

# Pacific Journal of Mathematics

**RATIONAL APPROXIMATION TO  $x^n$**

DONALD J. NEWMAN AND A. R. REDDY

## RATIONAL APPROXIMATION TO $x^n$

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This note is concerned with the approximations of  $x^n$  on  $[0, 1]$  by polynomials and rational functions having only non-negative coefficients and of degree at most  $k$  ( $1 \leq k \leq n - 1$ ). It is shown that the best approximating polynomial of degree  $k$  on  $[0, 1]$  to  $x^n$  is of the form

$$p_k(x) = dx^k,$$

where  $d > 0$  and satisfies the assumption that

$$n(1 - d) = (n - k) \left( \frac{k}{n} \right)^{k/(n-k)} d^{n/(n-k)},$$

with an error  $\varepsilon_k = 1 - d$ , for each fixed  $k = 1, 2, 3, \dots, n - 1$ . It is also shown that  $dx^k$  is a best approximating rational function of degree  $k$  to  $x^n$  on  $[0, 1]$ .

More than one hundred years ago Chebyshev showed that  $x^n$  can be uniformly approximated on  $[-1, 1]$  by polynomials of degree at most  $(n - 1)$  with an error of exactly  $2^{-n+1}$ .

Just recently D. J. Newman [1] has shown that  $x^n$  can be uniformly approximated on  $[-1, 1]$  by rational functions of degree at most  $(n - 1)$  with an error roughly  $\sqrt{n}(3\sqrt{3})^{-n}$ .

If one looks carefully at the above results, then the following questions arise naturally.

Q.1: How close can one approximate  $x^n$  uniformly on  $[0, 1]$  by polynomials of degree  $(n - 1)$  having only non-negative coefficients?

Q.2: Is the error obtained by rational functions of degree  $(n - 1)$  having only nonnegative coefficients in approximating  $x^n$  on  $[0, 1]$  less than the error obtained by polynomials of degree  $(n - 1)$  having only nonnegative coefficients?

We answer these questions in this note.

Let

$$(1) \quad \varepsilon_k = \inf_{p \in \pi_k^*} \|x^n - p(x)\|_{L^\infty[0,1]}$$

where  $\pi_k^*$  ( $1 \leq k < n$ ) denotes the class of all algebraic polynomials of degree at most  $k$  having only nonnegative coefficients.

$$(1') \quad \theta_k = \inf_{p, q \in \pi_k^*} \left\| x^n - \frac{p(x)}{q(x)} \right\|_{L^\infty[0,1]}.$$

**THEOREM 1.** *If  $p_k(x) = dx^k$ ,  $1 \leq k < n$ , where  $d > 0$  and satisfies the assumption that*

$$(2) \quad n(1-d) = (n-k) \left(\frac{k}{n}\right)^{k/(n-k)} d^{n/(n-k)}$$

*then  $p_k(x)$  is a best approximating polynomial to  $x^n$  in the sense of (1). In fact, we get*

$$(3) \quad n\varepsilon_k = (n-k) \left(\frac{k}{n}\right)^{k/(n-k)} (1-\varepsilon_k)^{n/(n-k)}.$$

*Proof.* Let

$$(4) \quad p_k(x) = dx^k$$

then it is easy to see by finding a point where  $|x^n - p_k(x)|$  attains its maximum on  $[0, 1]$ , that

$$(5) \quad \varepsilon_k \leq \|x^n - p_k(x)\|_{L^\infty[0,1]} = \max \left\{ (1-d), \left(\frac{n-k}{n}\right) \left(\frac{k}{n}\right)^{k/(n-k)} d^{n/(n-k)} \right\}.$$

From (2), it is clear that

$$(6) \quad \varepsilon_k \leq \|x^n - p_k(x)\|_{L^\infty[0,1]} = (1-d).$$

So that, again by (2), we obtain

$$(7) \quad n\varepsilon_k \leq (1-\varepsilon_k)^{n/(n-k)} (n-k) \left(\frac{k}{n}\right)^{k/(n-k)}.$$

Now we get the lower bound to  $n\varepsilon_k$ .

From (1) and the nonnegativity of the coefficients we get

$$\begin{aligned} \varepsilon_k &\geq p(x) - x^n \geq [p(1)]x^k - x^n = [p(1) - 1]x^k + x^k - x^n \\ &\geq x^k(-\varepsilon_k + 1 - x^{n-k}) \end{aligned}$$

i.e.,

$$(8) \quad \varepsilon_k \geq \frac{x^k(1-x^{n-k})}{1+x^k}.$$

$\frac{(1-x^{n-k})x^k}{1+x^k}$  attains its maximum for values of  $x$  satisfying

$$x^{n-k} = \frac{k}{n} \left( \frac{1+x^n}{1+x^k} \right).$$

Hence for this value of  $x$ , we obtain

$$(9) \quad \varepsilon_k \geq x^k \left( \frac{n-k}{k} \right) x^{n-k} = \frac{x^n(n-k)}{k} = \frac{k-n}{k} x^{n-k} = 1 - \frac{n}{k} x^{n-k}.$$

From (9) we get

$$x^{n-k} \geq (1 - \varepsilon_k) \frac{k}{n}$$

i.e.,

$$(10) \quad x \geq \left[ (1 - \varepsilon_k) \frac{k}{n} \right]^{1/(n-k)}.$$

From (9) and (10) we obtain

$$(11) \quad \varepsilon_k \geq (1 - \varepsilon_k)^{n/(n-k)} \left( \frac{k}{n} \right)^{n/(n-k)} \left( \frac{n-k}{k} \right).$$

From (7) and (11) we get

$$n \varepsilon_k = (1 - \varepsilon_k)^{n/(n-k)} (n-k) \left( \frac{k}{n} \right)^{k/(n-k)}.$$

Hence,  $p_k(x) = d x^k$  is a best approximating polynomial in the sense of (1).

#### THEOREM 2.

$$(12) \quad \varepsilon_k = \theta_k \text{ for all } k(1 \leq k < n).$$

*Proof.* By definition, for a  $p(x)$  and  $q(x)$ , we have

$$(13) \quad \left\| x^n - \frac{p(x)}{q(x)} \right\|_{L^\infty[0,1]} = \theta_k.$$

From (13) we get as earlier

$$(14) \quad \begin{aligned} \theta_k &\geq \frac{p(x)}{q(x)} - x^n \geq \frac{p(1)x^k}{q(1)} - x^n \\ &= \left( \frac{p(1)}{q(1)} - 1 \right) x^k + x^k - x^n \geq x^k(1 - x^{n-k} - \theta_k). \end{aligned}$$

i.e.,

$$(15) \quad \theta_k \geq \frac{x^k(1 - x^{n-k})}{1 + x^k}$$

(8) and (15) being the same in terms of  $x$ ,  $n$  and  $k$ , we get

$$(16) \quad n \theta_k \geq (n-k) \left( \frac{k}{n} \right)^{k/(n-k)} (1 - \theta_k)^{n/(n-k)}.$$

From Theorem 1 and (16), we obtain

$$(17) \quad (1 - \varepsilon_k)^{n/(n-k)} \left(\frac{k}{n}\right)^{k/(n-k)} \geq \varepsilon_k \left(\frac{n}{n-k}\right) \geq \left(\frac{n}{n-k}\right) \theta_k \\ \geq (1 - \theta_k)^{n/(n-k)} \left(\frac{k}{n}\right)^{k/(n-k)} \geq (1 - \varepsilon_k)^{n/(n-k)} \left(\frac{k}{n}\right)^{k/(n-k)} .$$

(12) follows easily from (17). Hence the result is proved.

Remarks on Theorems 1 and 2. According to ([2], Theorem 6)  $p_k$  of our Theorem 1 is unique. Hence  $p_k$  is the best approximating polynomial in the sense of (1). (ii) As a result of Theorems 1 and 2 a best approximation to  $x^n$  in the sense of (1') is also

$$p_k(x) - dx^k ,$$

where  $d > 0$ , satisfies (2). (iii) Let us suppose  $\varepsilon_k < 1 - d$ , then from (2) and (3), we get  $\varepsilon_k > 1 - d$ . Similarly, assume  $\varepsilon_k > 1 - d$ , then we get from (2) and (3),  $\varepsilon_k < 1 - d$ . Hence we have from (2) and (3),

$$\varepsilon_k = 1 - d, \text{ for each fixed } k = 1, 2, \dots, n - 1 .$$

(iv) For the case  $k = n - 1$ , we get

$$\theta_{n-1} = \varepsilon_{n-1} \sim \frac{c}{n} ,$$

where  $c$  satisfies the equation  $ce^{c+1} = 1$ .

## REFERENCES

1. D. J. Newman, *Rational approximation to  $x^n$* , J. Approximation Theory, to appear.
2. J. A. Roulier and G. D. Taylor, *Uniform approximation having bounded coefficients*, Abhand. aus dem Math. Sem. der Univ. Hamburg band, **36** (1971), 126-135.

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