

# Pacific Journal of Mathematics

**RATIONAL APPROXIMATION TO  $x^n$**

DONALD J. NEWMAN AND A. R. REDDY

## RATIONAL APPROXIMATION TO $x^n$

DONALD J. NEWMAN AND A. R. REDDY

**This note is concerned with the approximations of  $x^n$  on  $[0, 1]$  by polynomials and rational functions having only non-negative coefficients and of degree at most  $k$  ( $1 \leq k \leq n - 1$ ). It is shown that the best approximating polynomial of degree  $k$  on  $[0, 1]$  to  $x^n$  is of the form**

$$p_k(x) = dx^k,$$

where  $d > 0$  and satisfies the assumption that

$$n(1 - d) = (n - k) \left( \frac{k}{n} \right)^{k/(n-k)} d^{n/(n-k)},$$

with an error  $\varepsilon_k = 1 - d$ , for each fixed  $k = 1, 2, 3, \dots, n - 1$ . It is also shown that  $dx^k$  is a best approximating rational function of degree  $k$  to  $x^n$  on  $[0, 1]$ .

More than one hundred years ago Chebyshev showed that  $x^n$  can be uniformly approximated on  $[-1, 1]$  by polynomials of degree at most  $(n - 1)$  with an error of exactly  $2^{-n+1}$ .

Just recently D. J. Newman [1] has shown that  $x^n$  can be uniformly approximated on  $[-1, 1]$  by rational functions of degree at most  $(n - 1)$  with an error roughly  $\sqrt[n]{n}(3\sqrt{3})^{-n}$ .

If one looks carefully at the above results, then the following questions arise naturally.

Q.1: How close can one approximate  $x^n$  uniformly on  $[0, 1]$  by polynomials of degree  $(n - 1)$  having only non-negative coefficients?

Q.2: Is the error obtained by rational functions of degree  $(n - 1)$  having only nonnegative coefficients in approximating  $x^n$  on  $[0, 1]$  less than the error obtained by polynomials of degree  $(n - 1)$  having only nonnegative coefficients?

We answer these questions in this note.

Let

$$(1) \quad \varepsilon_k = \inf_{p \in \pi_k^*} \|x^n - p(x)\|_{L^\infty[0,1]}$$

where  $\pi_k^*$  ( $1 \leq k < n$ ) denotes the class of all algebraic polynomials of degree at most  $k$  having only nonnegative coefficients.

$$(1') \quad \theta_k = \inf_{p, q \in \pi_k^*} \left\| x^n - \frac{p(x)}{q(x)} \right\|_{L^\infty[0,1]}.$$

**THEOREM 1.** *If  $p_k(x) = dx^k$ ,  $1 \leq k < n$ , where  $d > 0$  and satisfies the assumption that*

$$(2) \quad n(1-d) = (n-k) \left(\frac{k}{n}\right)^{k/(n-k)} d^{n/(n-k)}$$

then  $p_k(x)$  is a best approximating polynomial to  $x^n$  in the sense of (1). In fact, we get

$$(3) \quad n\varepsilon_k = (n-k) \left(\frac{k}{n}\right)^{k/(n-k)} (1-\varepsilon_k)^{n/(n-k)}.$$

*Proof.* Let

$$(4) \quad p_k(x) = dx^k$$

then it is easy to see by finding a point where  $|x^n - p_k(x)|$  attains its maximum on  $[0, 1]$ , that

$$(5) \quad \varepsilon_k \leq \|x^n - p_k(x)\|_{L^\infty[0,1]} = \max \left\{ (1-d), \left(\frac{n-k}{n}\right) \left(\frac{k}{n}\right)^{k/(n-k)} d^{n/(n-k)} \right\}.$$

From (2), it is clear that

$$(6) \quad \varepsilon_k \leq \|x^n - p_k(x)\|_{L^\infty[0,1]} = (1-d).$$

So that, again by (2), we obtain

$$(7) \quad n\varepsilon_k \leq (1-\varepsilon_k)^{n/(n-k)} (n-k) \left(\frac{k}{n}\right)^{k/(n-k)}.$$

Now we get the lower bound to  $n\varepsilon_k$ .

From (1) and the nonnegativity of the coefficients we get

$$\begin{aligned} \varepsilon_k &\geq p(x) - x^n \geq [p(1)]x^k - x^n = [p(1) - 1]x^k + x^k - x^n \\ &\geq x^k(-\varepsilon_k + 1 - x^{n-k}) \end{aligned}$$

i.e.,

$$(8) \quad \varepsilon_k \geq \frac{x^k(1-x^{n-k})}{1+x^k}.$$

$\frac{(1-x^{n-k})x^k}{1+x^k}$  attains its maximum for values of  $x$  satisfying

$$x^{n-k} = \frac{k}{n} \left(\frac{1+x^n}{1+x^k}\right).$$

Hence for this value of  $x$ , we obtain

$$(9) \quad \varepsilon_k \geq x^k \left( \frac{n-k}{k} \right) x^{n-k} = \frac{x^n(n-k)}{k} = \frac{k-nx^{n-k}}{k} = 1 - \frac{n}{k} x^{n-k}.$$

From (9) we get

$$x^{n-k} \geq (1 - \varepsilon_k) \frac{k}{n}$$

i.e.,

$$(10) \quad x \geq \left[ (1 - \varepsilon_k) \frac{k}{n} \right]^{1/(n-k)}.$$

From (9) and (10) we obtain

$$(11) \quad \varepsilon_k \geq (1 - \varepsilon_k)^{n/(n-k)} \left( \frac{k}{n} \right)^{n/(n-k)} \left( \frac{n-k}{k} \right).$$

From (7) and (11) we get

$$n \varepsilon_k = (1 - \varepsilon_k)^{n/(n-k)} (n-k) \left( \frac{k}{n} \right)^{k/(n-k)}.$$

Hence,  $p_k(x) = d x^k$  is a best approximating polynomial in the sense of (1).

#### THEOREM 2.

$$(12) \quad \varepsilon_k = \theta_k \text{ for all } k(1 \leq k < n).$$

*Proof.* By definition, for a  $p(x)$  and  $q(x)$ , we have

$$(13) \quad \left\| x^n - \frac{p(x)}{q(x)} \right\|_{L^\infty[0,1]} = \theta_k.$$

From (13) we get as earlier

$$(14) \quad \begin{aligned} \theta_k &\geq \frac{p(x)}{q(x)} - x^n \geq \frac{p(1)x^k}{q(1)} - x^n \\ &= \left( \frac{p(1)}{q(1)} - 1 \right) x^k + x^k - x^n \geq x^k(1 - x^{n-k} - \theta_k). \end{aligned}$$

i.e.,

$$(15) \quad \theta_k \geq \frac{x^k(1 - x^{n-k})}{1 + x^k}$$

(8) and (15) being the same in terms of  $x$ ,  $n$  and  $k$ , we get

$$(16) \quad n \theta_k \geq (n-k) \left( \frac{k}{n} \right)^{k/(n-k)} (1 - \theta_k)^{n/(n-k)}.$$

From Theorem 1 and (16), we obtain

$$(17) \quad (1 - \varepsilon_k)^{n/(n-k)} \left(\frac{k}{n}\right)^{k/(n-k)} \geq \varepsilon_k \left(\frac{n}{n-k}\right) \geq \left(\frac{n}{n-k}\right) \theta_k \\ \geq (1 - \theta_k)^{n/(n-k)} \left(\frac{k}{n}\right)^{k/(n-k)} \geq (1 - \varepsilon_k)^{n/(n-k)} \left(\frac{k}{n}\right)^{k/(n-k)} .$$

(12) follows easily from (17). Hence the result is proved.

Remarks on Theorems 1 and 2. According to ([2], Theorem 6)  $p_k$  of our Theorem 1 is unique. Hence  $p_k$  is the best approximating polynomial in the sense of (1). (ii) As a result of Theorems 1 and 2 a best approximation to  $x^n$  in the sense of (1') is also

$$p_k(x) - dx^k ,$$

where  $d > 0$ , satisfies (2). (iii) Let us suppose  $\varepsilon_k < 1 - d$ , then from (2) and (3), we get  $\varepsilon_k > 1 - d$ . Similarly, assume  $\varepsilon_k > 1 - d$ , then we get from (2) and (3),  $\varepsilon_k < 1 - d$ . Hence we have from (2) and (3),

$$\varepsilon_k = 1 - d, \text{ for each fixed } k = 1, 2, \dots, n - 1 .$$

(iv) For the case  $k = n - 1$ , we get

$$\theta_{n-1} = \varepsilon_{n-1} \sim \frac{c}{n} ,$$

where  $c$  satisfies the equation  $ce^{c+1} = 1$ .

#### REFERENCES

1. D. J. Newman, *Rational approximation to  $x^n$* , J. Approximation Theory, to appear.
2. J. A. Roulier and G. D. Taylor, *Uniform approximation having bounded coefficients*, Abhand. aus dem Math. Sem. der Univ. Hamburg band, **36** (1971), 126-135.

Received May 26, 1976 and in revised form July 21, 1976.

YESHIVA UNIVERSITY  
AND  
INSTITUTE FOR ADVANCED STUDY

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

**RICHARD ARENS** (Managing Editor)  
University of California  
Los Angeles, California 90024

**J. DUGUNDJI**  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

**R. A. BEAUMONT**  
University of Washington  
Seattle, Washington 98105

**D. GILBARG AND J. MILGRAM**  
Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

**E. F. BECKENBACH**

**B. H. NEUMANN**

**F. WOLF**

**K. YOSHIDA**

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

# Pacific Journal of Mathematics

Vol. 67, No. 1

January, 1976

Gregory Wayne Brumfiel and John W. Morgan, <i>Homotopy theoretic consequences of N. Levitt's obstruction theory to transversality for spherical fibrations</i> . . . . .	1
Jacob Burbea, <i>Total positivity of certain reproducing kernels</i> . . . . .	101
Wai-Mee Ching, <i>The structure of standard <math>C^*</math>-algebras and their representations</i> . . . . .	131
Satya Deo, <i>The cohomological dimension of an <math>n</math>-manifold is <math>n + 1</math></i> . . . . .	155
Masahiko Fujiwara and Masaki Sudo, <i>Some forms of odd degree for which the Hasse principle fails</i> . . . . .	161
Mikihiro Hayashi, <i>Smoothness of analytic functions at boundary points</i> . . . .	171
Rebecca A. Herb, <i>A uniqueness theorem for tempered invariant eigendistributions</i> . . . . .	203
David Alan Legg, <i>Orlicz space convergence of martingales of Radon-Nikodým derivatives given a <math>\sigma</math>-lattice</i> . . . . .	209
D. B. McAlister, <i><math>v</math>-prehomomorphisms on inverse semigroups</i> . . . . .	215
Bruno J. Mueller, <i>Localization in fully bounded Noetherian rings</i> . . . . .	233
Donald J. Newman and A. R. Reddy, <i>Rational approximation to <math>x^n</math></i> . . . . .	247
Abraham Ziv, <i>Inclusion relations between power methods of limitation</i> . . . . .	251