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MEROMORPHIC IN THE UNIT DISK WITH A SPIRAL
ASYMPTOTIC VALUE**

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The object in this paper is to examine the value distribution of functions $f(z)$ nonconstant and meromorphic in the unit disk which have an asymptotic value α , finite or infinite, along a spiral boundary path. The main result which we prove is that if $\Delta(r)$ is a component of the set of values z such that $|f(z) - \alpha| < r$, $r > 0$, which contains a boundary path on which $f(z)$ tends to α as $|z| \rightarrow 1$, then $f(z)$ assumes every value in $|w - \alpha| < r$ infinitely often in $\Delta(r)$ except for at most two values (if $\Delta(r)$ is simply-connected, then there is at most one exceptional value).

1. Introduction. A boundary path $S: z = s(t)$, $0 \leq t < 1$, in $|z| < 1$ shall be called a spiral if $\arg s(t) \rightarrow +\infty$ as $t \rightarrow 1$ or $\arg s(t) \rightarrow -\infty$ as $t \rightarrow 1$. We shall denote by (S) the class of functions, nonconstant and meromorphic in $|z| < 1$, which have an asymptotic value α , finite or infinite, along a spiral S . The object in this paper is to examine the value distribution of the functions in class (S) . In §3 we have the main result which is a version of Picard's theorem localized to a neighborhood of a transcendental singularity for functions of class (S) . This result is applied in §4 to show that, in some sense, the number of direct transcendental singularities of functions of class (S) cannot be too large. These results extend some earlier work of K. Noshiro [7] (see also [9, p. 163-167]).

2. Components of functions of class (S) . Let $f(z) \in (S)$. Then there is a complex value α , finite or infinite, and a spiral $S: z = s(t)$, $0 \leq t < 1$, in $|z| < 1$ such that $\lim_{t \rightarrow 1} f(s(t)) = \alpha$. Let $r > 0$ and ω be a complex number. We form the open set

$$G = \{z \mid |f(z) - \omega| < r\}$$

if ω is finite, and

$$G = \{z \mid |f(z)| > r\}$$

if ω is infinite. Since functions of class (S) are of unbounded characteristic [5, p. 172], the global cluster set $C(f)$ of $f(z)$ in $|z| < 1$ is total; hence, $G \neq \emptyset$. We denote by $\Delta(r)$ a nonempty open component of G . If $\text{Fr}(\Delta(r)) \cap \{|z| = 1\} = \emptyset$, where $\text{Fr}(A)$ denotes the set of frontier points of the set A , we shall call $\Delta(r)$ a finite domain.

If, however, $\text{Fr}(\mathcal{A}(r)) \cap \{|z| = 1\} \neq \emptyset$, we shall call $\mathcal{A}(r)$ an infinite domain.

By the minimum principle, each finite domain $\mathcal{A}(r)$ contains a zero of $f(z) - \omega$. Also, by Rouché's theorem, a finite domain $\mathcal{A}(r)$ contains the same number of roots, counting multiplicities, for $f(z) - \beta$ for each value β , $|\beta - \omega| < r$.

If $\mathcal{A}(r)$ is an infinite domain, then $\mathcal{A}(r)$ is not, in general, simply-connected. However, an easy application of the maximum principle shows $\mathcal{A}(r)$ to be simply-connected if either $\omega \neq \infty$ and $f(z)$ is holomorphic in $|z| < 1$, i.e., $f(z)$ omits ∞ in $|z| < 1$, or $\omega = \infty$ and $f(z)$ omits 0 in $|z| < 1$. An infinite domain $\mathcal{A}(r)$ with the property that the portion of its boundary $\text{Fr}(\mathcal{A}(r))$ which lies within $|z| < 1$ consists entirely of closed analytic curves shall be called an annular domain. An infinite domain $\mathcal{A}(r)$ with the property that there exists a spiral S^* in $|z| < 1$ such that $\mathcal{A}(r) \subseteq \{|z| < 1\} - S^*$ shall be called a spiral domain.

THEOREM 1. *If $\mathcal{A}(r)$ is an infinite domain for $f(z) \in (S)$, then $\mathcal{A}(r)$ is one of the following: (i) a spiral domain, (ii) an annular domain, or (iii) a subset of an annular domain.*

Proof. Suppose $\mathcal{A}(r)$ is neither annular domain nor a spiral domain. Let $r_1 > r$. Let $\mathcal{A}(r_1)$ be the component of the open set $\{z \mid |f(z) - \omega| < r_1\}$ such that $\mathcal{A}(r) \subseteq \mathcal{A}(r_1)$. Suppose $\mathcal{A}(r_1)$ is not an annular domain. Then, applying methods found in [3], we can find a boundary path L' in $|z| < 1$ on which $|f(z) - \omega| = r_1$. Since $\mathcal{A}(r)$ is not a spiral domain, the spiral S intersects $\mathcal{A}(r)$ in $1 - \delta < |z| < 1$ for each $\delta > 0$. Thus, $|\alpha - \omega| \leq r$, and there exists t_0 , $0 < t_0 < 1$, such that the spiral $S^*: z = s(t)$, $t_0 \leq t < 1$, is disjoint from L' . Thus, L' is also a spiral and $\mathcal{A}(r) \subseteq \mathcal{A}(r_1) \subseteq \{|z| < 1\} - L'$. But this implies that $\mathcal{A}(r)$ is a spiral domain, contrary to our assumption. Therefore, $\mathcal{A}(r_1)$ is an annular domain.

THEOREM 2. *If $\mathcal{A}(r)$ is an infinite domain for $f(z) \in (S)$ and if L is a boundary path in $\mathcal{A}(r)$ on which $f(z) \rightarrow \omega$ as $|z| \rightarrow 1$, then $\mathcal{A}(r)$ is either a spiral domain or an annular domain.*

Proof. Recall that α is the asymptotic value of $f(z)$ along the spiral S . Suppose $\omega \neq \alpha$. Then, in this case, the boundary path L must be a spiral. Suppose $\mathcal{A}(r)$ is not an annular domain. As in the proof above we apply the methods found in [3] to find a boundary path L' in $|z| < 1$ on which $|f(z) - \omega| = r$. Since L is a spiral, L' is a spiral and $\mathcal{A}(r) \subseteq \{|z| < 1\} - L'$.

Suppose $\omega = \alpha$. Then, there exists t_0 , $0 < t_0 < 1$, such that

$|f(s(t)) - \alpha| < r$ for all t , $t_0 \leq t < 1$. Let $S^*: z = s(t)$, $t_0 < t < 1$. Clearly, either $S^* \subseteq \mathcal{A}(r)$ or $\mathcal{A}(r) \subseteq \{|z| < 1\} - S^*$. If $S^* \subseteq \mathcal{A}(r)$, then we can argue as above to show that $\mathcal{A}(r)$ is a spiral domain.

Let $z = \phi(w)$ denote the inverse function of $w = f(z) \in (S)$. The domain of $z = \phi(w)$ is a Riemann surface Φ . We shall write $Q(w; w_0)$ to denote a functional element with center $w = w_0$ for $z = \phi(w)$. Let

$$\mathcal{A}: q(t) = Q(w; w(t)), \quad 0 \leq t < 1,$$

with $\lim_{t \rightarrow 1} w(t) = \omega$, be a curve on the Riemann surface Φ . This curve \mathcal{A} is said to define a transcendental singularity Ω for $z = \phi(w)$ on Φ , with projection $w = \omega$, if (i) for every positive number δ , $\delta < 1$, the system of functional elements $Q(w; w(t))$, $0 \leq t \leq \delta$, defines an analytic continuation (possibly, of algebraic character), but (ii) for any functional element $Q(w; \omega)$, rational or algebraic, with center at $w = \omega$, the system $Q(w; w(t))$, $0 \leq t \leq 1$, where $w(1) = \omega$, never defines an analytic continuation. A theorem due to the work of Iversen [6, p. 13] and Noshiro [7, p. 53] states that there is a one-to-one correspondence between the asymptotic paths of $w = f(z)$ and the transcendental singularities of $z = \phi(w)$, the inverse function of $w = f(z)$. In view of this result, we shall say that two asymptotic boundary paths L_1 and L_2 for $f(z) \in (S)$ are equivalent if L_1 and L_2 both correspond (in the sense of the Iversen-Noshiro theorem) to the same transcendental singularity Ω on the Riemann surface Φ of $z = \phi(w)$, and, we shall indicate this equivalence by the notation $[L_1] = [L_2]$. We refer the reader to [2] and [3] where the notions of equivalent and nonequivalent asymptotic paths are analyzed in greater detail.

THEOREM 3. *If $f(z) \in (S)$ and $\mathcal{A}(r)$ is an annular domain for all $r > 0$, then the inverse function $z = \phi(w)$ of $w = f(z)$ has exactly one transcendental singularity and it lies above $w = \omega$.*

Proof. Suppose $z = \phi(w)$ has at least two transcendental singularities Ω_1 and Ω_2 . Let S_1 and S_2 be the asymptotic boundary paths in $|z| < 1$ for $f(z)$ which correspond to Ω_1 and Ω_2 , respectively. Since $f(z) \in (S)$, S_1 and S_2 are spirals. Let $r > 0$. Since $\mathcal{A}(r)$ is an annular domain, S_1 and S_2 intersect $\mathcal{A}(r) \cap \{\delta < |z| < 1\}$ for each δ , $0 < \delta < 1$. Since r is an arbitrary positive number, we have that ω is the asymptotic value on S_1 and S_2 . By [3, Theorem 1], $[S_1] \neq [S_2]$ implies that there exists $b > 0$ and a boundary path L' , necessarily a spiral, on which $|f(z) - \omega| = b$. But, then $\mathcal{A}(r_0) \subseteq \{|z| < 1\} - L'$ for r_0 , $0 < r_0 < b$, which contradicts our hypothesis that $\mathcal{A}(r_0)$ is an

annular domain. Hence, $z = \phi(w)$ has exactly one transcendental singularity.

In [10] Valiron offers a construction of a function which shows that the converse of Theorem 3 is false. Valiron's construction is very difficult to follow and prompts the need for another approach to the construction of such a function.

3. Value distribution of functions of class (S) in $\Delta(r)$. We denote by $n(\phi, a)$ the number of functional elements $Q(w; a)$, with center $w = a$, for $z = \phi(w)$ the inverse of $f(z) \in (S)$, where an algebraic functional element is counted k times if its order of ramification is $k - 1$. Noshiro [7, p. 60] proved the following: Let $z = \phi_D(w)$ denote the branch of $z = \phi(w)$ obtained by continuing $Q(w; a)$, $a \in D$, inside D with algebraic elements, where D is an arbitrary domain of the w -plane. If $z = \phi_D(w)$ has no transcendental singularity with projection inside D , then $n(\phi_D, w)$ is a finite or infinite constant in D .

THEOREM 4. *If $\Delta(r)$ is an annular domain for $f(z) \in (S)$ and if the transcendental singularities of $z = \phi(w)$ lying above $|w - \omega| < r$ have the property that they lie above at most a finite set of points w_1, w_2, \dots, w_k in $|w - \omega| < r$, then every value of $|w - \omega| < r$ is assumed infinitely often by $f(z)$ in $\Delta(r)$, except possibly the values w_1, w_2, \dots, w_k .*

Proof. Let $D' = \{|w - \omega| < r\} - \cup_{j=1}^k \{w_j\}$. Then, the branch $z = \phi_D(w)$ of $z = \phi(w)$ has no transcendental singularity with projection inside D' . By Noshiro's theorem above, the function $n(\phi_D, w)$ is constant throughout D' . If $\Delta(r)$ had at most finitely many holes, then, since $\Delta(r)$ is an annular domain, the global cluster set $C(f)$ of $f(z)$ would be contained in the closed disk $|w - \omega| \leq r$. But this contradicts functions of class (S) having total global cluster sets. Thus, $\Delta(r)$ has infinitely many holes. Each hole is bounded by a closed analytic curve whose image under $f(z)$ covers the circumference $|w - \omega| = r$ completely. Thus, $n(\phi_D, w) = +\infty$ throughout D' and we are done.

THEOREM 5. *If $\Delta(r)$ is a spiral domain for $f(z) \in (S)$, then each value β , $|\beta - \omega| < r$, omitted by $f(z)$ in $\Delta(r)$ is an asymptotic value along a spiral contained in $\Delta(r)$.*

Proof. Since $\Delta(r)$ is a spiral domain, there exists a spiral S' in $|z| < 1$ such that $\Delta(r) \subseteq \{|z| < 1\} - S'$. Let $\zeta = \zeta(z)$ be a one-to-one conformal map of the simply-connected region $\{|z| < 1\} - S'$ onto

$|\zeta| < 1$ such that the prime end P of $\{|z| < 1\} - S'$ whose impression $I(P)$ is $|z| = 1$ corresponds to $\zeta = 1$. We use $z = z(\zeta)$ to denote the inverse map of $\zeta = \zeta(z)$. Let $\mathcal{A}'(r)$ be the image of $\mathcal{A}(r)$ in $|\zeta| < 1$ under $\zeta = \zeta(z)$. Since $\mathcal{A}(r)$ is a spiral domain, we have that $1 \in \text{Fr}(\mathcal{A}'(r))$.

The function $F(\zeta) = f(z(\zeta))$ is holomorphic in $\mathcal{A}'(r)$ and continuous in $\overline{\mathcal{A}'(r)}$, with the exception of $\zeta = 1$. In fact, $|F(\zeta) - \omega| < r$ for $\zeta \in \mathcal{A}'(r)$ and $|F(\zeta) - \omega| = r$ for $\zeta \in \text{Fr}(\mathcal{A}'(r))$, $\zeta \neq 1$.

Suppose $f(z)$ omits β in $\mathcal{A}(r)$, $|\beta - \omega| < r$. Then, $F(\zeta)$ omits β in $\mathcal{A}'(r)$. Let

$$F_2(\zeta) = \frac{F_1(\zeta) - \beta^*}{1 - \overline{\beta^*} F_1(\zeta)}$$

where $F_1(\zeta) = 1/r(F(\zeta) - \omega)$ and $\beta^* = 1/r(\beta - \omega)$. Then, $F_2(\zeta)$ is holomorphic in $\mathcal{A}'(r)$ with $|F_2(\zeta)| < 1$ in $\mathcal{A}'(r)$, $|F_2(\zeta)| = 1$ for $\zeta \in \text{Fr}(\mathcal{A}'(r))$, $\zeta \neq 1$, and $F_2(\zeta) \neq 0$ in $\mathcal{A}'(r)$. If $\mathcal{A}(r)$ were simply-connected, then $\mathcal{A}'(r)$ would be simply-connected and we could, then, apply results of functions belonging to Seidel's class (U) [8, p. 32] to prove our theorem. Unfortunately, $\mathcal{A}(r)$ may not be simply-connected, in general. An argument of Doob [4] helps to surmount this difficulty.

Since $F_2(\zeta)$ omits 0 in $\mathcal{A}'(r)$, $1/F_2(\zeta)$ is holomorphic in $\mathcal{A}'(r)$. Suppose $1/F_2(\zeta)$ is bounded in $\mathcal{A}'(r)$. Then there exists a number $K > 0$ such that $1/|F_2(\zeta)| < K$ in $\mathcal{A}'(r)$. Let σ be a number such that $0 < \sigma < 1$. Choose a definite branch of the function $(1/2(\zeta - 1))^\sigma$ in $\mathcal{A}'(r)$. This branch is holomorphic in $\mathcal{A}'(r)$ with $|1/2(\zeta - 1)| \leq 1$ in $\mathcal{A}'(r)$. The function

$$\phi_\sigma(\zeta) = \frac{\left(\frac{1}{2}(\zeta - 1)\right)^\sigma}{F_2(\zeta)}$$

is holomorphic and bounded in $\mathcal{A}'(r)$. Furthermore,

$$|\phi_\sigma(\zeta)| \leq \frac{1}{|F_2(\zeta)|} = 1$$

for $\zeta \in \text{Fr}(\mathcal{A}'(r))$, $\zeta \neq 1$. Also,

$$\lim_{\substack{\zeta \rightarrow 1 \\ \zeta \in \mathcal{A}'(r)}} \phi_\sigma(\zeta) = 0.$$

By the maximum principle, $|\phi_\sigma(\zeta)| \leq 1$ for $\zeta \in \mathcal{A}'(r)$, for every $\sigma > 0$. If we let $\sigma \rightarrow 0$, we have that $|F_2(\zeta)| \geq 1$ in $\mathcal{A}'(r)$, and this is a contradiction. Thus, we have that 0 is a cluster value for F_2 at $\zeta = 1$ in $\mathcal{A}'(r)$. But, clearly, 0 does not belong to the set of boundary cluster values for F_2 at $\zeta = 1$ in $\mathcal{A}'(r)$. Since 0 is omitted by F_2 in

$\mathcal{A}'(r)$, by the Gross-Iversen theorem [8, p. 23-24], there exists a path L in $\mathcal{A}'(r)$ terminating at $\zeta = 1$ on which $F_2(\zeta) \rightarrow 0$ as $|\zeta| \rightarrow 1$. Thus, on L $f(z(\zeta)) \rightarrow \beta$ as $|\zeta| \rightarrow 1$. We simply notice that the image of L is a spiral lying in $\mathcal{A}(r)$ on which $f(z) \rightarrow \beta$ as $|z| \rightarrow 1$.

THEOREM 6. *Let $\mathcal{A}(r)$ be an infinite domain for $f(z) \in (S)$ such that $\mathcal{A}(r)$ contains a boundary path L on which $f(z) \rightarrow \omega$ as $|z| \rightarrow 1$. Let $\mathcal{A}_\tau = \mathcal{A}(r) \cap \{|z| < \tau\}$, $0 < \tau < 1$, and let $A(\tau)$ denote the area of the Riemannian image of \mathcal{A}_τ under $w = f(z)$. Then*

$$\lim_{\tau \rightarrow 1} A(\tau) = +\infty.$$

Proof. Suppose for all r_1 , $0 < r_1 < r$, $\mathcal{A}(r_1)$ is an annular domain (here it is understood, of course, that $\mathcal{A}(r_1)$ is that component which contains the end part of L). Then, by Theorem 3, the inverse $z = \phi(w)$ has exactly one transcendental singularity which lies above $w = \omega$. By Theorem 4, every value in the disk $|w - \omega| < r$, except possibly $w = \omega$, is assumed infinitely often by $f(z)$ in $\mathcal{A}(r)$. Thus, in this case,

$$\lim_{\tau \rightarrow 1} A(\tau) = +\infty.$$

Let us, next, consider the case that for some r_0 , $0 < r_0 < r$, $\mathcal{A}(r_0)$ is a spiral domain. Let $\mathcal{A}_\tau^0 = \mathcal{A}(r_0) \cap \{|z| < \tau\}$, $0 < \tau < 1$, and let $A_0(\tau)$ denote the area of the Riemannian image of \mathcal{A}_τ^0 under $f(z)$. Since $\mathcal{A}(r_0) \subseteq \mathcal{A}(r)$, it follows that $A_0(\tau) \leq A(\tau)$. In view of this, it suffices to assume that $\mathcal{A}(r)$ is a spiral domain. Since $\mathcal{A}(r) \subseteq \{|z| < 1\} - S'$ for some spiral S' in $|z| < 1$, we again map $\mathcal{A}(r)$ onto $\mathcal{A}'(r)$ using the one-to-one conformal map $\zeta = \zeta(z)$ of $\{|z| < 1\} - S'$ onto $|\zeta| < 1$ that we used in the proof of Theorem 5. Then, the image L' of the spiral L is a path in $\mathcal{A}'(r)$ which terminates at $\zeta = 1$.

Let $\mathcal{A}''(r)$ denote the image of $\mathcal{A}'(r)$ in the t -plane under the map $t = (\zeta - \zeta_0)/(\zeta - 1)$, where ζ_0 is the initial point of the path L' . The image L'' of L' under this map is a path which begins at the interior point $t = 0$ of $\mathcal{A}''(r)$ and terminates at the boundary point $t = \infty$ of $\mathcal{A}''(r)$. We define

$$G(t) = f\left(z\left(\frac{t - \zeta_0}{t - 1}\right)\right)$$

in $\mathcal{A}''(r)$. Let $\mathcal{A}_\tau'' = \mathcal{A}''(r) \cap \{|t| < \tau\}$, $0 < \tau < +\infty$. Let $A''(\tau)$ denote the area of the Riemannian image of the open set \mathcal{A}_τ'' under $G(t)$. Since the range of $G(t)$ in $\mathcal{A}''(r)$ is identical to that of $f(z)$ in $\mathcal{A}(r)$ and since $\mathcal{A}''(r)$ is linked to $\mathcal{A}(r)$ by means of a one-to-one conformal map, it suffices to show that

$$\lim_{\tau \rightarrow +\infty} A''(\tau) = +\infty.$$

Let $\tau_0 > 0$ be fixed so that $|t| < \tau_0$ contains at least one boundary point of $\mathcal{A}''(r)$. Denote by L'_τ the part of the path L'' which runs from the last point of intersection t_τ of L'' with $|t| = \tau$, counting from $t = 0$. Since $G(t) \rightarrow \omega$ on L'' as $|t| \rightarrow +\infty$, there exists a number τ_1 , $\tau_1 > \tau_0 > 0$, such that (i) $|G(t) - \omega| < (1/2)r$ for all $t \in L''_{\tau_1}$, and (ii) for any $\tau > \tau_1$, if γ_τ denotes the collection of component arcs of $|t| = \tau$ which fall into $\mathcal{A}''(r)$ (there can be at most finitely many such arcs since the boundary of $\mathcal{A}''(r)$ is an analytic curve), then γ_τ contains a crosscut of $\mathcal{A}''(r)$, call it λ_τ , such that the point $t_\tau \in \lambda_\tau$ and the endpoints of λ_τ lie on $\text{Fr}(\mathcal{A}''(r))$. Since the image of the arc λ_τ under $G(t)$ is a curve which starts from a point on $|w - \omega| = r$, passes through a point lying in $|w - \omega| < (1/2)r$, and, finally, terminates at a point on $|w - \omega| = r$, we have that the length of the image of λ_τ under $G(t)$ is greater than or equal to r .

Let $L(\tau)$ be the total length of the image of γ_τ under $G(t)$; let $l(\tau)$ be the total length of γ_τ . Then for $t = \tau e^{i\theta}$,

$$L(\tau) = \int_{\gamma_\tau} |G'(t)| \tau d\theta.$$

By the Schwarz inequality,

$$(L(\tau))^2 \leq \left(\int_{\gamma_\tau} |G'(t)|^2 \tau d\theta \right) \left(\int_{\gamma_\tau} \tau d\theta \right) = l(\tau) \int_{\gamma_\tau} |G'(t)|^2 \tau d\theta.$$

Hence,

$$(1) \quad \frac{(L(\tau))^2}{l(\tau)} \leq \int_{\gamma_\tau} |G'(t)|^2 \tau d\theta,$$

and, for $\tau > \tau_1$,

$$(2) \quad \int_{\tau_1}^{\tau} \frac{(L(\tau))^2}{l(\tau)} d\tau \leq \int_{\tau_1}^{\tau} \int_{\gamma_\tau} |G'(t)|^2 \tau d\tau d\theta \leq A''(\tau) - A''(\tau_1).$$

Since $l(\tau) \leq 2\pi\tau$ and $L(\tau) \geq r$ for all τ , $\tau_1 \leq \tau \leq +\infty$,

$$\int_{\tau_1}^{\tau} \frac{(L(\tau))^2}{l(\tau)} d\tau \geq \frac{r^2}{2\pi} \int_{\tau_1}^{\tau} \frac{d\tau}{\tau} \longrightarrow +\infty$$

as $\tau \rightarrow +\infty$. Therefore,

$$\lim_{\tau \rightarrow +\infty} A''(\tau) = +\infty.$$

THEOREM 7. *Under the hypothesis of Theorem 6,*

$$\liminf_{\tau \rightarrow 1} \frac{L(\tau)}{A(\tau)} = 0 ,$$

where $L(\tau)$ denotes the length of the images of the collection of arcs of $|z| = \tau$ which fall into $\Delta(r)$ under $f(z)$.

Proof. By (2)

$$\frac{(L(\tau))^2}{l(\tau)} \leq \frac{dA''(\tau)}{d\tau} .$$

Hence,

$$\frac{d\tau}{l(\tau)} \leq \frac{dA''(\tau)}{(L(\tau))^2} .$$

Let $E = \{\tau | \tau > \tau_0, L(\tau) \geq A''(\tau)^{1/2+\varepsilon}\}$, $0 < \varepsilon < 1/2$. Since $l(\tau) \leq 2\pi\tau$,

$$\frac{1}{2\pi} \int_E \frac{d\tau}{\tau} \leq \int_E \frac{d\tau}{l(\tau)} .$$

Thus,

$$\begin{aligned} \frac{1}{2\pi} \int_E \frac{d\tau}{\tau} &\leq \int_E \frac{dA''(\tau)}{(L(\tau))^2} \leq \int_E \frac{dA''(\tau)}{(A''(\tau)^{1/2+\varepsilon})^2} \\ &\leq \int_{A''(\tau_0)}^{\infty} \frac{dt}{t^{1+2\varepsilon}} < +\infty . \end{aligned}$$

Thus, there exists a sequence of positive numbers $\{\tau_n\}$ such that $\tau_n \rightarrow +\infty$ as $n \rightarrow +\infty$ with $\tau_n \notin E$ for all n . Therefore,

$$\frac{L(\tau_n)}{A''(\tau_n)} \leq \frac{A''(\tau_n)^{1/2+\varepsilon}}{A''(\tau_n)} = \frac{1}{A''(\tau_n)^{1/2-\varepsilon}}$$

and, by Theorem 6, we have

$$(3) \quad \liminf_{\tau \rightarrow +\infty} \frac{L(\tau)}{A''(\tau)} = 0 .$$

For $f(z) \in (S)$, the Riemannian image Φ_r of the disk $|z| < r$, $0 < r < 1$, under $f(z)$ is a finite covering of the Riemann sphere of diameter 1 and tangent to the w -plane endowed with the spherical distance as metric. The Riemann images Φ_r exhaust the surface Φ of the inverse $z = \phi(w)$ of $w = f(z)$ [8, p. 90]. Let $A(r)$ denote the spherical area of Φ_r and let $L(r)$ denote the spherical length of the boundary of Φ_r .

COROLLARY. For $f(z) \in (S)$,

$$(i) \quad \lim_{r \rightarrow 1} A(r) = +\infty ,$$

and

$$(ii) \quad \liminf_{r \rightarrow 1} \frac{L(r)}{A(r)} = 0.$$

Proof. Follows immediately from Theorems 6 and 7.

A Riemann surface Φ which satisfies condition (ii) in the above corollary is called regularly exhaustible. For a thorough discussion of regularly exhaustible surfaces and the value distribution theory connected with them we refer the reader to [9, p. 152-170]. In view of this, it is appropriate to state the following corollary.

COROLLARY. *The Riemannian image of a function of class (S) is regularly exhaustible.*

The next theorem is a Picard theorem localized to a transcendental singularity of the inverse function $z = \phi(w)$ of $f(z) \in (S)$. The idea behind the proof comes from the work of K. Noshiro [7].

THEOREM 8. *Let $\Delta(r)$ be an infinite domain for $f(z) \in (S)$ such that $\Delta(r)$ contains a boundary path L on which $f(z) \rightarrow \omega$ as $|z| \rightarrow 1$. Then, the function takes every value of $|w - \omega| < r$ infinitely often in $\Delta(r)$, except for at most two values. In particular, if $\Delta(r)$ is simply-connected, then $f(z)$ assumes every value of $|w - \omega| < r$ infinitely often in $\Delta(r)$ with one possible exception.*

Proof. We assume, first, that $\Delta(r)$ is simply-connected. Let $\Delta''(\tau)$, $A''(\tau)$, γ_τ , $L(\tau)$, and $G(t)$ be as found in the proof of Theorem 6. Suppose that w_1 and w_2 are distinct values in $|w - \omega| < r$ such that $G(t)$ omits w_1 and w_2 in $\Delta''(r) \cap \{|t| > \tau_0\}$ for some $\tau_0 > 0$. Consider the open set $\Delta'_\tau = \Delta''(r) \cap \{\tau_0 < |t| < \tau\}$. Since each component of Δ'_τ contains at least one arc of either γ_{τ_0} or γ_τ , we must have that Δ'_τ consists of finitely many simply-connected components

$$\Delta''_\tau(1), \Delta''_\tau(2), \dots, \Delta''_\tau(m), m = m(\tau).$$

The Riemannian image of Δ'_τ under $G(t)$ consists of m simply-connected covering surfaces $\Phi''_\tau(j)$ corresponding to $\Delta''_\tau(j)$, $j = 1, 2, \dots, m$, of the base surface

$$B = \{|w - \omega| < r\} - \{w_1, w_2\}.$$

The Euler characteristic of B is 1 [5, p. 136].

Applying Ahlfors' theorem of covering surfaces [5, p. 137] to $\Delta''_\tau(j)$ and $\Phi''_\tau(j)$, we have $S^j \leq hL^j$, $j = 1, 2, \dots, m$, where S^j denotes the ratio between the area of $\Phi''_\tau(j)$ and the area of B , and L^j

denotes the length of the boundary of $\Phi_r''(j)$ relative to B , h being a constant dependent on B . Thus,

$$\sum_{j=1}^m S^j \leq h \sum_{j=1}^m L^j,$$

and

$$S(\tau) \leq h(L(\tau) + L(\tau_0))$$

where $S(\tau) = A''(\tau)/\pi r^2$. Thus, $(L(\tau) + L(\tau_0))/S(\tau) \geq 1/h$ for all τ , $\tau > \tau_0$. Therefore,

$$\lim_{\tau \rightarrow +\infty} \frac{L(\tau)}{A''(\tau)} \geq \frac{\pi r^2}{h} > 0.$$

But this contradicts (3). Thus, $G(t)$ in $\mathcal{A}''(r)$ and, clearly, $f(z)$ in $\mathcal{A}(r)$ assumes all values of $|w - \omega| < r$, except possibly one, infinitely many times.

Suppose $\mathcal{A}(r)$ is an arbitrary infinite domain which contains L ; $\mathcal{A}(r)$ may not be simply-connected. Assume that there are three distinct values w_1, w_2, w_3 of $|w - \omega| < r$ which are assumed only finitely many times by $f(z)$ in $\mathcal{A}(r)$. We draw a simple closed analytic curve L in $|w - \omega| < r$ which encloses ω, w_1, w_2 , and passes through w_3 . Since $f(z) \rightarrow \omega$ on L as $|z| \rightarrow 1$, we may assume that the image of L under $f(z)$ lies entirely in the interior H of L .

Let \mathcal{A}_H be the component of $\{z | f(z) \in H\}$ which contains the path L . Then, clearly, $\mathcal{A}_H \subseteq \mathcal{A}(r)$. Choose r_0 , $0 < r_0 < 1$, such that $f(z)$ omits w_1, w_2 , and w_3 in $\mathcal{A}(r) \cap \{r_0 < |z| < 1\}$. Let \mathcal{A}_H^* be the component of $\mathcal{A}_H \cap \{r_0 < |z| < 1\}$ which contains the end part of L . Then, $\mathcal{A}_H^* \subseteq \mathcal{A}_H$, and \mathcal{A}_H^* is simply-connected. Indeed, if it were not, then on the boundary of each hole $f(z)$ would assume the value w_3 . But by the construction of \mathcal{A}_H^* this is impossible. Also, \mathcal{A}_H^* is clearly a spiral domain.

We can, now, apply the above argument to the simply-connected spiral domain \mathcal{A}_H^* to show that it cannot omit two values. Thus, our theorem is proved.

THEOREM 9. *Each exceptional value of Theorem 8 is an asymptotic value for $f(z)$ along a spiral contained in $\mathcal{A}(r)$.*

Proof. Suppose $\beta \neq \omega$ is assumed by $f(z)$ in $\mathcal{A}(r)$ only finitely many times. Then, there exists δ , $0 < \delta < 1$, such that $f(z)$ omits β in $\mathcal{A}(r) \cap \{\delta < |z| < 1\}$. Choose δ_1 , $\delta < \delta_1 < 1$, so that $f(z) \neq \beta$ on $|z| = \delta_1$. Let

$$\varepsilon = \min |f(z) - \beta| \quad \text{for} \quad |z| = \delta_1,$$

and, let $\rho = 1/4 \min(\varepsilon, r - |\beta - \omega|, |\beta - \omega|)$. Clearly, $\rho > 0$. By Theorem 8, there exists a value z_0 such that $z_0 \in \Delta(r) \cap \{\delta_1 < |z| < 1\}$ and $|f(z_0) - \beta| < \rho$. Let $\Delta_\beta(\rho)$ be that component of $\{z \mid |f(z) - \beta| < \rho\}$ which contains z_0 . By the choice of ρ , we have $\Delta_\beta(\rho) \subseteq \{\delta_1 < |z| < 1\}$ and $\Delta_\beta(\rho) \subseteq \Delta(r)$. Since $f(z)$ omits β in $\Delta_\beta(\rho)$, we have that $\Delta_\beta(\rho)$ is an infinite domain. We, also, point out that the end part of the spiral on which $f(z) \rightarrow \omega$ as $|z| \rightarrow 1$ is disjoint from $\Delta_\beta(\rho)$. Thus, $\Delta_\beta(\rho)$ is a spiral domain in which β is an omitted value of $f(z)$. By Theorem 5, β is an asymptotic value along a spiral contained in $\Delta_\beta(\rho)$.

4. Direct transcendental singularities. Let $f(z) \in (S)$ and $\Delta(r)$ be an infinite domain in $|z| < 1$ such that $\Delta(r)$ contains a boundary path L on which $f(z) \rightarrow \omega$. If $f(z)$ omits ω in $\Delta(r)$ for $r > 0$ sufficiently small, then the transcendental singularity Ω of $z = \phi(w)$ which corresponds to L is said to be a direct transcendental singularity.

THEOREM 10. *Let $f(z) \in (S)$ and let $z = \phi(w)$ be its inverse function. Then, the set of values ω in the w -plane which are projections of direct transcendental singularities of $z = \phi(w)$ is at most countable.*

Proof. Let $\{\omega_n\}$ be the rational points in the w -plane and let $\{r_n\}$ be the rationals of the interval $(0, 1)$. Let $G_n = \{z \mid |f(z) - \omega_n| < r_n\}$. We set $H = \bigcup_{n=1}^{\infty} H_n$, where H_n is the set of points of $|w - \omega_n| < r_n$ which are not covered by the image of at least one component of G_n under $f(z)$. By Theorem 9, H_n is at most countable, and, hence, so is H .

Suppose $f(z) \rightarrow \omega$ on a spiral S and S corresponds to a direct transcendental singularity for $z = \phi(w)$. Then, there exists $r > 0$ such that $S \subset \Delta(r)$ and $f(z)$ omits ω in $\Delta(r)$. But there exists an integer n such that a component Δ_n of G_n is contained in $\Delta(r)$ with $|\omega - \omega_n| < r_n$. Therefore, $\omega \in H_n$ and the theorem is proven.

We, next, present an example of a holomorphic function $f(z) \in (S)$ such that its inverse function has uncountably many transcendental singularities above $w = \infty$, and we note that since $f(z)$ is holomorphic these are direct transcendental singularities. This example places the last result in clearer perspective.

Let A be the extended complex plane and let M be the Cantor set on the interval $0 \leq \theta \leq 2\pi$ with $\{I_n\}$ denoting the sequence of the open middle-third intervals of $0 \leq \theta \leq 2\pi$ which are deleted to construct M . The order of the sequence $\{I_n\}$ is as follows:

$$I_1 = \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right), \quad I_2 = \left(\frac{2\pi}{9}, \frac{4\pi}{9}\right),$$

$$I_3 = \left(\frac{14\pi}{9}, \frac{16\pi}{9}\right), \quad I_4 = \left(\frac{2\pi}{27}, \frac{4\pi}{27}\right), \dots$$

Let $\{J_n\}$ be the sequence of open intervals of the sequence of open sets $[0, 2\pi] - \bar{I}_1, [0, 2\pi] - \overline{I_1 \cup I_2}, \dots$, with the ordering

$$J_1 = \left(0, \frac{2\pi}{3}\right), \quad J_2 = \left(\frac{4\pi}{3}, 2\pi\right), \quad J_3 = \left(0, \frac{2\pi}{9}\right),$$

$$J_4 = \left(\frac{4\pi}{9}, \frac{2\pi}{3}\right), \quad J_5 = \left(\frac{4\pi}{3}, \frac{14\pi}{9}\right), \dots$$

Since A is an analytic set and M is a closed nowhere dense set we can apply a theorem of Bagemihl and Seidel [1, p. 198-199] to claim the existence of a function $f(z)$, holomorphic in $|z| < 1$, with the following properties:

(i) for every $\theta \in M$, $\lim_{r \rightarrow 1} f(re^{i(\theta+1/(1-r))}) = w_\theta$ exists (possibly infinite);

(ii) if I is any subinterval of $0 \leq \theta \leq 2\pi$ such that $I \cap M \neq \emptyset$, then $A = \{w_\theta | \theta \in I \cap M\}$, and for every $\alpha \in A$, there are uncountably many values of $\theta \in I \cap M$ for which $w_\theta = \alpha$.

By (ii) every value of the extended complex plane is an asymptotic value on uncountably many spiral paths $S_\theta: z = s_\theta(r) = re^{i(\theta+1/(1-r))}$, $0 \leq r < 1$, for $\theta \in M$. Let $\theta_1, \theta_2 \in M$, $\theta_1 < \theta_2$ such that $f(z) \rightarrow \omega$ on S_{θ_1} and S_{θ_2} as $|z| \rightarrow 1$. We can find two intervals I_{n_1} and I_{n_2} of $\{I_n\}$ such that $I_{n_1} \subseteq (\theta_1, \theta_2)$ and $I_{n_2} \subseteq [0, 2\pi] - [\theta_1, \theta_2]$. Thus, for n'_1, n'_2 sufficiently large, there exist intervals $J_{n'_1}$ abutting I_{n_1} and $J_{n'_2}$ abutting I_{n_2} such that $J_{n'_1} \subseteq (\theta_1, \theta_2)$ and $J_{n'_2} \subseteq [0, 2\pi] - [\theta_1, \theta_2]$. But since $J_{n'_1} \cap M \neq \emptyset$ and $J_{n'_2} \cap M \neq \emptyset$, by (ii), there exist spirals separating S_{θ_1} and S_{θ_2} on which $f(z)$ has asymptotic values different from $w = \omega$. Thus $[S_{\theta_1}] \neq [S_{\theta_2}]$. Since ω is an arbitrary value of the extended complex plane, we see that the inverse $z = \phi(w)$ has uncountably many transcendental singularities above every value of the extended w -plane. In particular, since $f(z)$ omits $w = \infty$, $z = \phi(w)$ has uncountably many direct transcendental singularities above $w = \infty$.

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