RATIONAL APPROXIMATION TO $x^n$

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This note is concerned with the approximations of $x^n$ on $[0, 1]$ by polynomials and rational functions having only non-negative coefficients and of degree at most $k(1 \leq k \leq n - 1)$. It is shown that the best approximating polynomial of degree $k$ on $[0, 1]$ to $x^n$ is of the form

$$p_k(x) = dx^k,$$

where $d > 0$ and satisfies the assumption that

$$n(1 - d) = (n - k)\left(\frac{k}{n}\right)^{k/(n-k)} d^{n/(n-k)},$$

with an error $\epsilon_k = 1 - d$, for each fixed $k = 1, 2, 3, \ldots, n - 1$. It is also shown that $dx^k$ is a best approximating rational function of degree $k$ to $x^n$ on $[0, 1]$.

More than one hundred years ago Chebyshev showed that $x^n$ can be uniformly approximated on $[-1, 1]$ by polynomials of degree at most $(n - 1)$ with an error of exactly $2^{-n+1}$.

Just recently D. J. Newman [1] has shown that $x^n$ can be uniformly approximated on $[-1, 1]$ by rational functions of degree at most $(n - 1)$ with an error roughly $\sqrt{n}(3\sqrt{3})^{-n}$.

If one looks carefully at the above results, then the following questions arise naturally.

**Q.1:** How close can one approximate $x^n$ uniformly on $[0, 1]$ by polynomials of degree $(n - 1)$ having only non-negative coefficients?

**Q.2:** Is the error obtained by rational functions of degree $(n - 1)$ having only nonnegative coefficients in approximating $x^n$ on $[0, 1]$ less than the error obtained by polynomials of degree $(n - 1)$ having only nonnegative coefficients?

We answer these questions in this note.

Let

$$\varepsilon_k = \inf_{p \in \pi_k^*} \|x^n - p(x)\|_{L^\infty[0, 1]}$$

where $\pi_k^*(1 \leq k < n)$ denotes the class of all algebraic polynomials of degree at most $k$ having only nonnegative coefficients.

Let

$$\theta_k = \inf_{p, q \in \pi_k^*} \left\| \frac{x^n - p(x)}{q(x)} \right\|_{L^\infty[0, 1]}.$$

We have

$$\varepsilon_k \leq \theta_k \leq \sqrt{n}(3\sqrt{3})^{-n}.$$

Thus, for $k = 1, 2, \ldots, n - 1$, we have

$$\theta_k \geq 1 - \sqrt{n}(3\sqrt{3})^{-n}.$$
THEOREM 1. If \( p_k(x) = dx^k, 1 \leq k < n \), where \( d > 0 \) and satisfies the assumption that

\[
n(1 - d) = (n - k) \left( \frac{k}{n} \right)^{k/(n-k)} d^{n/(n-k)}
\]

then \( p_k(x) \) is a best approximating polynomial to \( x^n \) in the sense of (1). In fact, we get

\[
n \varepsilon_k = (n - k) \left( \frac{k}{n} \right)^{k/(n-k)} (1 - \varepsilon_k)^{n/(n-k)}.
\]

Proof. Let

\[
p_k(x) = dx^k
\]

then it is easy to see by finding a point where \( |x^n - p_k(x)| \) attains its maximum on \([0, 1]\), that

\[
\varepsilon_k = \|x^n - p_k(x)\|_{L^\infty[0,1]} = \max \left\{ (1 - d), \left( \frac{n - k}{n} \right)^{k/(n-k)} d^{n/(n-k)} \right\}.
\]

From (2), it is clear that

\[
\varepsilon_k \leq \|x^n - p_k(x)\|_{L^\infty[0,1]} = (1 - d).
\]

So that, again by (2), we obtain

\[
n \varepsilon_k \leq (1 - \varepsilon_k)^{n/(n-k)} (n - k) \left( \frac{k}{n} \right)^{k/(n-k)}.
\]

Now we get the lower bound to \( n \varepsilon_k \).

From (1) and the nonnegativity of the coefficients we get

\[
\varepsilon_k \geq p(x) - x^n \geq [p(1)]x^k - x^n = [p(1) - 1]x^k + x^k - x^n
\]

\[
\geq x^k(-\varepsilon_k + 1 - x^{n-k})
\]

i.e.,

\[
\varepsilon_k \geq \frac{x^k(1 - x^{n-k})}{1 + x^k}
\]

\[
\frac{1 - x^{n-k}}{1 + x^k}
\]

attains its maximum for values of \( x \) satisfying

\[
x^{n-k} = \frac{k}{n} \left( \frac{1 + x^n}{1 + x^k} \right)
\]

Hence for this value of \( x \), we obtain
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(9)  \[ \varepsilon_k \geq x^k \left( \frac{n-k}{k} \right)^{x^{-k}} = \frac{x^n(n-k)}{k} = \frac{k-n}{k} x^{n-k} = 1 - \frac{n}{k} x^{n-k}. \]

From (9) we get
\[ x^{n-k} \geq (1 - \varepsilon_k) \frac{k}{n}, \]
i.e.,
\[ x \geq \left[ (1 - \varepsilon_k) \frac{k}{n} \right]^{1/(n-k)}. \]

From (9) and (10) we obtain
\[ \varepsilon_k \geq (1 - \varepsilon_k)^{n/(n-k)} \left( \frac{k}{n} \right)^{n/(n-k)} \left( \frac{n-k}{k} \right). \]

From (7) and (11) we get
\[ n \varepsilon_k = (1 - \varepsilon_k)^{n/(n-k)}(n-k) \left( \frac{k}{n} \right)^{k/(n-k)}. \]

Hence, $p_k(x) = d x^k$ is a best approximating polynomial in the sense of (1).

**THEOREM 2.**

(12)  \[ \varepsilon_k = \theta_k \text{ for all } k(1 \leq k < n). \]

**Proof.** By definition, for a $p(x)$ and $q(x)$, we have
\[ \left\| x^n - \frac{p(x)}{q(x)} \right\|_{L_\infty[0,1]} = \theta_k. \]

From (13) we get as earlier
\[ \theta_k \geq \frac{p(x)}{q(x)} - x^n \geq \frac{p(1)x^k}{q(1)} - x^n \]
\[ = \left( \frac{p(1)}{q(1)} - 1 \right)x^k + x^k - x^n \geq x^k(1 - x^{n-k} - \theta_k). \]
i.e.,
\[ \theta_k \geq \frac{x^k(1 - x^{n-k})}{1 + x^k}. \]

(8) and (15) being the same in terms of $x$, $n$ and $k$, we get
\[ n \theta_k \geq (n-k) \left( \frac{k}{n} \right)^{k/(n-k)} (1 - \theta_k)^{n/(n-k)}. \]
From Theorem 1 and (16), we obtain
\[(1 - \varepsilon_k)^{n/(n-k)} \left( \frac{k}{n} \right)^{k/(n-k)} \geq \varepsilon_k \left( \frac{n}{n-k} \right) \theta_k \geq (1 - \theta_k)^{n/(n-k)} \left( \frac{k}{n} \right)^{k/(n-k)} \geq (1 - \varepsilon_k)^{n/(n-k)} \left( \frac{k}{n} \right)^{k/(n-k)}.\]

(12) follows easily from (17). Hence the result is proved.

**Remarks on Theorems 1 and 2.** According to ([2], Theorem 6) \(p_k\) of our Theorem 1 is unique. Hence \(p_k\) is the best approximating polynomial in the sense of (1). (ii) As a result of Theorems 1 and 2 a best approximation to \(x^n\) in the sense of (1') is also
\[p_k(x) - dx^k,
\]
where \(d > 0\), satisfies (2). (iii) Let us suppose \(\varepsilon_k < 1 - d\), then from (2) and (3), we get \(\varepsilon_k > 1 - d\). Similarly, assume \(\varepsilon_k > 1 - d\), then we get from (2) and (3), \(\varepsilon_k < 1 - d\). Hence we have from (2) and (3),
\[\varepsilon_k = 1 - d, \text{ for each fixed } k = 1, 2, \ldots, n - 1.
\]
(iv) For the case \(k = n - 1\), we get
\[\theta_{n-1} = \varepsilon_{n-1} \sim \frac{c}{n},
\]
where \(c\) satisfies the equation \(ce^{c+1} = 1\).

**References**


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