RATIONAL APPROXIMATION TO $x^n$

Donald J. Newman and A. R. Reddy
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DONALD J. NEWMAN AND A. R. REDDY

This note is concerned with the approximations of $x^n$ on $[0,1]$ by polynomials and rational functions having only non-negative coefficients and of degree at most $k(1 \leq k \leq n - 1)$. It is shown that the best approximating polynomial of degree $k$ on $[0,1]$ to $x^n$ is of the form

$$p_k(x) = dx^k,$$

where $d > 0$ and satisfies the assumption that

$$n(1 - d) = (n - k) \left( \frac{k}{n} \right)^{k/(n-k)} d^{n/(n-k)},$$

with an error $\epsilon_k = 1 - d$, for each fixed $k = 1, 2, 3, \ldots, n - 1$. It is also shown that $dx^k$ is a best approximating rational function of degree $k$ to $x^n$ on $[0,1]$.

More than one hundred years ago Chebyshev showed that $x^n$ can be uniformly approximated on $[-1,1]$ by polynomials of degree at most $(n - 1)$ with an error of exactly $2^{-n+1}$.

Just recently D. J. Newman [1] has shown that $x^n$ can be uniformly approximated on $[-1,1]$ by rational functions of degree at most $(n - 1)$ with an error roughly $\sqrt{n/3^n}$.

If one looks carefully at the above results, then the following questions arise naturally.

Q.1: How close can one approximate $x^n$ uniformly on $[0,1]$ by polynomials of degree $(n - 1)$ having only non-negative coefficients?

Q.2: Is the error obtained by rational functions of degree $(n - 1)$ having only nonnegative coefficients in approximating $x^n$ on $[0,1]$ less than the error obtained by polynomials of degree $(n - 1)$ having only nonnegative coefficients?

We answer these questions in this note.

Let

$$\epsilon_k = \inf_{p \in \pi_k^*} \|x^n - p(x)\|_{L^\infty[0,1]},$$

where $\pi_k^*(1 \leq k < n)$ denotes the class of all algebraic polynomials of degree at most $k$ having only nonnegative coefficients.

Let

$$\theta_k = \inf_{p, q \in \pi_k^*} \|x^n - \frac{p(x)}{q(x)}\|_{L^\infty[0,1]}.$$
Theorem 1. If \( p_k(x) = dx^k, 1 \leq k < n \), where \( d > 0 \) and satisfies the assumption that
\[
(n(1 - d) = (n - k) \left( \frac{k}{n} \right)^{k/(n-k)}d^{n/(n-k)}
\]
then \( p_k(x) \) is a best approximating polynomial to \( x^n \) in the sense of (1). In fact, we get
\[
n \varepsilon_k = (n - k) \left( \frac{k}{n} \right)^{k/(n-k)}(1 - \varepsilon_k)^{n/(n-k)}.
\]

Proof. Let
\[
p_k(x) = d x^k
\]
then it is easy to see by finding a point where \( |x^n - p_k(x)| \) attains its maximum on \([0, 1]\), that
\[
\varepsilon_k \leq ||x^n - p_k(x)||_{L_{\infty}[0,1]} = \max \left\{ (1 - d), \left( \frac{n - k}{n} \right)^{k/(n-k)}d^{n/(n-k)} \right\}.
\]
From (2), it is clear that
\[
\varepsilon_k \leq ||x^n - p_k(x)||_{L_{\infty}[0,1]} = (1 - d).
\]
So that, again by (2), we obtain
\[
n \varepsilon_k \leq (1 - \varepsilon_k)^{n/(n-k)}(n - k) \left( \frac{k}{n} \right)^{k/(n-k)}.
\]
Now we get the lower bound to \( n \varepsilon_k \).
From (1) and the nonnegativity of the coefficients we get
\[
\varepsilon_k \geq p(x) - x^n \geq [p(1)x^k - x^n = [p(1) - 1]x^k + x^k - x^n
\geq x^k(-\varepsilon_k + 1 - x^{n-k})
\]
i.e.,
\[
\varepsilon_k \geq \frac{x^k(1 - x^{n-k})}{1 + x^k}.
\]
\[
\frac{(1 - x^{n-k})x^k}{1 + x^k}
\]
attains its maximum for values of \( x \) satisfying
\[
x^{n-k} = \frac{k}{n} \left( \frac{1 + x^n}{1 + x^k} \right).
\]
Hence for this value of \( x \), we obtain
\[ (9) \quad \varepsilon_k \geq x^k \left( \frac{n - k}{k} \right) x^{n-k} = \frac{x^n(n - k)}{k} = \frac{k}{k} x^{n-k} = 1 - \frac{n - k}{k} x^{n-k}. \]

From (9) we get

\[ x^{n-k} \geq (1 - \varepsilon_k) \frac{k}{n} \]

i.e.,

\[ (10) \quad x \geq \left[ (1 - \varepsilon_k) \frac{k}{n} \right]^{1/(n-k)}. \]

From (9) and (10) we obtain

\[ (11) \quad \varepsilon_k \geq (1 - \varepsilon_k)^{n/(n-k)} \left( \frac{k}{n} \right)^{n/(n-k)} \left( \frac{n - k}{k} \right). \]

From (7) and (11) we get

\[ n \varepsilon_k = (1 - \varepsilon_k)^{n/(n-k)} (n - k) \left( \frac{k}{n} \right)^{b/(n-k)}. \]

Hence, \( p_h(x) = dx^k \) is a best approximating polynomial in the sense of (1).

**Theorem 2.**

\[ (12) \quad \varepsilon_k = \theta_k \text{ for all } k(1 \leq k < n). \]

**Proof.** By definition, for a \( p(x) \) and \( q(x) \), we have

\[ (13) \quad \left\| x^n - \frac{p(x)}{q(x)} \right\|_{L^\infty[0,1]} = \theta_k. \]

From (13) we get as earlier

\[ (14) \quad \theta_k \geq \frac{p(x)}{q(x)} - x^n \geq \frac{p(1)x^k}{q(1)} - x^n \]

\[ = \left( \frac{p(1)}{q(1)} - 1 \right)x^k + x^k - x^n \geq x^k(1 - x^{n-k} - \theta_k). \]

i.e.,

\[ (15) \quad \theta_k \geq \frac{x^k(1 - x^{n-k})}{1 + x^k}. \]

(8) and (15) being the same in terms of \( x, n \) and \( k \), we get

\[ (16) \quad n \theta_k \geq (n - k) \left( \frac{k}{n} \right)^{b/(n-k)} (1 - \theta_k)^{a/(n-k)}. \]
From Theorem 1 and (16), we obtain

\[
(1 - \varepsilon_k)^{\frac{n}{(n-k)}} \left( \frac{k}{n} \right)^{\frac{k}{(n-k)}} \geq \varepsilon_k \left( \frac{n}{n-k} \right) \geq \left( \frac{n}{n-k} \right) \theta_k
\]

\[
\geq (1 - \theta_k)^{\frac{n}{(n-k)}} \left( \frac{k}{n} \right)^{\frac{k}{(n-k)}} \geq (1 - \varepsilon_k)^{\frac{n}{(n-k)}} \left( \frac{k}{n} \right)^{\frac{k}{(n-k)}} .
\]

(12) follows easily from (17). Hence the result is proved.

Remarks on Theorems 1 and 2. According to ([2], Theorem 6) $p_k$ of our Theorem 1 is unique. Hence $p_k$ is the best approximating polynomial in the sense of (1). (ii) As a result of Theorems 1 and 2 a best approximation to $x^n$ in the sense of (1') is also

\[
p_d(x) - dx^k,
\]

where $d > 0$, satisfies (2). (iii) Let us suppose $\varepsilon_k < 1 - d$, then from (2) and (3), we get $\varepsilon_k > 1 - d$. Similarly, assume $\varepsilon_k > 1 - d$, then we get from (2) and (3), $\varepsilon_k < 1 - d$. Hence we have from (2) and (3),

\[
\varepsilon_k = 1 - d, \text{ for each fixed } k = 1, 2, \ldots, n - 1 .
\]

(iv) For the case $k = n - 1$, we get

\[
\theta_{n-1} = \varepsilon_{n-1} \sim \frac{c}{n} ,
\]

where $c$ satisfies the equation $ce^{c+1} = 1$.

REFERENCES


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