ON CERTAIN ALGEBRAIC INTEGERS AND APPROXIMATION BY RATIONAL FUNCTIONS WITH INTEGRAL COEFFICIENTS

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Let \( A \) be a finite set of integers \( \{a_1, a_2, \ldots, a_l\} \) and (possibly) \( \infty \). Let \( X \) be a nonempty closed subset of \( \mathbb{C} \cup \{\infty\} \), the field of complex numbers together with \( \infty \), under the topology of the Riemann sphere. Suppose that \( X \) is symmetric with respect to the field of real numbers \( \mathbb{R} \) (i.e. if \( z \in X \) then \( z \in X \)) and disjoint from \( A \). We are interested in the following two problems:

I. Under what conditions do there exist, for each neighborhood \( N \) of \( X \), infinitely many algebraic numbers \( \theta \) such that \( 1/(\theta - a_1), 1/(\theta - a_2), \ldots, 1/(\theta - a_l) \) are algebraic integers and, if \( \infty \in A, \theta \) is itself an algebraic integer, such that all of the (algebraic) conjugates of \( \theta \) lie in \( N \)?

II. If \( X \) has empty interior and connected complement, then the polynomials are dense in the ring of continuous functions of \( X \). What is the uniform closure of the polynomials with integral coefficients in \( 1/(x-a_1), 1/(x-a_2), \ldots, 1/(x-a_l) \), and if \( \infty \in A, x \) itself?

Problem I was investigated by Raphael Robinson [10]; however instead of requiring the \( 1/(\theta - a_i) \) to be algebraic integers, he required that the \( b_i/(\theta - a_i) \) be algebraic integers, where the \( b_i \) are integers satisfying \( (a_i - a_j)|b_j \) for each \( j \neq i \). Our methods are similar to those of Robinson; there are, however, significant differences.

Throughout the remainder of this paper, \( A \) will denote a nonempty finite set consisting of real numbers \( a_1, a_2, \ldots, a_l \) and (possibly) \( \infty \). We assume that \( |a_i - a_j| \geq 1 \) if \( i \neq j \). In \( \S \S \ 2, 3, 4 \), we shall assume that the \( a_i \) are integers. If \( \infty \in A \), we shall sometimes denote it by \( a_0 \). By a symmetric closed (SC) \( A \)-set \( X \), we shall mean a nonempty closed subset of the Riemann sphere, symmetric with respect to the \( x \)-axis, satisfying \( A \cap X = \emptyset \).

If \( P(z) \) is a polynomial, we shall denote the leading coefficient of \( P(z) \) by \( P(\infty) \).

1. Classification of SC \( A \)-sets. A rational function with real coefficients \( \varphi(z) \) is said to be an \( A \)-function if it is regular except possibly for poles at \( a_i \in A \). Such a function can be written uniquely in the form \( P(z)/D(z) \) where \( P(z) \) is a polynomial, \( D(z) = \prod_{i=1}^{l} (z-a_i)^{r_i} \) where the \( r_i \geq 0 \) and \( P(a_i) \neq 0 \) when \( r_i > 0 \), for \( 1 \leq i \leq l \). If
If \( \infty \in A \) put \( r = \sum_{i=1}^{l} r_i \), while if \( \infty \in A \), put \( r = \max(\deg P(z), \sum_{i=1}^{l} r_i) \) and \( r_i = r - \sum_{i=1}^{l} r_i \). Thus, in either case, \( r \) is the number of poles (counting multiplicity) of \( \varphi(z) \). We call \( \{r_1, r_2, \ldots, r_l\} \) (or if \( \infty \in A \), \( \{r_0, r_1, r_2, \ldots, r_l\} \)) the degree sequence of \( \varphi(z) \) (with respect to \( A \)). We shall say that the \( A \)-function \( \varphi(z) \) is an upper \( A \)-function if all \( r_i \) are positive and \( |P(a)| \geq 1 \) for each \( a \in A \). (Recall that by our convention \( P(\infty) \) is the leading coefficient of \( P(z) \).) We shall say that the \( A \)-function \( \varphi(z) \) is a lower \( A \)-function if it is both upper and lower; i.e. if all \( r_i \) are positive and \( |P(a)| = 0 \) for all \( a \in A \). We shall say that the \( A \)-function \( \varphi(z) \) is an integral \( A \)-function if \( P(z) \) has integral coefficients.

An SC \( A \)-set \( X \) is said to be \( A \)-small if there exists an upper \( A \)-function \( \varphi(z) \) with \( \|\varphi\|_X < 1 \). (Here and throughout \( \| \|_X \) denotes the sup norm over \( X \).) The set \( X \) is said to be \( A \)-large if for each neighborhood \( N \) of \( X \) there exists a lower \( A \)-function \( \varphi(z) \) satisfying \( \{z: |\varphi(z)| = 1\} \subset N \) and \( X \subset \{z: |\varphi(z)| < 1\} \). Note that if \( A = \{\infty\} \) then an \( A \)-small set is simply a set with transfinite diameter \( < 1 \) and an \( A \)-large set is one with transfinite diameter \( \geq 1 \) [3, Theorem 1].

**Theorem 1.1.** Suppose \( A' \) is a non-empty subset of \( A \). No SC \( A \)-set \( X \) is both \( A \)-large and \( A' \)-small.

**Proof.** Suppose \( X \) is both \( A \)-large and \( A' \)-small. Let \( f(z) \) be an upper \( A' \)-function with \( \rho = \|f(z)\|_X < 1 \). Choose \( \sigma \) satisfying \( \rho < \sigma < 1 \). The set \( N_\sigma = \{z: |f(z)| < \sigma\} \) is an open neighborhood of \( X \). Since \( X \) is \( A \)-large there exists a lower \( A \)-function \( g(z) \) such that \( \{z: |g(z)| = 1\} \subset N_\sigma \). Then, for any \( z \), \( |g(z)| = 1 \) implies \( |f(z)| < \sigma \). Now suppose that \( \infty \in A \); the proof is similar and simpler if \( \infty \in A \). Let \( \{r_0, r_1, \ldots, r_l\} \) be the degree sequence of \( f \) (with respect to \( A \)) and let \( \{s_0, s_1, \ldots, s_l\} \) be the degree sequence of \( g \). Clearly all \( s_i \) are \( > 0 \). Choose \( h \) so that \( r_h/s_h = \max_i (r_i/s_i); r_h \) is \( > 0 \). Put \( g_i(z) = g(z)^{r_h} \) and \( f_i(z) = f(z)^{r_h} \). The degree sequence of \( f_i \) is \( \leq \) (componentwise) the degree sequence of \( g_i \), with equality at the \( h \)-th component. Put \( u(z) = f_i(z)/g_i(z); u(z) \) is regular for all \( z \) for which \( g_i(z) \neq 0 \); in particular \( u(z) \) is regular in \( D = \{z: |g_i(z)| > 1\} \). On \( |g_i(z)| = 1 \), the boundary of \( D \), \( |f_i(z)| < \sigma^h < 1 \), hence \( |u(z)| < 1 \); by the maximum principle this holds for all \( z \in D \). But at \( z = a_h \in D \), \( |u(z)| \geq 1 \), since \( f_i(z) \) is an upper \( A' \)-function and \( g_i(z) \) is a lower \( A \)-function. This contradiction completes the proof.

(The author would like to thank the referee for providing this elegant short proof; the original was much longer and more complicated.) We shall need the following.
LEMMA 1.2. Suppose $B = (\beta_{ij})$ is a matrix with real entries whose off-diagonal elements are nonnegative. Then either (a) there exists a nonzero vector $x \geq 0$ such that $Bx \geq 0$, or else (b) $B$ is invertible and $B^{-1}$ is $\leq 0$.

Proof. Choose $\mu$ so that $B + \mu I$ is $\geq 0$ and let $\lambda$ be the largest eigenvalue of $B + \mu I$. By an extension of the Perron-Frobenius Theorem [4, Chapter XIII, Theorem 3, p. 66], $-B^{-1} = (\mu I - (B + \mu I)^{-1})$ exists if $\mu > \lambda$ and when that is so is $\geq 0$, while if $\mu \leq \lambda$, then $B + \mu I$ has a nonnegative eigenvector $x$ satisfying $(B + \mu I)x = \lambda x$ or $Bx = (\lambda - \mu)x \geq 0$.

The following is closely related to the main result of § 4 of [11].

THEOREM 1.3. Let $X$ be an SC $A$-set. Then either $X$ is $A$-large or there exists a nonempty subset $A'$ of $A$ such that $X$ is $A'$-small. If $X$ is $A$-large then for every neighborhood $N$ of $X$ there exists a normal $A$-function $\varphi(z)$ and $R > 1$ such that $\{z: |\varphi(z)| = R\} \subset N$ and $X \subset \{z: |\varphi(z)|| < R\}$. Finally, if all finite $a_i \in A$ are rational, we may choose $\varphi(z)$ so that its numerator has rational coefficients.

Proof. We shall prove this when $\infty \in A$. The case when $\infty \in A$ is simpler. The complement of $X$ in the Riemann sphere is a union of components. Let $C_0, C_1, C_2, \cdots, C_s$ be those components which have a nonempty intersection with $A$, and suppose they are numbered so that $\infty \in C_0$. Put $A_k = A \cap C_k$ and put $I_k = \{i: a_i \in C_k\}$. Denote by $X_k$ the complement of $C_k$ (in the Riemann sphere). Let $N_k$ be a neighborhood of $X_k$ disjoint from $A$. Suppose that $a_j \in A_k$.

By Theorem G of [3], there exist polynomials $f_j(z)$ with real coefficients such that

\begin{align}
(1) & \{X_k \subset \{z: |f_j(1/(z - a_j))| < 1\} \text{ if } j > 0\} \\
(2) & \{\{z: |f_j(1/(z - a_j))| \leq 1\} \subset N_k \text{ if } j > 0\}.
\end{align}

Since each $N_k \cap A$ is empty, $|f_k(a_j)| > 1$ and $|f_k(1/(a_j - a_k))| > 1$ for all $k > 0$ and $j \neq k$. By replacing each $f_j$ by a positive integral power of itself, if necessary, we may assume that the $f_j$ all have the same degree, say $d$, and that $d > l$. We are going to construct a function $\varphi(z)$ of the form
\[ \varphi(z) = f_0(z)^{t_0} \prod_{j \in I_0} f_j(1/(z - a_j))^{t_j} \]

\[ + \sum_{k=1}^{s} \prod_{j \in I_k} f_j(1/(z - a_j))^{t_j}, \]

where here, and throughout this proof, \( I'_0 = I_0 - \{0\}. \) We can write \( \varphi(z) \) in the form

\[ \varphi(z) = \frac{P(z)}{\prod_{j=1}^{s} (z - a_j)^{t_j}}, \]

where \( P(z) \) is a polynomial of degree \( d(t_0 + t_1 + \cdots + t_s) \), and is explicitly given by

\[ P(z) = f_0(z)^{t_0} \prod_{j \in I_0} [(z - a_j)^{t_j} f_j(1/(z - a_j))]^{t_j} \prod_{j \in I_0} (z - a_j)^{t_j} \]

\[ + \sum_{k=1}^{s} \prod_{j \in I_k} [(z - a_j)^{t_j} f_j(1/(z - a_j))]^{t_j} \prod_{j \in I_k} (z - a_j)^{t_j}. \]

Then

\[ P(\infty) = f_0(\infty)^{t_0} \prod_{j \in I_0} f_j(0)^{t_j}, \]

\[ P(a_i) = f_0(a_i)^{t_0} \prod_{j \in I_0} [(a_i - a_j)^{t_j} f_j(1/(a_i - a_j))]^{t_j} \]

\[ \times f_i(\infty)^{t_i} \prod_{j \in I_0} (a_i - a_j)^{t_j} \text{ if } i \in I'_0, \]

\[ P(a_i) = \prod_{j \in I_k} [(a_i - a_j)^{t_j} f_j(1/(a_i - a_j))]^{t_j} \]

\[ \times f_i(\infty)^{t_i} \prod_{j \in I_k} (a_i - a_j)^{t_j} \text{ if } i \in I_k, \ k \neq 0. \]

Put

\[ \beta_{i0} = \log |f_i(\infty)|, \]

\[ \beta_{ij} = \left\{ \begin{array}{ll} \log |f_j(0)| & \text{if } j \in I'_0 \\ 0 & \text{if } j \in I_0 \end{array} \right\}, \]

\[ \beta_{i0} = \left\{ \begin{array}{ll} \log |f_0(a_i)| & \text{if } i \in I'_0 \\ 0 & \text{if } i \in I_0 \end{array} \right\}; \]

if \( i, j \geq 0 \) then put

\[ \beta_{ij} = \left\{ \begin{array}{ll} \log |(a_i - a_j)^{t_j} f_j(1/(a_i - a_j))| & \text{if } i \sim j, \ i \neq j \\ \log |f_i(\infty)| & \text{if } i = j \\ \log |(a_i - a_j)^{t_i}| & \text{if } i \not\sim j \end{array} \right\}. \]
Here \( i \sim j \) means \( i \) and \( j \) are in the same \( I_k \) and \( i \not\sim j \) means that this is not so. Since \(|a_i - a_j| \geq 1\) if \( i \neq j, \) \( i, j > 0\), we see that if \( i \neq j \) then \( \beta_{ij} \geq 0 \); moreover if \( A \) lies in the one component \( C_0 \) and \( i \neq j \), \( \beta_{ij} > 0 \). We have that

\[
(5) \quad \log |P(a_i)| = \sum_{j=0}^{1} \beta_{ij}t_j.
\]

We now apply Lemma 1.2 to the matrix \( B = (\beta_{ij}) \). If case (a) holds, then there exist real \( t_0, t_1, \ldots, t_l \geq 0 \), not all 0, such that all of the sums \( \sum_j \beta_{ij}t_j \) are \( \geq 0 \). Let \( A' \) be the union of those \( A_k \) for which there exist \( i \in I_k \) such that \( t_j > 0 \). Put \( I' = \{ i : a_i \in A' \} \). By replacing each \( f_i \) by \( \lambda f_i \), where \( \lambda > 1 \) is small enough that (1) and (2) are still satisfied, we increase \( \beta_{ij} \) when \( i \sim j \). Hence if \( i \in I' \), we increase at least one coefficient of a positive \( t_j \) in the linear form \( \sum_j \beta_{ij}t_j \). Thus we may assume that the linear forms \( \sum_j \beta_{ij}t_j \) are positive when \( i \in I' \). By modifying the positive \( t \) for which \( j \in I' \) slightly to make them positive rationals and then multiplying through by a common denominator, we may assume the \( t_j \) are positive integers, and \( \sum_j \beta_{ij}t_j > 0 \) when \( i \in I' \). We can multiply the \( t_j \) by such a large positive integer that if \( i \in I_k, i \neq 0 \), then \( |f_i(1/(z-a_i))t_i| < 1/(s + 1) \) for all \( z \in X_k \) and is \( > 1 \) for all \( z \) outside of \( N_k \). Similarly we will have \( |f_i(z)^n| < 1/(s + 1) \) for \( z \in X_k \) and \(|f_i(z)^n| > 1 \) for \( z \in N_k \). Now, construct \( \varphi \) as in (3) substituting \( A' \) for \( A \) and using the same \( N_k \) and \( f_j \). Then \( \varphi \) is an upper \( A' \)-function and it is easy to see that \( X \subset \{ z : |\varphi(z)| < 1 \} \) so that \( X \) is \( A' \)-small.

Next suppose that case (b) of Lemma 1.2 holds. Then \( B^{-1} \leq 0 \). Put

\[
(6) \quad t = (t_0, t_1, \ldots, t_l)^* = B^{-1}(-1, -1, \ldots, -1)^*.
\]

Then \( \sum_j \beta_{ij}t_j = -1 \) and each component \( t_j \) of \( t \) is \( > 0 \), for clearly \( t_j \geq 0 \) and if \( t_j = 0 \), then the \( j^{th} \) row of \( B^{-1} \) would be 0, which is not possible. There is a unique polynomial \( g_d(z) \) of degree \( \leq l - 1 \) such that \( z^d + g_d(z) = 0 \) for \( z = a_1, a_2, \ldots, a_l \). Since \( d > l \) the polynomial \( f_i(z) + \delta_i(z^d + g_d(z)) \) has leading coefficient \( f_i(\infty) + \delta_i \) and takes the same values at \( z = a_1, a_2, \ldots, a_l \) as \( f_i(z) \). Thus replacing \( f_i(z) \) by \( f_i(z) + \delta_i(z^d + g_d(z)) \) would change \( \beta_{ij} \) but none of the other \( \beta_{ij} \). If \( \delta_i \) is small enough then (1) and (2) would remain satisfied. Similar comments apply to \( f_i, f_{i+1}, \ldots, f_l \). Thus there exists \( \delta > 0 \) such that each \( f_j \) can be modified in such a way that \( \beta_{ij} \) is unchanged if \( i \neq j \), while \( \beta_{ij} \) varies over an interval of length \( 2\delta \), and at the same time (1) and (2) remain valid. Choose positive rational \( t'_j \) so close to \( t_j \), \( 0 \leq j \leq l \), that \( |\sum_{i=0}^{l} \beta_{ij}(t_j - t'_j)|t'_j| < \varepsilon \) for \( 0 \leq i \leq l \).

Now put \( \beta'_{ij} = \beta_{ij} \) if \( i \neq j \) and choose \( \beta'_{ii} \) so that \( \sum_{j=0}^{l} \beta'_{ij}t_j = -1 \)
for $0 \leq i \leq l$. Then $|\beta'_{ii} - \beta_{ii}| = |\sum_{j=0}^l \beta_{ij}(t_j - t'_j)| < \varepsilon$ for $0 \leq i \leq l$. Now modify the $t_i$ slightly so that the $\beta_{ij}$ are replaced by the $\beta'_{ij}$ and the $t_i$ by the $t'_i$, still preserving (1) and (2). Thus, after this replacement we may assume that the $t_i$ are all positive rational numbers. Now multiply the $t_i$ by such a large positive integer $n$ that they become integers and such that if $i \in I_k$, $i \neq 0$, then $|f_i(1/(z - a_i))|^{n|z|}$ is $<1/(s + 1)$ for $z \in X_k$ and $>2$ for $z \in N_k$. Similarly $|f_i(z)|^{n|z|}$ is $<1/(s + 1)$ for $z \in X_0$ and $>2$ for $z \in N_0$. Then $\phi$ as defined in (3) is a lower $A$-function and all of the $|P(a_j)|$ are equal to $1/e^\alpha$. By replacing $\phi$ by $\phi^2$, we obtain a lower $A$-function $\phi$ with $P(a_j) = 1/e^{2\alpha}$ for $0 \leq j \leq l$. For $z \in N_k$ all but one of the terms in (3) have absolute value $<1/(s + 1)$ while the remaining term has value $>2$. Thus $|\phi(z)|$ is $>1$ outside of each $N_k$. If $z \in X$, however, then each term in (3) has absolute value $<1/(s + 1)$ and $|\phi(z)| < 1$. Let $Y$ be the union of $X$ and those components of the complement of $X$ which are disjoint from $A$; i.e., $Y$ is obtained from $X$ by filling in those holes which contain no $a_i$. If $N$ is any neighborhood of $X$, then there exist neighborhoods $N_k$ of $X_k$ such that $\bigcap_k N_k \subset N \cup Y$. The $\phi$, as modified above and corresponding to this choice of the $N_k$ is lower, $X \subset \{z: |\phi(z)| < 1\}$, and $\{z: |\phi(z)| = 1\} \subset N$. Thus $X$ is $A$-large. Then $\phi(z) = \phi(z)e^{2\alpha}$ is $A$-normal and $X \subset \{z: |\phi(z)| < e^{2\alpha}\}$ and $\{z: |\phi(z)| = e^{2\alpha}\} \subset N$.

Finally, suppose the $a_i$ are rational, and $\phi(z) = P(z)/D(z)$; then $P_i(a_i) = 1$, $0 \leq i \leq l$. We can choose a polynomial $C(z)$ of degree $<\deg(P(z)) - l$ and with arbitrarily small coefficients such that $P_i(z) = P_i(z) + C(z) \prod_{i=1}^l (z - a_i)$ has all coefficients of terms of degree $\geq l$ rational. Since $P_i(a_i) = 1$, $0 \leq i \leq l$, the remaining coefficients are rational. If $C(z)$ is small enough then $\phi_i(z) = P_i(z)/D(z)$ meets the requirements of the theorem.

**Remark 1.4.** The $A'$ in the above theorem is a union of some of the $A_k$. In particular if all $a_i$ lie in one component of the complement of $X$, then either $X$ is $A$-large or $X$ is $A$-small.

We shall need the following theorem in § 4.

**Theorem 1.5.** A finite SC $A$-set $X$ is $A$-small.

**Proof.** By standard interpolation theory results, there exists a monic polynomial $P$ which vanishes at each element of $X$ and is 1 at each finite element of $A$. We may choose $P$ to have degree $\geq l + 1$, and then $\phi(z) = P(z)/\prod_{i=1}^l (z - a_i)$ is a normal $A$-function which has absolute value $<1$ on $X$. 

**Remark 1.4.**
THEOREM 1.6. If $X$ is an $A$-small set, then there exists a normal $A$-function $\varphi(z)$ such that $\|\varphi(z)\|_X < 1$.

Proof. We shall prove this in the case when $\infty \in A$. The case when $\infty \not\in A$ is simpler. By definition there exists an upper $A$-function $Q(z)/D(z)$ such that $\|Q(z)/D(z)\|_X < 1$. Suppose $D(z) = \prod_{i=1}^l (z - a_i)^{r_i}$. Since $z$ and each of the functions $1/(z - a_i)$ is bounded on $X$, there exists an integer $n \geq 1$ so large that $\|Q(z)^n/D(z)^{n-1}\|_X < 1/(l + 1)$ and $\|Q(z)^n/(z - a_1)^{r_j}D(z)^{n-1}\|_X < 1/(l + 1)$ for $1 \leq j \leq l$. Now put

$$P(z) = (a_0D(z) + \sum_{j=1}^l a_jD(z)/(z - a_j)^{r_j})Q(z)^n$$

where the $a_i$ will be chosen later. Then $P(\infty) = a_0Q(\infty)^n$ and $P(a_i) = a_i \prod_{j=1}^l (a_i - a_j)^{r_j}Q(a_i)^n$. Thus there exist unique choices for the $a_i$ so that $P(\infty) = 1$ and all $P(a_i) = 1$, and the $a_i$ will have absolute value $\leq 1$. Put $\varphi(z) = P(z)/D(z)^n$; $\varphi(z)$ is a normal $A$-function and

$$\|\varphi(z)\|_X \leq |a_0| \cdot \|Q(z)^n/D(z)^{n-1}\|_X$$

$$+ \sum_{j=1}^l |a_j| \cdot \|Q(z)^n/(z - a_j)^{r_j}D(z)^{n-1}\|_X$$

$$< 1.$$ 

If $N(z)$ is a nonconstant polynomial, then any power series $u(z)$ can be written uniquely in the form

$$u(z) = \sum_{i=0}^\infty c_i(z)N(z)^i$$

where the $c_i(z)$ are polynomials of degree $< \deg(N(z))$. This is the special case, $A = \{\infty\}$, of the next lemma. To extend to general sets $A$, we must replace $N(z)$ by a rational function which has poles at each $a_i \in A$, and allow the $c_i(z)$ to be rational functions with poles of bounded order at each $a_i \in A$. In the following lemma, $N(z)$ is replaced by $N(z)/D(z)$ and the $c_i(z)$ by the $c_i(z)/D(z)$.

LEMMA 1.7. Suppose $\infty \in A$ and $D(z) = \prod_{i=1}^l (z - a_i)^{r_i}$ where the $r_i$ are $>0$. Suppose $N(z)$ is a polynomial, relatively prime to $D(z)$, of degree $r = \sum_{i=0}^l r_i$ where $r_0$ is $>0$. If $u(z)$ is an $A$-function satisfying $u(a_i) \neq 0$, $0 \leq i \leq r$, we can write uniquely

$$u(z) = \sum_{i=0}^n c_i(z)N(z)^i/D(z)^{i+1}$$

where $n$ is the least integer $\geq 0$ such that

$$-\operatorname{ord}_\infty u(z) \leq (n + 1)r_0 - 1.$$
and
\[-\text{ord}_{\alpha_i} u(z) \leq (n + 1)r_i \quad \text{for} \quad 1 \leq i \leq l;\]
and where the $c_i(z)$ are polynomials of degree $< r$ and $c_n(z)$ is not 0.

Suppose $\infty \in A$ and $D(z) = \prod_{i=1}^{l} (z - a_i)^{r_i}$, where the $r_i$ are $> 0$. Suppose $N(z)$ is a polynomial, relatively prime to $D(z)$ and of degree $\leq r = \sum_{i=1}^{l} r_i$. If $u(z)$ is an $A$-function satisfying $u(a_i) \neq 0$, $1 \leq i \leq r$ and vanishing at $\infty$, we can write, uniquely,

$$u(z) = \sum_{i=0}^{n} c_i(z) \frac{N(z)^i}{D(z)^{i+1}}$$

where $n$ is the least integer $\geq 0$ such that
\[-\text{ord}_{\alpha_i} u(z) \leq (n + 1)r_i \quad \text{for} \quad 1 \leq i \leq l;\]
and where the $c_i(z)$ are polynomials of degree $< r$, and $c_n(z)$ is not 0.

**Proof.** We give the proof for the case $\infty \in A$, and it is by induction on $n$. The result is clear when $n = 0$, for then $D(z)u(z)$ is a polynomial of degree $< r$. If $n \geq 1$, choose the polynomial $c_0(z)$ of degree $< r$ and $\equiv u(z)D(z) \pmod{N(z)}$; then the polynomial $D(z)^n(u(z)D(z) - c_0(z))$ is divisible by $N(z)$. Note that this is the unique choice for $c_0(z)$. Then
\[-\text{ord}_{\alpha_i} (u(z)D(z) - c_0(z)) \frac{N(z)}{N(z)} \leq nr_0 - 1\]
and
\[-\text{ord}_{\alpha_i} (u(z)D(z) - c_0(z)) \frac{N(z)}{N(z)} \leq nr_i\]
for $1 \leq i \leq l$. Thus, inductively, we have, uniquely,

$$(u(z) - c_0(z)/D(z)) \frac{D(z)}{N(z)} = \sum_{i=0}^{n-1} c_{i+1}(z) \frac{N(z)^i}{D(z)^{i+1}}$$

and then

$$u(z) = \sum_{i=0}^{n} c_i(z) \frac{N(z)^i}{D(z)^{i+1}}.$$
then \( n = rs + t \), where \( 0 \leq t < r \), \( d_r(z) \) is a monic polynomial of degree \( t \), \( \| \theta_r(z) \|_x < M \lambda^r \), and \( |\theta_r(z)| < M |g(z)|^{1 + n/r} \) when \( |g(z)| > 1 \).

Suppose \( \infty \in A \) and \( X \) is an SC A-set. Suppose \( g(z) = P(z)/D(z) \) is a normal A-function where \( D(z) = \prod_{i=1}^{r} (z - a_i)^{r_i} \) and \( P(z) \) has degree \( < r = \sum_{i=1}^{r} r_i \). Put \( \lambda = \| g(z) \|_x^{1/r} \). Then there exists \( M > 0 \) and for each integer \( n \geq 0 \) an A-function \( \theta_n(z) \) such that when \( \theta_n(z) \) is expanded according to Lemma 1.7, using \( N(z) = 1 \),

\[
\theta_n(z) = \sum_{i=0}^{r} d_i(z)N(z)^i/D(z)^{i+1},
\]

then \( n = rs + t \) where \( 0 \leq t < r \) and \( d_r(z) \) is a monic polynomial of degree \( t \), \( \| \theta_n(z) \|_x < M \lambda^r \), and \( |\theta_n(z)| < M |g(z)|^{1 + n/r} \) when \( |g(z)| > 1 \).

Proof. Suppose first that \( \infty \in A \). Expand \( g(z) \) by Lemma 1.7:

\[
g(z)^n = \sum_{i=0}^{n} c_i(z)N(z)^i/D(z)^{i+1}.
\]

It is easy to verify that \( c_n(z) = D(z) \), hence is monic of degree \( \sum_{i=1}^{r} r_i \). Then \( g(z)^n \) will serve for \( \theta_{(m+1)r-r_0} \). The functions

\[
z^{r_0} \theta_{(m+1)r-r_0}, z^{r_0} \theta_{(m+1)r-r_0+1}, \ldots, z^{r_0} \theta_{(m+1)r-r_0+2}, \ldots,
\]

will serve for \( \theta_{(m+1)r-r_0+1}, \theta_{(m+1)r-r_0+2}, \ldots, \theta_{(m+1)r-r_0+2} \), respectively. The functions \( \theta_{(m+1)r-r_0}/(z - a_i), \theta_{(m+1)r-r_0}/(z - a_i)^2, \ldots, \theta_{(m+1)r-r_0}/(z - a_i)^{r_i} \) will serve for \( \theta_{(m+1)r-r_0+1}, \theta_{(m+1)r-r_0+2}, \ldots, \theta_{(m+1)r-r_0+2} \), respectively. Continuing in this way, dividing next by \( (z - a_i)^2 \), then \( (z - a_i)^3 \), and so forth will give the remaining functions. Since all of the functions \( z, 1/(z - a_i), \ldots, 1/(z - a_i)^{r_i} \) are bounded on \( X \) and \( z^{r_0}/g(z), 1/((z - a_i)^{r_0}/g(z)), \ldots, 1/((z - a_i)^{r_0}/g(z)) \) are bounded when \( |g(z)| > 1 \), there exists \( M > 0 \) as required for the Lemma. If \( \infty \in A \), use the above procedure with \( r_0 = 0 \), omitting \( z \) and \( z^{r_0}/g(z) \) when defining \( M \).

2. Classification of A-sets—Integral A. In this and succeeding sections we assume that the \( a_i \in A \) are integers and strengthen the results of § 1.

Theorem 2.1. If \( X \) is A-small there exists an integral, normal A-function \( \varphi(z) \) such that \( \| \varphi(z) \|_x < 1 \).

Proof. We give the proof in the case that \( \infty \in A \). There exists an A-normal function \( P(z)/D(z) \), where \( D(z) = \prod_{i=1}^{r} (z - a_i)^{r_i} \) and \( \| P(z)/D(z) \|_x < 1 \). Suppose \( P(z) \) has degree \( r = \sum_{i=0}^{r} r_i \) and put
\[ N(z) = z^rD(z) + 1. \] Choose \( m > 0 \). For any \( n > m \), the function 
\[ (P(z)/D(z))^n(1 + 1/((z - a_1)(z - a_2) \cdots (z - a_l))) \] is \( A \)-normal and by 
Lemma 1.7 can be written in the form \( \sum_{i=0}^{n} c_i(z)N(z)^i/D(z)^{i+1} \). It is 
easy to verify that \( c_i(z) = D(z)(1 + 1/((z - a_1)(z - a_2) \cdots (z - a_l))) \).

We can successively add \( \epsilon_0\theta_{n_0}(z), \epsilon_1\theta_{n_1}(z), \ldots, \epsilon_{n-m}\theta_{n-m}(z) \), where 
the \( \theta_i(z) \) are the functions defined in Lemma 1.8 and the \( \epsilon_i \) are real 
numbers in the interval \([-1/2, 1/2] \), so as to obtain a function 

\[ h_n(z) = \sum_{i=0}^{n} d_i(z)N(z)^i/D(z)^{i+1} \]

where \( d_n(z) = c^n(z) \) and \( d_m(z), d_{m+1}(z), \ldots, d_{n-1}(z) \) have integral 
coefficients. Furthermore, with \( M \) and \( \lambda \) as defined in Lemma 1.8,

\[ ||h_n(z)||_X < M' ||P(z)/D(z)||_X + M(\lambda^m + \lambda^{m+1} + \cdots + \lambda^{n-1}) < M''\lambda^m \]

where \( M' = ||(1 + 1/((z - a_1)(z - a_2) \cdots (z - a_l)))||_X \) and \( M'' = \max(M, M')/(1 - \lambda) \). We can choose \( m \) so large that \( M''\lambda^m < 1/3 \). For each 
\( n > m \), we obtain such a function \( h_n(z) \) and in the expansion (7), all 
of the \( d_i^{m+i}(z) \), except those with \( i < m \), have integral coefficients. 
We can find \( n_2 > n_1 > m \) so that all of the coefficients of the \( d_i^{m+i}(z) - d_i^{n_1}(z) \), for \( 0 \leq i \leq m - 1 \), are extremely small modulo 1. 

When this is the case put \( \varphi(z) = \sum_{i=0}^{n_2} e_i(z)N(z)^i/D(z)^{i+1} \) where 
\( e_i(z) \) is the polynomial with integral coefficients nearest to \( d_i^{n_2}(z) - d_i^{n_1}(z) \); here we put \( d_i^{n_1}(z) = 0 \) when \( i > n_1 \). If \( n_1 \) and \( n_2 \) were 
chosen appropriately, \( \varphi(z) \) will satisfy \( ||\varphi(z)||_X < 1 \) and since \( e_{n_2}(z) = (1 + 1/((z - a_1)(z - a_2) \cdots (z - a_l)))D(z) \), \( \varphi(z) \) is normal.

**Theorem 2.2.** Suppose \( X \) is \( A \)-large. Then for each neighborhood \( N \) of \( X \) there exists an integral normal \( A \)-function \( \varphi(z) \) and 
an integer \( S > 1 \) such that \( \{z: |\varphi(z)| = S\} \subset N \) and \( X \subset \{z: |\varphi(z)| < S\} \).

**Proof.** We give the proof for the case that \( \infty \in A \). By Theorem 
1.3, there exists a normal \( A \)-function \( g(z) \) with rational coefficients 
and \( R > 1 \) such that \( \{z: |g(z)| = R\} \subset N \) and \( X \subset \{z: |g(z)| < R\} \). We 
can write \( g(z) = N(z)/D(z) + c(z)/(hD(z)) \) where, as usual, \( D(z) = \Pi_{i=1}^{l}(z - a_i)^{r_i} \), 
\( N(z) = z^sD(z) + 1 \), \( r = \sum_{i=0}^{l} r_i \), \( c(z) \) is a polynomial of 
degree \( < r \) with integral coefficients satisfying \( c(a_i) = 0 \) for \( i \leq \)
\( i \leq l \), and \( h \) is a positive integer. We can write

\[ g(z)^n = (N(z) + c(z)/h)^n/D(z)^n \]

\[ = \sum_{i=0}^{n-1} N(z)^{n-i}c(z)^i \binom{n}{i} \left( \frac{1}{h^iD(z)^n} \right) \]

\[ + \sum_{i=n}^{n} N(z)^{n-i}c(z)^i \binom{n}{i} \left( \frac{1}{h^iD(z)^n} \right) , \]
where $m < n$ will be chosen later in this proof.

When the first sum is written as a rational function in $z$ with denominator $D(z)^r$, each coefficient of a power of $z$ in the numerator will be a polynomial in $n$ with rational coefficients. Since the polynomial $\binom{n}{i}$ in $n$ is divisible by $n$ for each $i > 0$, the numerator polynomial will have integral coefficients when $n$ is divisible by a certain fixed integer $n_0$.

Since $c(z)$ has degree $< r$, the second sum has a pole at $\infty$ of order $\leq (n-m)r + m(r-1)$. Since $c(z)$ vanishes at each $a_i$, the second sum has a pole at $a_i$ of order $\leq nr_i - m$. By Lemma 1.7, the second sum can be written in the form

$$\sum_{i=0}^{k} b_i N(z)^i / D(z)^{i+1}$$

where $k$ is the least integer $\geq 0$ satisfying

$$k + 1 \geq n - (m-1)/r_0$$

$$k + 1 \geq n - m/r_i; \quad 1 \leq i \leq l.$$

Put $j = (k+1)r - 1$. Let $\theta_0(z), \theta_1(z), \theta_2(z), \cdots$ be the functions constructed in Lemma 1.8 using $P(z)/D(z) = g(z)$. By adding successively $\varepsilon_j \theta_j(z), \varepsilon_{j-1} \theta_{j-1}(z), \cdots$ where the $\varepsilon_i$ are chosen appropriately from the interval $[-1/2, 1/2)$, to $g(z)^n$ we obtain an integral normal $A$-function $f_n(z)$. Choose $R_1$ and $R_2$ close to $R$ with $1 < R_1 < R < R_2$ such that $X \subset \{z : \|g(z)\| < R\}$ and $\{z : R_1 \leq \|g(z)\| \leq R_2\} \subset N$. Then $f_n(z)$ differs from $g(z)^n$ in the set $\{z : \|g(z)\| \leq R\}$ by less than $M' \|g(z)\|^{\lfloor j/r \rfloor} + \|g(z)\|^{\lfloor (j-1)/r \rfloor} + \cdots + 1$ or by less than $M' \|g(z)\|^{\lfloor j/r \rfloor}$ where $M' = M/(R_1^{\lfloor j/r \rfloor} - 1)$. Hence if $n/j$ is large enough, $f_n(z)$ does not vanish when $\|g(z)\| \geq R_0$. Similarly, if $\|g(z)\| \leq R_0$, then $f_n(z)$ differs from $g(z)^n$ by $\leq M' R_0^{\lfloor j/r \rfloor}$. Thus by the maximal principal, if $\|g(z)\| \geq R_0$, $\|f_n(z)\| \geq (1 - \delta)R_0^2$ and if $\|g(z)\| \leq R_0$, $\|f_n(z)\| \leq (1 + \delta)R_0^2$, where $\delta > 0$ can be made arbitrarily close to 0 by choosing $m$ large. If $n$ is large enough and divisible by $n_0$ there will be an integer $S$ in the interval $((1 + \delta)R_0^2, (1 - \delta)R_0^2)$; putting $\phi(z) = f_n(z)$ completes the proof.

3. $A$-integers. An algebraic number $\theta$ is said to be an $A$-integer if $1/(\theta - a_i)$ is an algebraic integer for each $a_i \in A$ and $\theta$ is an algebraic integer if $\infty \in A$.

**Lemma 3.1.** If $\phi(z) = P(z)/D(z)$ is an integral normal $A$-function and $\theta$ is a complex number such that $\phi(\theta) = \alpha$ is an algebraic
integer, then \( \theta \) is an \( A \)-integer.

**Proof.** The polynomial \( P(z) - \alpha D(z) \) has algebraic integer coefficients and is satisfied by \( \theta \). If \( \infty \in A \), then this polynomial is monic of degree \( r \) and hence \( \theta \) is an algebraic integer. Since \( P(a_i) - \alpha D(a_i) = P(a_i) = 1 \), the polynomial with algebraic integer coefficients satisfied by \( 1/(\theta - a_i) \) is monic and \( 1/(\theta - a_i) \) is an algebraic integer.

**Lemma 3.2.** If \( \varphi(z) \) is an integral \( A \)-function and \( \theta \) is an \( A \)-integer then \( \varphi(\theta) \) is an algebraic integer.

**Proof.** We first show that the ring generated by the functions \( 1, 1/(z - a_1), 1/(z - a_2), \ldots, 1/(z - a_r) \), and if \( \infty \in A \), the function \( z \), contains all integral \( A \)-functions. This is clear if \( \infty \in A \), so suppose \( \infty \notin A \). Suppose \( P(z)/D(z) \) is an integral \( A \)-function, \( D(z) = \sum_{i=1}^{r} (z - a_i)^{r_i} \), and \( r = \sum_{i=1}^{r} r_i \). We proceed by induction on \( r \). If \( r = 0 \), the result is clear. Otherwise some \( r_i \), say \( r_i \), is \( >0 \). Then \( P(z)/D(z) = (P(z) - P(a_i))/D(z) + P(a_i)/D(z) \). Clearly \( P(a_i)/D(z) \) is in the ring and since \( (z - a_i)| (P(z) - P(a_i)) \), \( (P(z) - P(a_i))/D(z) \) is in the ring by induction. Since each \( 1/(\theta - a_i) \) is an algebraic integer and if \( \infty \in A \), \( \theta \) is an algebraic integer, \( \varphi(\theta) \) is an algebraic integer.

We now give the basic results of this section.

**Theorem 3.3.** Let \( X \) be a set which is not \( A \)-large. Then there exists a neighborhood of \( X \) which contains only finitely many complete conjugate sets of \( A \)-integers.

**Proof.** By Theorems 1.3 and 2.1, \( A \) contains a nonempty subset \( A' \) for which there exists an integral \( A' \)-function \( \varphi(z) \) such that \( ||\varphi(z)||_x < 1 \). Pnt \( N = \{ z : |\varphi(z)| < 1 \} \). If \( \{ \theta_1, \theta_2, \ldots, \theta_m \} \) is a complete conjugate set of \( A \)-integers contained in \( N \), then \( \{ \varphi(\theta_1), \varphi(\theta_2), \ldots, \varphi(\theta_m) \} \) is a sequence of algebraic integers, consisting of repetitions of a complete conjugate set. Since each \( \varphi(\theta_i) \) has absolute value \( <1 \), the norm of each is \( <1 \), hence 0. Thus each \( \varphi(\theta_i) = 0 \) and so the total number of \( \theta_i \) is \( \leq r \), the degree of the numerator of \( \varphi(z) \).

**Theorem 3.4.** Let \( X \) be an \( A \)-large set. Then every neighborhood \( N \) of \( X \) contains infinitely many complete sets of conjugate \( A \)-integers.

**Proof.** Let \( N \) be a neighborhood of \( X \). By Theorem 2.2 there exists an integral normal \( A \)-function \( \varphi(z) \) and an integer \( S > 1 \) such that \( \{ z : |\varphi(z)| = S \} \subset N \). The solutions to \( \varphi(z)^n = S^n \) lie in \( N \) and
by Lemma 3.1 are $A$-integers.

It is probable that if $X$ is an $A$-large subset of $R$ then every real neighborhood of $X$ contains infinitely many complete sets of conjugate $A$-integers. In the case $A = \{\infty\}$ and $X$ is a finite union of closed intervals in $R$ this was shown by Robinson in [7] and [8], and in the case $X$ is a closed interval and $A = \{\infty, 0\}$ this was shown by Robinson in [9].

4. Approximation. Let $X$ be an SC set with empty interior and such that each component of the complement of $X$ in $C$ contains an element of $A$. A complex valued function $f$ on $X$ is called symmetric if $f(x) = \overline{f(x)}$ for all $x \in X$. We shall denote the ring of continuous symmetric functions on $X$ by $C_s(X)$. A theorem of Mergelyan [6, Theorem 2.3] asserts that the $A$-functions are dense, in the uniform norm, in $C_s(X)$. We are interested in investigating the uniform closure of the integral $A$-functions in $C_s(X)$. For the case $A = \{\infty\}$ see [1] and [5]. If $Y$ is an SC subset of $X$, we shall say that the symmetric function $f$ is matchable on $Y$ if there exists an integral $A$-function $p$ such that $p(y) = f(y)$ for all $y \in Y$ and we shall say that $f$ is approximable on $Y$ if for each $\varepsilon > 0$ there exists an integral $A$-function $p$ such that $\|p - f\|_U < \varepsilon$.

**Theorem 4.1.** If $X$ is $A$-large then the integral $A$-functions form a closed discrete subset of $C_s(X)$.

**Proof.** Suppose $\varphi_1$ and $\varphi_2$ are integral $A$-functions with $\|\varphi_1 - \varphi_2\|_X < 1$. If $\varphi_1 \neq \varphi_2$ then $\varphi_1 - \varphi_2$ is an upper $A'$-function for some nonempty $A' \subset A$. But this implies that $X$ is $A'$-small, contradicting Theorem 1.1.

Now define $J(X, A)$ to be the union of the complete sets of conjugate $A$-integers contained in $X$. Note that if $X$ is not $A$-large then, by Theorem 3.3, $J(X, A)$ is finite.

**Theorem 4.2.** If $X$ is $A'$-small for some non-empty $A' \subset A$ and each component of the complement of $X$ contains an element of $A'$ then $f \in C_s(X)$ is approximable on $X$ if and only if it is matchable on $J(X, A)$.

**Proof.** First observe that if $\varphi$ is an integral $A$-function which satisfies $|\varphi(x)| < 1$ for each $x \in J(X, A)$, then $\varphi(x) = 0$ for each $x \in J(X, A)$. Indeed $J(X, A)$ is the disjoint union of complete sets of conjugate $A$-integers. Let $x_1, x_2, \cdots, x_r$ be one such complete set.
Then \( \prod_{i=1}^{r} \varphi(x_i) \) is a rational integer with absolute value <1. Hence the product is 0, and so at least one of the \( \varphi(x_i) = 0 \), and since they are conjugate they are all 0, and \( \varphi \) vanishes on \( J(X, A) \). Now suppose \( f \) is approximable on \( X \) and that \( \| p_i - f \|_X < 1/2 \) and \( \| p_2 - f \|_X < 1/2 \). Then \( \| p_1 - p_2 \|_X < 1 \). By what we proved above \( p_i(x) = p_A(x) \) for all \( x \in J(X, A) \). Since \( \| p_2 - f \|_X \) can be chosen arbitrarily small, it follows that \( f(x) = p_i(x) \) for all \( x \in J(X, A) \); hence that \( f \) is matchable on \( J(X, A) \).

Assume \( \infty \in A' \). The proof is similar when \( \infty \notin A' \). Since \( X \) is \( A' \)-small, there exists a normal integral \( A' \)-function \( \varphi \) with \( \| \varphi \|_X < 1 \). Let \( K \) be the (finite) set of those zeros of \( \varphi \) contained in \( X \). Since \( \| \varphi \|_X < 1 \), \( |\varphi(x)| < 1 \) for all \( x \in J(X, A) \) and hence \( \varphi \) vanishes on \( J(X, A) \). Thus \( J(X, A) \subset K \).

Let \( m \) be a positive integer. By a standard extension of the Stone-Weierstrass theorem, the closed ideal generated by \( \varphi^m \) in \( C_s(X) \) consists of all functions \( g \in C_s(X) \) vanishing on \( K \). By our assumption about \( X \), the \( A' \)-functions are dense in \( C_s(X) \). Thus if \( \epsilon > 0 \) and \( g \in C_s(X) \) vanishes on \( K \), there exists an \( A' \)-function \( h(x) \) such that \( \| \varphi(x)^m h(x) - g(x) \|_X < \epsilon \). By Lemma 1.7, we can write

\[
\varphi(x)^m h(x) = \sum_{j=m}^n (h_i(x)/D(x)) \varphi(x)^j,
\]

where \( D(x) = \prod_{i=1}^{r'} (x - a_i)^{r_i} \) is the denominator of \( \varphi(x) \) with \( A' = \{a_0, a_1, \ldots, a_{r'}\} \), and where \( h_i(x) \) are polynomials of degree \( < r' = \sum_{i=0}^{r'} r_i \). Put \( M = \sum_{j=m}^{n} \| x^j D(x) \|_X \). If \( H_i(x) \) is the polynomial obtained from \( h_i(x) \) by replacing each coefficient of \( h_i(x) \) with its integral part, it is immediate that \( \|(h_i(x) - H_i(x))/D(x))\|_X < M \). Put \( p(x) = \sum_{j=m}^{n} (H_i(x)/D(x)) \varphi(x)^j \); \( p(x) \) is an integral \( A \)-function and

\[
\| p(x) - \varphi(x)^m h(x) \|_X \leq M \sum_{j=m}^n \| \varphi(x) \|_X^j
\]

and hence

\[
\| g(x) - p(x) \|_X \leq \epsilon + M \cdot \| \varphi(x) \|_X^T/(1 - \| \varphi(x) \|_X) .
\]

Thus if \( m \) is sufficiently large, \( \| g(x) - p(x) \|_X < 2\epsilon \) and hence \( g \) is approximable on \( X \). We have just shown that if \( g \) vanishes on \( K \) then \( g \) is approximable on \( X \). If \( \epsilon > 0 \) and \( g \in C_s(S) \) satisfies \( \| g \|_X < \epsilon \), then it is easy to find \( g \in C_s(X) \) vanishing on \( K \) and satisfying \( \| g - g \|_X < 2\epsilon \). It is immediate that if \( g \) is approximable on \( K \) then it is approximable on \( X \). Thus we must show that if \( g \) is matchable on \( J(X, A) \) it is approximable on \( K \). By replacing \( g \) by \( g - p \) where \( p \) is an appropriate integral \( A \)-function, we may assume that \( g \) vanishes on \( J(X, A) \). Now we must show
that if \( g \) vanishes on \( J(X, A) \), then it is approximable on \( K \). Choose \( \theta \in K - J(X, A) \). Let \( \theta = \theta_1, \theta_2, \ldots, \theta_m \) be the conjugates of \( \theta \) which are contained in \( K \). Since \( \theta \in J(X, A) \), either \( \theta \) is not an \( A \)-integer or \( \theta \) has a conjugate outside of \( X \). Suppose first that \( \theta \) is not an \( A \)-integer. By Theorem 1.5, the set \( \{\theta_1, \theta_2, \ldots, \theta_m\} \) is \( A \)-small and hence there exists a normal, integral \( A \)-function \( p \) such that \( |p(\theta_i)| < 1 \) for \( 1 \leq i \leq m \). Since \( \theta \) is not an \( A \)-integer, none of the \( p(\theta_i) \) are 0. Next suppose that at least one conjugate is outside of \( X \). Since \( m \) is less than the degree \( d \) of \( \theta \), there exist, by Minkowski’s Theorem on linear forms, integers \( b_0, b_1, \ldots, b_{d-1} \) not all 0 such that \( |\sum_{i=0}^{d-1} b_i \theta_i| < 1 \) for \( 1 \leq i \leq m \). If \( p(x) = \sum_{i=0}^{d-1} b_i x^i \) then \( p(\theta_1) \neq 0 \), for the degree of \( p(x) \) is less than the degree of \( \theta \). Thus in either case \( p(x) \) is an integral \( A \)-function with \( 0 < |p(\theta_i)| < 1 \) for \( 1 \leq i \leq m \).

By replacing \( p \) by \( p^h \) where \( n \) is a large enough integer and \( h \) is an appropriate integral \( A \)-function, we may assume in addition that \( p \) vanishes on all elements of \( K \) not conjugate to \( \theta \). Let \( p_1, p_2, \ldots, p_s \) be the functions obtained for each set of conjugate \( A \)-integers in \( K - J(X, A) \). If \( n \) is large enough, \( \varphi = p_1^* + p_2^* + \cdots + p_s^* \) will satisfy \( 0 < |\varphi(x)| < 1 \) for \( x \in K - J(X, A) \) and \( \varphi(x) = 0 \) for \( x \in J(X, A) \). By the earlier part of the proof applied to \( K \) instead of \( X \), any function in \( C_s(K) \) which vanishes on \( J(X, A) \) is approximable on \( K \). By the earlier comments, the proof is complete.

We now give a characterization of \( J(X, A) \).

**Theorem 4.3.** Suppose \( X \) is \( A' \)-small for some nonempty \( A' \subset A \) and that each component of the complement of \( X \) contains an element of \( A' \). There exists an integral \( A \)-function \( \varphi \) such that \( ||\varphi(x)|| < 1 \) and the zeros of \( \varphi \) in \( X \) form the set \( J(X, A) \).

**Proof.** Let \( q(x) \) be an integral \( A \)-function whose zeros are the elements of \( J(X, A) \). Choose \( h \in C_s(X) \) satisfying, for all \( x \in X \): (1) \( ||h||_x = 1 \); (2) \( h(x) = 1 \) if \( q(x) = 0 \); (3) \( |h(x)| < 1/(2 |q(x)|) \) if \( |q(x)| > 1/2 \); (4) \( h(x) \neq 0 \). Such an \( h \) is matchable by 1 on \( J(X, A) \), hence is approximable on \( X \). Any sufficiently good approximation, say, the integral \( A \)-function \( g \), satisfies, for all \( x \in X \), (1) \( ||g|| \leq 3/2 \); (2) \( |g(x)| < 2/(3 |q(x)|) \) if \( |q(x)| > 1/2 \); (3) \( g(x) \neq 0 \). Put \( \varphi = gq \) to complete the proof.

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