \psi \in L_2 \). From Corollary 4.1, we have

\[
G(t, \xi, \lambda) = \sum_{m=0}^\infty g_m(t, \xi, \lambda)
\]

where the sum is taken in the \( L_2 \)-norm topology.

At first we will prove that

\[
\sum_{m=0}^\infty g_m(t, \xi, \lambda) = \int_{-\infty}^{\infty} p_n^a(t, u - \xi)\psi(u)du \\
+ \sum_{m=0}^\infty (B) \int_0^t ds \left[ \int_{-\infty}^{\infty} p_n^a(t - s, u - \xi)\theta(s, u)g_m(s, u, \lambda)du \right]
\]

where both sums are taken in the \( L_2 \)-norm topology. In order to prove this, since

\[
g_0(t, \xi, \lambda) = \int_{-\infty}^{\infty} p_n^a(t, u - \xi)\psi(u)du,
\]

it suffices to prove that
We prove that the integrand of (5.3) is Bochner integrable with respect to the variable $s$ over $[0, t]$. By [5. Theorem 3.7.12 and the remark following], it holds that

$$Y(s, \xi) = \int_{-\infty}^{\infty} p^*_x(t - s, u - \xi)\theta(s, u)g_m(s, u, \lambda)du$$

for almost all $s \in [0, t]$ and hence by using [5. Theorem 3.7.6], we obtain for almost all $s \in [0, t]$,

$$\|Y\| \leq M^{m+1} \frac{t^m}{m!} \|\psi\|.$$

Furthermore it is necessary to show that $Y$ is strongly measurable in $s$. To show this, it is sufficient to show that $Y$ is weakly measurable in $s$ since $L_2$ is separable. Let $\varphi$ be in $L_2$. By a change of variables, we have

$$\int_{S_m(0,s)} (m) \int ds_1 \cdots ds_m \left\{ \int_{-\infty}^{\infty} \varphi(\xi) \left[ \int_{-\infty}^{\infty} p^*_x(t - s, u - \xi)\theta(s, u)du \right. \right.$$  

$$\left. \cdot \int_{-\infty}^{\infty} p^*_x(s, u_1 - u)\theta_1(s, u_1)du_1 \cdots \int_{-\infty}^{\infty} p^*_x(s_m - s_m - 1, u_m - u_m - 1) \right.$$  

$$\cdot \theta_1(s_m, u_m)du_m \int_{-\infty}^{\infty} p^*_x(s - s_m, u_m + 1 - u_m)\psi(u_m + 1)du_m \right\} d\xi$$

$$= \int_{S_m(\tau_t, 1)} (m) \int d\tau_2 \cdots d\tau_{m+1} \left\{ \int_{-\infty}^{\infty} \varphi(\xi) \left[ \int_{-\infty}^{\infty} p^*_x(\tau_1, u - \xi)\theta_1(\tau_1, u)du \right. \right.$$  

$$\left. \cdot \int_{-\infty}^{\infty} p^*_x(\tau_2 - \tau_1, u_1 - u)\theta_1(\tau_2, u_1)du_1 \cdots \int_{-\infty}^{\infty} p^*_x(\tau_{m+1} - \tau_m, u_m - u_m - 1) \right.$$  

$$\cdot \theta_1(\tau_{m+1}, u_m)du_m \int_{-\infty}^{\infty} p^*_x(t - \tau_{m+1}, u_m + 1 - u_m)\psi(u_m + 1)du_m \right\} d\xi.$$
where let \( t - s = \tau_1, \ s_1 = \tau_2 - \tau_1, \ s_2 = \tau_3 - \tau_1, \cdot \cdot \cdot, \ s_m = \tau_{m+1} - \tau_1 \). The integrand with the variables \( \tau_1, \tau_2, \cdot \cdot \cdot, \tau_{m+1} \) of the right side of (5.4) is measurable in \((\tau_1, \cdot \cdot \cdot, \tau_{m+1})\) on \( S_{m+1}(0, t) \) as shown in the proof of Theorem 3.1 and integrable with respect to \( \tau_1, \cdot \cdot \cdot, \tau_{m+1} \) over \( S_{m+1}(0, t) \). Therefore by the Fubini Theorem, the right side of (5.4) is measurable in \( \tau_1 \), hence we can say that

\[
(b) \quad Y(s, \xi) \text{ is strongly measurable in } s.
\]

From (a) and (b), it follows that

\[
(c) \quad Y(s, \xi) \text{ is Bochner integrable with respect to } s \text{ over } [0, t].
\]

Therefore by [5. Theorem 3.7.12 and the remark following] and the Fubini Theorem,

\[
\int_{-\infty}^{\infty} \overline{\varphi(\xi)} \left\{ (B) \int_0^t ds \left[ \int_{-\infty}^{\infty} p_s^*(t - s, u - \xi) \theta(s, u) g_m(s, u, \lambda) du \right] \right\} d\xi
\]

\[
= \int_0^t ds \int_{-\infty}^{\infty} \overline{\varphi(\xi)} \left[ \int_{-\infty}^{\infty} p_s^*(t - s, u - \xi) \theta(s, u) g_m(s, u, \lambda) du \right] d\xi
\]

\[
= \int_0^t ds \left\{ \left[ \int_{S_{m+1}(0, t)} (m) \int ds_1 \cdot \cdot \cdot ds_m \left[ \int_{-\infty}^{\infty} \varphi(\xi) \left[ \int_{-\infty}^{\infty} p_s^*(t - s, u - \xi) \theta(s, u) g_m(s, u, \lambda) du \right] d\xi \right] \right] \right\}
\]

\[
= \int_{S_{m+1}(0, t)} (m + 1) \int d\tau_1 \cdot \cdot \cdot d\tau_{m+1} \left\{ \int_{-\infty}^{\infty} \overline{\varphi(\xi)} \left[ \int_{-\infty}^{\infty} p_s^*(t - s, u - \xi) \theta_1(\tau_1, u) du \right] \right\}
\]

\[
= \int_{-\infty}^{\infty} \overline{\varphi(\xi)} g_{m+1}(t, \xi, \lambda) d\xi.
\]

From this fact, (5.3) follows. Therefore (5.2) is valid.
From Corollary 4.1, Theorem 1.1 and from the boundedness of \( \theta \), we obtain that

\[
\sum_{m=0}^{\infty} \int_{-\infty}^{\infty} p_{\lambda}^{n}(t-s, u-\xi) \theta(s, u) g_{m}(s, u, \lambda) \, du
\]

(5.5)

\[
\rightarrow \int_{-\infty}^{\infty} p_{\lambda}^{n}(t-s, u-\xi) \theta(s, u) G(s, u, \lambda) \, du
\]

in the \( L_{2} \)-norm topology as \( n \to \infty \) for almost all \( s \in [0, t] \). Since the left side of (5.5) is also Bochner integrable in \( s \) over \([0, t]\) by (c), it follows from (a) and [5, Theorem 3.7.9] that the right side of (5.5) is Bochner integrable in \( s \) over \([0, t]\). Furthermore, we have

\[
\sum_{m=0}^{\infty} \left( B \right) \int_{0}^{t} ds \int_{-\infty}^{\infty} p_{\lambda}^{n}(t-s, u-\xi) \theta(s, u) g_{m}(s, u, \lambda) \, du
\]

\[
= \lim_{n \to \infty} \left( B \right) \int_{0}^{t} ds \left[ \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} p_{\lambda}^{n}(t-s, u-\xi) \theta(s, u) g_{m}(s, u, \lambda) \, du \right]
\]

\[
= \left( B \right) \int_{0}^{t} ds \left[ \int_{-\infty}^{\infty} p_{\lambda}^{n}(t-s, u-\xi) \theta(s, u) G(s, u, \lambda) \, du \right].
\]

where both infinite sums are to be taken in the sense of the norm topology. By this fact, we see that \( G(t, \xi, \lambda) \) is a solution of the integral equation (5.1).

References


Received July 18, 1975 and in revised form May 12, 1976. The author is grateful to the referee for his many helpful suggestions.

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