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Let G be a real reductive Lie group and π a tempered invariant eigendistribution on G. Given a natural ordering on the set of conjugacy classes of Cartan subgroups of G, π is called extremal if it has a unique maximal element in its support. T. Hirai has proved for a restricted class of real simple Lie groups that if π is extremal and satisfies certain regularity conditions, it is uniquely determined by its restriction to the maximal element in its support. The purpose of this paper is to show that Hirai's theorem is true without restriction of the type of G.

1. Introduction. Let G be a connected, acceptable, real reductive Lie group with compact center. Let π be a tempered invariant eigendistribution on G. Then π can be realized as a locally summable function, π', analytic on the dense open set G' of regular elements of G. [1] This function is uniquely determined by its restrictions to a complete set of representatives of $\mathrm{Car}(G)$, the set of conjugacy classes of Cartan subgroups of G.

The function π' can be quite complicated on the various Cartan subgroups. However, there is a natural ordering on $\mathrm{Car}(G)$ such that, if $[H] \in \mathrm{Car}(G)$ is a maximal element for which $\pi'|_H \neq 0$, then there exist functions ε_R^H and Δ^H on H such that $k^H = \varepsilon_R^H \Delta^H \pi'|_H$ is analytic on all of H and is given by a simple Weyl character type formula [5, vol. II, p. 60–62]. Thus it is useful to know when π is uniquely determined by the restrictions of π' to maximal Cartan subgroups in its support.

For example, if π is the character of a discrete series representation of G, then G has a compact Cartan subgroup B which belongs to the unique maximal conjugacy class in $\mathrm{Car}(G)$. Harish-Chandra proved in [2] that π is the unique tempered invariant eigendistribution with given eigenvalue and given formula on B.

In two recent papers [3, 4], Hirai studies the space of tempered invariant eigendistributions on G, and proves a theorem which is a natural generalization of the theorem of Harish-Chandra. However, there is a crucial lemma for this theorem which is only stated for G a simple real Lie group of classical type or of type G_2. Hirai doesn't give a proof of this lemma, but claims it is proved by long but elementary case by case arguments.

The purpose of this paper is to give a general proof of Hirai's

lemma valid for any reductive real Lie group. As a consequence, Hirai's theorem on uniqueness of tempered invariant eigendistributions can be proved without restriction of the type of G.

Section 2 of this paper gives the details of Hirai's results and their extensions. Most notation and definitions follow [3, 4]. Section 3 gives the proof of Lemma 5 of § 2.

2. Uniqueness of tempered invariant eigendistributions. Let G be as in § 1, with Lie algebra $\mathfrak{G}$. Let $A_T(\lambda)$ be the set of all tempered invariant eigendistributions on G with eigenvalue λ. Thus, an element π of $A_T(\lambda)$ is a distribution on $C_c^\infty(G)$ such that:

(i) π is invariant under inner automorphisms of G;

(ii) $Z\pi = \lambda(Z)\pi$ where $Z \in \mathfrak{Z}$, the center of the universal enveloping algebra of the complexification $\mathfrak{G}_c$ of $\mathfrak{G}$ and λ is a homomorphism from $\mathfrak{Z}$ to $\boldsymbol{C}$;

(iii) π extends continuously to a distribution on $\mathcal{E}(G)$, the Schwartz space of G.

Let Car(G) and Car($\mathfrak{G}$) denote the sets of all conjugacy classes of Cartan subgroups of G and Cartan subalgebras of $\mathfrak{G}$ respectively. Then there is an order on Car($\mathfrak{G}$), which can be lifted to an order of Car(G), given as follows. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Cartan subalgebras of $\mathfrak{G}$. We say $[\mathfrak{A}] < [\mathfrak{B}]$ if there is a real root α of $\mathfrak{A}$ such that $\nu(\mathfrak{A})_c = \mathfrak{B}_c$, where $\nu = \nu_\alpha = \exp(-\pi/4\sqrt{-1}\ ad(X_\alpha + X_{-\alpha}))$, and X_α, $X_{-\alpha}$ are root vectors for α and $-\alpha$ in $\mathfrak{G}$ satisfying $[X_\alpha, X_{-\alpha}] = 2H_\alpha/\|\alpha\|^2$. In this case $[\mathfrak{A}] \neq [\mathfrak{B}]$, and $\nu\alpha$ is a singular imaginary root of $\mathfrak{B}$. Extending the order $<$ transitively we get an order on Car[$\mathfrak{G}$] with unique maximal and minimal elements, the Cartan subalgebras with maximal compact and vector parts respectively.

For any tempered invariant eigendistribution π, let $C(\pi)$ be the subset of Car(G) consisting of conjugacy classes $[H]$ for which $\pi'|_H \neq 0$. Any Cartan subgroup H such that $[H]$ is a maximal element of $C(\pi)$ is called a highest Cartan subgroup of π, and the function $k^H = \varepsilon_R^H \Delta^H \pi'|_H$ on H is called one of the highest parts of π. π is called extremal of height $[H]$ if $[H]$ is the unique maximal element of $C(\pi)$.

Let λ be a homomorphism of $\mathfrak{Z}$ into $\boldsymbol{C}$. Let H be a Cartan subgroup of G with Lie algebra $\mathfrak{H}$, and complexification $\mathfrak{H}_c$. There is an isomorphism $\gamma: \mathfrak{Z} \to I(\mathfrak{H}_c)$, where $I(\mathfrak{H}_c)$ denotes the invariants of the full Weyl group, $W(\mathfrak{G}_c, \mathfrak{H}_c)$, in $S(\mathfrak{H}_c)$, the symmetric algebra of $\mathfrak{H}_c$[5, vol, I, p. 168]. Then $\lambda \cdot \gamma^{-1} = \lambda^\flat$ is a homomorphism of $I(\mathfrak{H}_c)$ into $\boldsymbol{C}$ and is induced by a linear form μ on $\mathfrak{H}_c$. We denote this relationship between λ and μ by $\lambda^\flat = \lambda_\mu$. Clearly $\lambda_\mu = \lambda_{w\mu}$ for all $w \in W(\mathfrak{G}_c, \mathfrak{H}_c)$.

For H and λ as above, let $B_T(H, \lambda)$ be the set of all analytic functions ξ on H such that:

(i) $D\xi = \lambda^{\$}(D)\xi$, $D \in I(\mathfrak{H}_c)$;

(ii) $\xi(wh) = \varepsilon(w, h)\xi(h)$, $w \in W(G, H) = N_G(H)/C_G(H)$, $h \in H'$, and $\varepsilon(w, h) = \pm 1$ is defined by $\varepsilon_R^H(wh)\Delta^H(wh) = \varepsilon(w, h)\varepsilon_R^H(h)\Delta^H(h)$;

(iii) $\sup_{h \in H} (1 + \sigma(h))^{-s} |\xi(h)| < +\infty$ for some positive number s, where σ is the function on G defined in [5, vol. II, p. 66].

The following are proved by Hirai in [4].

THEOREM 1. *Let $\pi \in A_T(\lambda)$, and H one of its highest Cartan subgroups. Then the highest part k^H of π on H is a nonzero element of $B_T(H, \lambda)$. Conversely, for any nonzero ξ in $B_T(H, \lambda)$, there exists at least one extremal element in $A_T(\lambda)$ with highest part ξ. The space $A_T(\lambda)$ is spanned by extremal elements.*

COROLLARY 2. *Suppose $B_T(H, \lambda) \neq \{0\}$. Every extremal element of $A_T(\lambda)$ with height $[H]$ is uniquely determined by its highest part if and only if $B_T(H', \lambda) = \{0\}$ for any $[H'] \in \mathrm{Car}(G)$ such that $[H'] < [H]$.*

LEMMA 3. *In order that $B_T(H, \lambda) \neq \{0\}$ it is necessary that:*

(i) *there exist $\mu \in \sqrt{-1}\mathfrak{H}^*$ such that $\lambda^{\$} = \lambda_\mu$;*

(ii) *$\exp X \to \exp(\mu(X))$ is a well-defined unitary character of the connected component of the identity in H;*

(iii) *$\langle \mu, \beta \rangle \neq 0$ for any compact root β of $\mathfrak{H}$.*

Further, in order that $B_T(H, \lambda) \neq \{0\}$ it is sufficient that in addition to (i) and (ii), we have:

(iii)′ μ is I-regular; that is, $\langle \mu, \beta \rangle \neq 0$ for any imaginary root β of $\mathfrak{H}$.

LEMMA 4. *Let $[\mathfrak{H}] \in \mathrm{Car}(\mathfrak{G})$, $[\mathfrak{H}]$ not minimal. Assume that $\mu \in \sqrt{-1}\mathfrak{H}^*$, $\lambda_\mu = \lambda^{\$}$, $B_T(H, \lambda) \neq \{0\}$. Then in order that every extremal element of $A_T(\lambda)$ of height $[H]$ be uniquely determined by its highest part, it is necessary that μ be I-regular.*

Finally, Hirai states the following lemma in the case that $\mathfrak{G}$ is a simple real Lie algebra of classical type or of type G_2.

LEMMA 5. *Let $\mathfrak{G}$ be a reductive real Lie algebra, and $\mathfrak{H}$ a Cartan subalgebra, not minimal with respect to the ordering on $\mathrm{Car}(\mathfrak{G})$. Let $\mu \in \sqrt{-1}\mathfrak{H}^*$ be I-regular, and let $\mathfrak{H}'$ be a Cartan subalgebra of $\mathfrak{G}$ such that $[\mathfrak{H}'] < [\mathfrak{H}]$. Let ν' be any inner automorphism*

of $\mathfrak{g}_c$ *such that* $\nu\mathfrak{H}_c = \mathfrak{H}'_c$. *Then* $\nu\mu \in (\mathfrak{H}'_c)^*$ *is never pure imaginary on* $\mathfrak{H}'$.

The main result of this paper is that the above lemma is true as stated without restriction on the type of $\mathfrak{g}$, and can be proved by a general argument. Using this lemma, together with Corollary 2 and Lemmas 3 and 4, we obtain the following theorem.

THEOREM 6. *Let* G *be a connected, acceptable, real reductive Lie group with compact center. Let* H *be a Cartan subgroup of* G *with Lie algebra* $\mathfrak{H}$, $[\mathfrak{H}]$ *not minimal in* $\mathrm{Car}(\mathfrak{g})$. *Assume that* $B_T(H, \lambda) \neq \{0\}$, *where* $\lambda^{\mathfrak{h}} = \lambda_\mu$, $\mu \in \sqrt{-1}\mathfrak{H}^*$. *Then any extremal element* π *in* $A_T(\lambda)$ *of height* $[H]$ *is uniquely determined by its highest part if and only if* μ *is* I*-regular.*

3. **Proof of Lemma 5.** For any Lie algebra $\mathfrak{G}$ and Cartan subalgebra $\mathfrak{H}$, let $\Phi = \Phi(\mathfrak{G}_c, \mathfrak{H}_c)$ denote the set of roots of the complexification, Φ_R, Φ_I, and Φ_{CPX} the subsets of Φ consisting of roots which are purely real, imaginary, or neither on $\mathfrak{H}$ respectively. For $\alpha \varepsilon \Phi$, H_α denotes the element of $\mathfrak{H}_c$ associated to α by the Killing form on $\mathfrak{H}_c$, $\mathscr{H}_\alpha$ the hyperplane in $\mathfrak{H}_c^*$ perpendicular to α, and s_α the reflection through $\mathscr{H}_\alpha$. For $\gamma \in \Phi_{CPX}$, γ^σ denotes the conjugate of γ, $\gamma^\sigma(H) = \overline{\gamma(\sigma(H))}$, $H \in \mathfrak{H}_c$, where σ is the conjugation of $\mathfrak{G}_c$ with respect to $\mathfrak{G}$. γ_I and γ_R denote the imaginary and real parts of γ respectively.

Let $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ be a Cartan decomposition of $\mathfrak{G}$ with $\mathfrak{K}$ a maximal compact subalgebra for which $\mathfrak{H} = (\mathfrak{H} \cup \mathfrak{K}) + (\mathfrak{H} \cup \mathfrak{P}) = \mathfrak{H}_k + \mathfrak{H}_p$. Set $\sqrt{-1}\mathfrak{H}_k^* = \{\delta \in \sqrt{-1}\mathfrak{H}^* : \delta|_{\mathfrak{H}_p} = 0\}$, and $\mathfrak{H}_p^* = \{\delta \in \mathfrak{H}^* : \delta|_{\mathfrak{H}_k} = 0\}$.

Lemma 5 is proved by a series of reductions. First, by analyzing the inner automorphism $\nu : \mathfrak{H}_c \to \mathfrak{H}'_c$, $[\mathfrak{H}'] < [\mathfrak{H}]$, we reduce the problem to a question just involving the root system $\Phi(\mathfrak{G}_c, \mathfrak{H}_c)$. We must prove that if $\mu \in \sqrt{-1}\mathfrak{H}_k^*$ is I-regular, and if $w\mu \in \sqrt{-1}\mathfrak{H}_k^*$ for some $w \in W = W(\mathfrak{G}_c, \mathfrak{H}_c)$, then $w\mu$ is also I-regular.

If $w \in W$ stabilizes Φ_I, then clearly $w\mu$ is I-regular if and only if μ is. We examine the types of hyperplanes which can separate μ and $w\mu$. Since μ, $w\mu \in \sqrt{-1}\mathfrak{H}_k^*$, they cannot be separated by hyperplanes $\mathscr{H}_\alpha$, $\alpha \in \Phi_R$. We show that hyperplanes of the form $\mathscr{H}_\beta$, $\beta \in \Phi_I$, $\mathscr{H}_\gamma$, $\gamma \in \Phi_{CPX}$, $\langle \gamma, \gamma^\sigma \rangle \geqq 0$, and certain hyperplanes $\mathscr{H}_\gamma$, $\langle \gamma, \gamma^\sigma \rangle < 0$, can be crossed by elements of W which stabilize Φ_I. Thus we may as well assume that μ and $w\mu$ are separated only by hyperplanes $\mathscr{H}_\gamma$, $\langle \gamma, \gamma^\sigma \rangle < 0$, such that $\gamma + \gamma^\sigma = \alpha \in \Phi_R$ is orthogonal to every other real root.

This allows us to reduce the problem to a root system Φ' for which Φ'_R is of type $(A_1)^r$ for some integer r. We then show that the assumption that $w\mu$ is not I-regular leads to a contradiction.

Let $\mu \in \sqrt{-1}\mathfrak{H}^*$ be I-regular, and write $\mu = \mu_k + \sqrt{-1}\mu_p$, $\mu_k \in \sqrt{-1}\mathfrak{H}_k^*$, $\mu_p \in \mathfrak{H}_p^*$. Then μ_k is also I-regular. For any $[\mathfrak{H}'] < [\mathfrak{H}]$ and inner automorphism ν such that $\nu\mathfrak{H}_c = \mathfrak{H}'_c$, $\nu = \nu_0 w$ for some $w \in W$ and some inner automorphism ν_0 of $\mathfrak{G}_c$ satisfying $\nu_0\mathfrak{H}_c = \mathfrak{H}'_c$, $\nu_0\beta_0$ is a real root of $\mathfrak{H}'$ for some singular imaginary root β_0 of $\mathfrak{H}$, and $\nu_0\mathfrak{H}_p \subset \mathfrak{H}'$.

Suppose for some $\mathfrak{H}'$ and some inner automorphism $\nu = \nu_0 w$ as above, $\nu\mu \in \sqrt{-1}\mathfrak{H}'^*$. Then $\nu\mu_k \in \sqrt{-1}\mathfrak{H}'^*$ also. Since $\mu_k \in \sqrt{-1}\mathfrak{H}_k^* + \mathfrak{H}_p^*$, the real subspace of $\mathfrak{H}_c^*$ spanned by Φ, $w\mu_k \in \sqrt{-1}\mathfrak{H}_k^* + \mathfrak{H}_p^*$ also. But $\nu_0\mathfrak{H}_p \subset \mathfrak{H}'$, so if $w\mu_k$ is nonzero on $\mathfrak{H}_p$, $\nu_0 w\mu_k = \nu\mu_k$ takes nonzero real values on $\mathfrak{H}'$. Thus $w\mu_k \in \sqrt{-1}\mathfrak{H}_k^*$.

Since $w\mu_k$, $\beta_0 \in \sqrt{-1}\mathfrak{H}_k^*$, $\langle w\mu_k, \beta_0\rangle$ is real. But $\langle w\mu_k, \beta_0\rangle = w\mu_k(H_{\beta_0}) = \nu\mu_k(H_{\nu_0\beta_0})$ where $H_{\nu_0\beta_0} \in \mathfrak{H}'$ since $\nu_0\beta_0$ is a real root of $\mathfrak{H}'$. Thus $\langle w\mu_k, \beta_0\rangle = 0$. Thus to prove that $\nu\mu$ is never pure imaginary on $\mathfrak{H}'$, it is sufficient to prove that $w\mu_k$ is I-regular for any $w \in W$ for which $w\mu_k \in \sqrt{-1}\mathfrak{H}_k^*$, and we may as well assume that $\mu = \mu_k \in \sqrt{-1}\mathfrak{H}_k^*$.

Assume that for $w \in W$, $w\mu \in \sqrt{-1}\mathfrak{H}_k^*$. We can pick $w' \in W_I$, the subgroup of W generated by s_β, $\beta \in \Phi_I$, such that $w'\mu$ and $w\mu$ are separated by no hyperplanes $\mathscr{H}_\beta$, $\beta \in \Phi_I$. Further, $w'\mu \in \sqrt{-1}\mathfrak{H}_k^*$ and is I-regular since W_I stabilizes $\sqrt{-1}\mathfrak{H}_k^*$ and Φ_I.

For $\gamma \in \Phi_{CPX}$, $\langle \gamma, \delta\rangle = -\langle \gamma^\sigma, \delta\rangle$ for any $\delta \in \sqrt{-1}\mathfrak{H}_k^*$, so that $\mathscr{H}_\gamma$ separates $w'\mu$ and $w\mu$ if and only if $\mathscr{H}_{\gamma^\sigma}$ does. If $\langle \gamma, \gamma^\sigma\rangle > 0$, $\gamma - \gamma^\sigma \in \Phi_I$ and $\mathscr{H}_\gamma$ and $\mathscr{H}_{\gamma\sigma}$ separate $w'\mu$ and $w\mu$ if and only if $\mathscr{H}_{\gamma-\gamma^\sigma}$ does. Thus $w'\mu$ and $w\mu$ are not separated by hyperplanes $\mathscr{H}_\gamma$, $\langle \gamma, \gamma^\sigma\rangle > 0$.

Suppose they are separated by hyperplanes $\mathscr{H}_\gamma$, $\langle \gamma, \gamma^\sigma\rangle = 0$. For such γ, $s_\gamma s_{\gamma^\sigma}$ stabilizes $\sqrt{-1}\mathfrak{H}_k^*$ and Φ_I. Therefore we can choose $\gamma_1, \cdots, \gamma_k \in \Phi_{CPX}$ such that $\langle \gamma_i, \gamma_i^\sigma\rangle = 0$, $i = 1, \cdots, k$, and $w\mu$ and $s_{\gamma_1}s_{\gamma_1^\sigma} \cdots s_{\gamma_k}s_{\gamma_k^\sigma} w'\mu = w''w'\mu$ are separated only by hyperplanes $\mathscr{H}_\gamma$, $\langle \gamma, \gamma^\sigma\rangle < 0$. Again, $w''w'\mu \in \sqrt{-1}\mathfrak{H}_k^*$ and is I-regular. Thus we may assume that μ and $w\mu$ are separated only by hyperplanes $\mathscr{H}_\gamma$, $\langle \gamma, \gamma^\sigma\rangle < 0$.

For such a root γ, $\gamma + \gamma^\sigma = \alpha \in \Phi_R$, so $\gamma = \alpha/2 + \gamma_I$, $\|\gamma_I\|^2 > \|\alpha\|^2/4$. Using the facts that $2\langle \gamma, \gamma^\sigma\rangle/\langle \gamma, \gamma\rangle$ and $2\langle \alpha, \gamma\rangle/\langle \gamma, \gamma\rangle$ are integers, we find that $\|\alpha\|^2 = \|\gamma\|^2$. Suppose there is $\alpha_1 \in \Phi_R$ such that $\langle \alpha, \alpha_1\rangle < 0$, $\alpha_1 \neq -\alpha$. Then $\langle \alpha_1, \gamma\rangle < 0$, so $\gamma_1 = \alpha_1 + \gamma \in \Phi$. If $\|\alpha\|^2 \neq \|\alpha_1\|^2$, $2\langle \alpha_1, \gamma\rangle/\langle \alpha_1, \alpha_1\rangle$ or $2\langle \alpha_1, \gamma\rangle/\langle \gamma, \gamma\rangle$ is not an integer. Thus $\|\alpha\|^2 = \|\alpha_1\|^2$ so that $\langle \gamma_1, \gamma_1^\sigma\rangle = 0$. For any $\delta \in \sqrt{-1}\mathfrak{H}_k^*$, $\langle \gamma_1, \delta\rangle = \langle \gamma, \delta\rangle$, so that if $\mathscr{H}_\gamma$ separates μ and $w\mu$, so does $\mathscr{H}_{\gamma_1}$. But this contradicts the assumption that μ and $w\mu$ are separated by no

complex hyperplanes $\mathscr{H}_{\gamma_1}$, $\langle \gamma_1, \gamma_1^\sigma \rangle = 0$. Thus if $\mathscr{H}_\gamma$ separates μ and $w\mu$, $\gamma_R = \alpha/2$, α is orthogonal to every other positive real root.

Pick $\gamma_1, \cdots, \gamma_k$, a subset of the set of all γ such that $\mathscr{H}_\gamma$ separates μ and $w\mu$, for which $w\mu = s_{\gamma_1} \cdots s_{\gamma_k} \mu$. Let $\alpha_1, \cdots \alpha_r$ be the distinct real roots which appear as $\gamma_i + \gamma_i^\sigma$, $i = 1, \cdots, k$. Then $\mathfrak{H}_p = \sum_{i=1}^r \boldsymbol{R} H_{\alpha_i} + \mathfrak{H}_p^1$ where $\mathfrak{H}_p^1 = \{H \in \mathfrak{H}_p : \alpha_i(H) = 0,\ i = 1, \cdots r\}$.

Let $\mathfrak{M}$ be the centralizer of $\mathfrak{H}_p^1$ in $\mathfrak{G}$. Then $\mathfrak{M}$ is a reductive Lie algebra with Cartan subalgebra $\mathfrak{H}$, and $s_{\gamma_1} \cdots s_{\gamma_k}$ stabilizes $\Phi' = \Phi(\mathfrak{M}_c, \mathfrak{H}_c)$. Further, $\Phi'_R = \{\pm\alpha_1, \cdots, \pm\alpha_k\}$ and $\Phi'_I = \Phi_I$. Let $\mathfrak{Z}$ be the centralizer in $\mathfrak{M}$ of $H_{\sqrt{-1}\mu}$. Then $\Phi(\mathfrak{Z}_c, \mathfrak{H}_c) = \Phi'(\mu) = \{\alpha \in \Phi' : \langle \alpha, \mu \rangle = 0\}$, and $s_{\gamma_1} \cdots s_{\gamma_k} \Phi'(\mu) = \Phi'(w\mu) = \{\alpha \in \Phi' : \langle \alpha, w\mu \rangle = 0\}$. If $w\mu$ is not I-regular, i.e., $\langle w\mu, \beta \rangle = 0$ for some $\beta \in \Phi_I$, then $\Phi'(w\mu)$ contains at least $r + 1$ mutually orthogonal roots, namely $\alpha_1 \cdots, \alpha_r$, β, and hence so does $\Phi'(\mu)$.

A real root system is called σ-normal if $\alpha - \alpha^\sigma \notin \Phi$ for every $\alpha \in \Phi$ [3]. $\Phi'(\mu)$ contains $\pm\alpha_1, \cdots \pm\alpha_r$, some complex roots, but no imaginary roots since μ is I-regular. Since $\alpha - \alpha^\sigma$ takes purely imaginary values on $\mathfrak{H}$ for any $\alpha \in \Phi'(\mu)$, $\alpha - \alpha^\sigma \notin \Phi'(\mu)$, and so $\Phi'(\mu)$ is a σ-normal real root system.

Write $\Phi'(\mu) = \Phi^1 \cup \cdots \cup \Phi^s$ where Φ^i is simple, $1 \leqq i \leqq s$ and $s \leqq r$. Then each Φ^i is a simple, σ-normal real root system with Φ_R^i of type $(A_1)^{r_i}$ and no imaginary roots. The possible simple σ-normal real root systems are classified, and we see by scanning a table such as [5, vol I, p. 30-32] that Φ^i can only be a real form of type A_{2r_i} or A_{2r_i-1}. In either case, Φ^i contains exactly r_i mutually orthogonal roots, and thus $\Phi'(\mu)$ contains only r mutually orthogonal roots. Therefore, $w\mu$ must be I-regular.

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