INTEGRALS OF FOLIATIONS ON MANIFOLDS WITH A GENERALIZED SYMPLECTIC STRUCTURE

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INTEGRALS OF FOLIATIONS ON MANIFOLDS WITH A GENERALIZED SYMPLECTIC STRUCTURE

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Let \( M \) be a \( C^\infty \) manifold of dimension \( m \) and \( E \) an integrable subbundle (foliation) of the tangent bundle \( TM \). We are interested in structures on the set of all local integrals of \( E \). For example, if \( M \) is a symplectic manifold then the Poisson brackets operation on the set \( C^\infty_0 \) of all local functions of \( M \) defines an algebraic structure on \( C^\infty_0 \). Earlier authors have called such structures "function groups." In particular, if \( X_H \) is a nonvanishing Hamiltonian vector field, then \( X_H \) defines a foliation \( E \) of \( M \) and the set of all local integrals of \( E \) is also a function group.

The Poisson brackets operation can be defined on manifolds with somewhat less restrictive requirements than that of being symplectic. Other authors such as S. Lie and C. Carathéodory [4] have studied this more general notion of Poisson brackets in the classical local setting. Hermann [9, p. 31] has indicated how to extend the definition of Poisson brackets to functions on manifolds having a closed 2-form \( \omega \) of constant rank (Recall that \( M \) is called symplectic if \( \omega_p \) has rank \( m \) for each \( p \in M \)).

The paper is largely self-contained, but does require the use of the following basic identities:

\[
L_X Y = [X, Y], \quad L_X = i_X d + di_X, \quad L_X i_Y - i_Y L_X = i_{[X,Y]}.
\]

The proofs of these identities may be found in Chapter IV of the first volume of [7]. Other undefined terms appear either in [1] or [7].

1. Generalized symplectic structures on manifolds. Let \( M \) be a \( C^\infty \) manifold of dimension \( m \) and let \( \omega \) be a closed 2-form on \( M \). Recall that the kernel of a 2-form \( \omega \) can be defined at each point \( p \in M \) by

\[
ker \omega_p = \{ v \in M_p | \omega(v, M_p) = 0 \}
\]

\[
= \{ v \in M_p | \omega(M_p, v) = 0 \}.
\]

The rank of \( \omega \) at \( p \) is defined to be the rank of the bilinear map \( \omega_p: M_p \times M_p \rightarrow R \). Of course, since \( \omega_p \) is a skew-symmetric bilinear map its rank is the even integer \( m - \dim (ker \omega_p) \).

Let \( \Gamma \) denote the set of sections of \( TM \) and \( \Gamma^* \) the set of sections of \( T^*M \). Define \( \alpha: \Gamma \rightarrow \Gamma^* \) by
\[ \alpha_x = i_x \omega \, . \]

Let \( \Gamma_\omega = \{ X \in \Gamma \mid i_X \omega = 0 \} = \ker \alpha \).

If we fix \( p \in M \) then we may regard \( \alpha = \alpha_p \) as a map from \( T_p M \) into \( T^*_p M \). Since \( T_p M \) is finite dimensional, \( T_p M \cong T^*_p M \) and we may apply the standard duality theorems of linear algebra. Thus, if we use the usual pairing between \( T_p M \) and \( T^*_p M \) we have, for \( x, y \in T_p M \),

\[ \langle \alpha(y), x \rangle = \alpha(y)(x) = \omega_p(y, x) = -\omega_p(x, y) = \langle y^{**}, -\alpha(x) \rangle . \]

Thus \( \alpha \) is skew adjoint: \( \alpha^* = -\alpha \), and

\[ \text{im} (\alpha^*) = \text{im} (\alpha) = \ker (\alpha) \]

where \( \ker (\alpha)^\perp \) is the annihilator of \( \ker (\alpha) \) in \( T^*_p M \).

From this we see that if \( \Gamma^*_\omega = \{ \beta \in \Gamma^* \mid \beta(\Gamma_\omega) = 0 \} \), then \( \Gamma^*_\omega = \ker (\alpha)^\perp \subseteq \Gamma^* \). From these remarks it follows that \( \Gamma^*_\omega = \text{im} (\alpha) \).

If \( \text{inv} (\Gamma) \) is defined by \( \text{inv} (\Gamma) = \{ X \in \Gamma \mid L_X \Gamma_\omega \subseteq \Gamma_\omega \} \) then \( \text{inv} (\Gamma) \) is the normalizer of \( \Gamma_\omega \) in \( \Gamma \) and thus is a Lie subalgebra of \( \Gamma \). Moreover, it is immediate from the definitions any subalgebra of a Lie algebra is always an ideal in its normalizer, thus \( \Gamma_\omega \) is an ideal in \( \text{inv} (\Gamma) \). We summarize all these remarks as a proposition.

**Proposition 1.1.** The image of the map \( \alpha: \Gamma \rightarrow \Gamma^* \) is precisely

\[ \Gamma^*_\omega = \{ \beta \in \Gamma^* \mid \beta(\Gamma_\omega) = 0 \} \, . \]

Moreover, \( \text{inv} (\Gamma) = \{ X \in \Gamma \mid L_X \Gamma_\omega \subseteq \Gamma_\omega \} \) is a Lie subalgebra of \( \Gamma \) which contains \( \Gamma_\omega \) as an ideal.

We now want to show that \( \alpha|_{\text{inv} (\Gamma)} \) is a Lie algebra antihomomorphism from \( \text{inv} (\Gamma) \) onto the set \( \text{inv} (\Gamma^*_\omega) \subseteq \Gamma^* \) where \( \text{inv} (\Gamma^*_\omega) \) is defined by

\[ \text{inv} (\Gamma^*_\omega) = \{ \beta \in \Gamma^*_\omega \mid L_Z \beta = 0 \text{ for all } Z \in \Gamma_\omega \} \, . \]

Before doing this we need to define a Lie algebra structure on \( \text{inv} (\Gamma^*_\omega) \). For this we need a lemma.

**Lemma 1.2.** If \( Z \in \Gamma_\omega \), then \( L_Z \Gamma^*_\omega \subseteq \Gamma^*_\omega \). In fact, \( L_Z \alpha_X = \alpha_{L_Z X} \)

for each \( X \in \Gamma \).

**Proof.** Since \( L_Z \omega = (i_Z d + di_Z)\omega = 0 \), \( L_Z \alpha_X = L_Z i_X \omega = i_X L_Z \omega + i_{(x, x)} \omega = \alpha_{[z, x]} = \alpha_{L_Z x} \).

**Corollary 1.3.** \( \alpha(\text{inv} \Gamma) = \text{inv} (\Gamma^*_\omega) \).

**Proof.** From Proposition 1.1, we know that \( \text{inv} (\Gamma^*_\omega) \) is contained
INTEGRALS OF FOLIATIONS ON MANIFOLDS

in \text{im}(\alpha)$. By the lemma above, for $Z \in \Gamma_\omega$, $L_Z\alpha_x = \alpha_{L_Zx}$; thus $\alpha_x \in \text{inv}(\Gamma_\omega^*)$ iff $L_ZZ \in \Gamma_\omega$ for all $Z \in \Gamma_\omega$. It follows that $\alpha(\text{inv}(\Gamma)) = \text{inv}(\Gamma_\omega^*)$.

The map $\alpha$ is a linear transformation from $\text{inv}(\Gamma)$ onto $\text{inv}(\Gamma_\omega^*)$ with kernel $\Gamma_\omega$. Thus $\text{inv}(\Gamma_\omega^*) \cong \text{inv}(\Gamma)/\Gamma_\omega$ as vector spaces. Since $\Gamma_\omega$ is a Lie ideal in $\text{inv}(\Gamma)$, the quotient $\text{inv}(\Gamma)/\Gamma_\omega$ is a Lie algebra. We impose this Lie structure on $\text{inv}(\Gamma_\omega^*)$ via the vector space isomorphism induced by $\alpha$.

**Proposition 1.4.** The set $\text{inv}(\Gamma_\omega^*)$ of all invariant elements of $\Gamma_\omega^*$ is a Lie algebra under $\{,\}$ where $\{,\}$ is defined by

$$\{\alpha_x, \alpha_y\} = -\alpha_{[x,y]}.$$

The map $\alpha: \text{inv}(\Gamma) \rightarrow \text{inv}(\Gamma_\omega^*)$ is a Lie algebra antihomomorphism with kernel $\Gamma_\omega$, thus the sequence

$$0 \rightarrow \Gamma_\omega \rightarrow \text{inv}(\Gamma) \xrightarrow{\alpha} \text{inv}(\Gamma_\omega^*) \rightarrow 0,$$

is an exact sequence of Lie algebras.

**Remark.** It is easy to see that for $\alpha, \beta \in \text{inv}(\Gamma_\omega^*)$ one has

$$\{\alpha, \beta\}_{|_U} = \{\alpha_{|_U}, \beta_{|_U}\}$$

for open subsets $U$ of $M$.

**Remark.** We now call attention to certain identities which have proven useful in our work. If $\beta$ and $\gamma$ are closed 1-forms in $\Gamma_\omega^*$ and $X$ and $Y$ are vector fields such that $\beta = \alpha_x$, $\gamma = \alpha_y$, then

$$\{\beta, \gamma\} = -i_{[x,y]}\omega = -L_XY = L_Y\beta = d(2\omega(X,Y)).$$

Note, in particular, that $\{\beta, \gamma\}$ is exact.

To see that the above identities hold, observe that

$$\{\beta, \gamma\} = \{\alpha_x, \alpha_y\} = -\alpha_{[x,y]}$$

$$= -i_{[x,y]}\omega = -L_Xi_Y\omega + i_YL_X\omega$$

$$= -L_X\alpha_y + i_Y(d\alpha_X + i_Xd\alpha) = -L_X\alpha_y + i_Y(d\alpha_X)$$

$$= -L_X\gamma = -(d\alpha_X + i_Xd\gamma) = -d(i_X\gamma)$$

$$= 2d(\omega(X,Y)).$$

Let $C^\omega(\omega)$ denote the set of all invariant functions of $\ker\omega$, i.e.

$$C^\omega(\omega) = \{f \mid L_Zf = df(Z) = 0 \text{ for all } Z \in \Gamma_\omega\}.$$  

We now define the Poisson bracket $\{,\}$ for pairs of invariant functions of $\ker\omega$:...
\[ \{f, g\} = 2\omega(X_f, X_g) \]
where \(X_f\) and \(X_g\) are any two vector fields such that
\[ dh = i_{X_h}\omega \]
for \(h = f, g\). Clearly \(\{\ , \\}\) is well-defined.

**Proposition 1.5.** If \(f, g \in C^\infty(\omega)\) the following statements are true:

1. \(\{f, g\} = -L_{X_f}(g) = L_{X_g}(f)\)
2. \(d\{f, g\} = \{df, dg\}\).

Moreover, \(C^\infty(\omega)\) is a Lie algebra with respect to \(\{\ , \\}\) and

3. \(X_{\{f, g\}} + [X_f, X_g] \in \Gamma_\omega\).

**Proof.** If \(f, g \in C^\infty(\omega)\) then (1) follows from \(\{f, g\} = 2\omega(X_f, X_g) = df(X_g) - L_{X_g}(f)\). By the above remark we have \(d\{f, g\} = d(2\omega(X_f, X_g)) = \{df, dg\}\) and thus (2) follows. The statement (3) is immediate from definitions.

**Proposition 1.6.** If \(f, g \in C^\infty(\omega)\) and \(dg = i_{X_g}\omega\) then \(f\) is constant on integral curves of \(X_g\) iff \(\{f, g\} = 0\).

**Proof.** \(X_g(f) = L_{X_g}(f) = \{f, g\} = 0\).

2. **Function groups.** Let \(M\) be a connected \(C^\infty\)-manifold of dimension \(m\) with a 2-form \(\omega\) of constant rank \(\rho \leq m\). In this case \(\ker\omega\) is locally trivial, i.e., \(\ker\omega\) is a subbundle of \(TM\). Moreover, \(\ker\omega\) is actually an integrable subbundle of \(TM\) and thus is a foliation of \(M\). To see this observe that for \(X \in \Gamma_\omega\),
\[ L_X\omega = i_X(d\omega) + d(i_X\omega) = 0. \]
Thus for \(X, Y\) in \(\Gamma_\omega\),
\[ i_{[X,Y]}\omega = L_X(i_Y\omega) - i_Y(L_X\omega) = 0. \]

A function \(f\) is called a local \(C^\infty\) function on \(M\) iff the domain \(U = \text{dom}(f)\) of \(f\) is an open subset of \(M\) and \(f \in C^\infty(U)\). Let \(C^\infty_{\text{loc}}(M)\) denote the set of all local \(C^\infty\) functions of \(M\). Let \(C^\infty_{\text{loc}}(\omega)\) denote the set of all local integrals of the foliation \(\ker\omega\), i.e.,
\[ C^\infty_{\text{loc}}(\omega) = \{f \in C^\infty_{\text{loc}} \mid df(\ker(\omega_p)) = 0 \text{ for all } p \in \text{dom } f\}. \]
Note that in the symplectic case \(C^\infty_{\text{loc}}(\omega) = C^\infty_{\text{loc}}\).

Recall that a function \(f \in C^\infty_{\text{loc}}\) is said to be \(C^\infty\)-dependent on \(f_1, f_2, \ldots, f_r \in C^\infty_{\text{loc}}\) at \(p \in M\) provided that there is a neighborhood \(U\)
of $p$ and a function $F \in C^\infty_c(R^r)$ such that
(1) the functions $f, f_1, f_2, \ldots, f_r$ are all defined on $U$, and
(2) $f(x) = F(f_1(x), f_2(x), \ldots, f_r(x))$ for each $x \in U$.

If $f, g \in C^\infty_c(\omega)$ and $U = \text{dom } f \cap \text{dom } g \neq \emptyset$, then $U$ can be regarded as a manifold with $\omega|_U$ a 2-form of constant rank on $U$. Thus \{f, g\} = \{f|_U, g|_U\}$ is a well-defined element of $C^\infty(\omega|_U)$. It follows that $X_f$ and $X_g$ have domains $\text{dom } f$ and $\text{dom } g$ respectively and thus $[X_f, X_g]$ and $X_{\langle f, g \rangle}$ are well-defined vector fields on $U$. Similarly, $\{df, dg\}$ is a well-defined 1-form on $U$.

**Definition 2.1.** A nonvoid subset $\mathcal{S}$ of $C^\infty_c(\omega)$ is called a function group iff the following conditions hold:
(1) $M = \bigcup_{f \in \mathcal{S}} \text{dom } (f)$,
(2) if $f \in \mathcal{S}$ and $U$ is an open subset of $\text{dom } f$ then $f|_U \in \mathcal{S}$,
(3) if $f, g \in \mathcal{S}$ and $\text{dom } (f) \cap \text{dom } (g) \neq \emptyset$, then \{f, g\} $\in \mathcal{S}$,
(4) if $f_1, f_2, \ldots, f_k$ are elements of $\mathcal{S}$ and $f$ is $C^\infty$-dependent on $f_1, f_2, \ldots, f_k$ then $f \in \mathcal{S}$,
(5) Let $U = \bigcup_j U_j$, where $U_j$ is an open subset of $M$ for each $j$. If $f \in C^\infty(U)$ and $f|_{U_j} \in \mathcal{S}$, for each $j$, then $f \in \mathcal{S}$.

A function group is said to be of rank $r$ at a point $p \in M$ provided that there are $r$ functions $f_1, f_2, \ldots, f_r$ in $\mathcal{S}$ such that
(1) there is a neighborhood $U$ of $p$ contained in the domain of each of the functions $f_1, f_2, \ldots, f_r$ such that for each $q \in U$
$$df_1, df_2, \ldots, df_r$$
are independent elements of $M^*_q$, and
(2) for each $f \in \mathcal{S}$, with $p \in \text{dom } f$, $f$ is $C^\infty$-dependent on $f_1, f_2, \ldots, f_r$ on some neighborhood of $p$.

In case $f_1, f_2, \ldots, f_r$ satisfy (1) and (2) we say that $f_1, f_2, \ldots, f_r$ generate $\mathcal{S}$ at $p$.

**Remark.** If $f_1, f_2, \ldots, f_r$ generate $\mathcal{S}$ at $p$ and $g_1, g_2, \ldots, g_s$ generate $\mathcal{S}$ at $p$, then $r = s$. To see this observe that the definition implies that there exists functions $F_i \in C^\infty_c(R^r)$, $G_j \in C^\infty_c(R^r)$ such that for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, r$
$$g_i = F_i(f_1, \ldots, f_r) \quad \text{and} \quad f_j = G_j(g_1, \ldots, g_s).$$

Then the chain rule applied to the equalities
$$g_i = F_i(G_1(g_1, \ldots, g_s), \ldots, G_r(g_1, \ldots, g_s)),
\quad f_j = G_j(F_1(f_1, \ldots, f_r), \ldots, F_s(f_1, \ldots, f_r))$$
implies that $(\partial F_i/\partial f_j)$ and $(\partial G_k/\partial g_l)$ are inverse matrices. Hence $r = s$. 

REMARK. If $\mathcal{G}$ is a function group of rank $r$ at $p \in M$, then one can easily show that if $h_1, h_2, \ldots, h_r$ are elements of $\mathcal{G}$ such that $dh_{1p}, dh_{2p}, \ldots, dh_{rp}$ are independent in $M^*_p$ then they generate $\mathcal{G}$ at $p$.

A function group is said to be of rank $r$ iff it is of rank $r$ at each point of $M$.

The following is an example to show that a function group may not have the same rank at each point of $M$. Let $M = \mathbb{R}^2$ and $\omega = dx \wedge dy$. Let $f \in C^\infty(\mathbb{R})$ such that

$$f(x) = 0, x \leq 0 \quad \text{and} \quad f(x) > 0, x > 0.$$  

Define functions $F$ and $G$ on $\mathbb{R}^2$ by $F(x, y) = x$ and $G(x, y) = f(x)y$. Let $\mathcal{G}$ denote the set of all functions of the form

$$(x, y) \longrightarrow \Phi(F(x, y), G(x, y))$$

where $\Phi$ is any element of $C^\infty_{loc}(\mathbb{R}^2)$. Then $\mathcal{G}$ is a function group which has rank 2 at points $(x, y)$ where $x > 0$ and rank 1 at points $(x, y)$ where $x < 0$.

We describe the relation between function groups of rank $r$ and foliations.

**Theorem 2.2.** Let $\mathcal{G}$ be a function group of rank $r$ and let $E_p = \{X_p | 2\omega_p(X_p, \cdot) = df(\cdot) \text{ for } f \in \mathcal{G}\}$ for each $p \in M$. Then $E = \bigcup_{p \in M} E_p \subseteq TM$ is an integrable subbundle of $TM$ which contains $ker(\omega)$.

**Proof.** We show $E$ is locally trivial. Choose $p \in M$, $U$ a neighborhood of $p$, and $f_1, \ldots, f_r$ in $\mathcal{G}$ as in the definition of a generating set for $\mathcal{G}$ at $p$. Let $X_i = X_{f_i}$. If $q \in U$ and $v \in E_q$ then $v = (X_h)_q$ for some $h \in \mathcal{G}$. Since $df_{1q}, \ldots, df_{rq}$ are independent we know that there exists $F \in C^\infty_{loc}(\mathbb{R}^r)$ such that

$$h = F(f_1, \ldots, f_r)$$

on a neighborhood $V$ of $q$. One sees that

$$X_h - \sum_1^r \frac{\partial F}{\partial x_i} X_i \in \Gamma(ker(\omega|V))$$

and thus $v = (X_h)_q \in \langle X_{1q}, \ldots, X_{rq} \rangle + ker(\omega_q)$. Therefore $E$ is a subbundle of $TM$.

We show $E$ is integrable. Let $X, Y$ belong to $\Gamma(E)$ and let $p \in M$. On a neighborhood $U$ of $p$ both $X$ and $Y$ are of the form

$$\Sigma \lambda_iX_i + Z$$
for $\lambda_i \in \mathcal{C}^\infty(U)$, $Z \in \Gamma(\omega \mid \sigma)$, and $X_i = X_{f_i}$. Then $[X, Y]$ will be in $\Gamma(E)$ provided that for $1 \leq i, j \leq r$, $[X_i, X_j] \in \Gamma(E)$ and for $Z \in \Gamma(\omega \mid U)$, $[X_i, Z] \in \Gamma(E)$. Since $\mathcal{S}$ is a function group, $(f_i, f_j) \in \mathcal{S}$ and $X_{(f_i, f_j)} \in \Gamma(E \mid U)$. By (3) of Proposition 1.5 it follows that $[X_i, X_j] \in \Gamma(E \mid U)$. Moreover, $2\omega([Z, X_i], Y) = (i_{[Z, X_i]}\omega)(Y) = L_Z(i_{X_i}\omega)(Y) = L_Z(df_j)(Y) = d(i_Z df_j)(Y) = 0$ for all $Y \in \Gamma$. Thus $[Z, X_i] \in \Gamma \omega$ for each $Z \in \Gamma \omega$ and consequently $E$ is integrable.

Hereafter the foliation $E$ described above will be called the foliation determined by $\mathcal{S}$. If $\mathcal{S}$ is a function group then the reciprocal of $\mathcal{S}$ is defined to be the set of all $g \in \mathcal{C}^\infty(\omega)$ such that $(f, g) = 0$ for all $f \in \mathcal{S}$ such that $\text{dom } f \cap \text{dom } g \neq \emptyset$. We denote the reciprocal of $\mathcal{S}$ by $\mathcal{S}''$. The fact that $\mathcal{S}''$ is a function group is somewhat trivial. To see that $\mathcal{S}''$ is closed under $\{,\}$ one uses the Jacobi identity. To see that (4) of Definition 2.1 holds we need an identity which is useful in subsequent sections of our paper: for arbitrary $h_1, h_2, \ldots, h_n \in \mathcal{C}^\infty(\omega)$ and $F \in \mathcal{C}^\infty(\mathbb{R}^n)$, then

$$(2.4) \quad (f, F(h_1, h_2, \ldots, h_n)) = \sum_i \frac{\partial F}{\partial h_i}(h_1, h_2, \ldots, h_n)(f, h_i).$$

Part (4) follows immediately from this identity. To prove 2.4 observe that

$$(f, F(h_1, h_2, \ldots, h_n)) = -2\omega(X_F, X_f) = -dF(X_f)$$

$$= -\sum_i \frac{\partial F}{\partial h_i}(h_1, f) = \sum_i \frac{\partial F}{\partial h_i}(f, h_i).$$

**Remark.** It is obvious that $\mathcal{S} \subseteq \mathcal{S}''$ for any function group $\mathcal{S}$. Observe that if $\mathcal{S}$ has rank $r$, then $\mathcal{S} = \mathcal{S}''$.

If $\mathcal{S}$ is a function group then $\mathcal{F}$ is a subgroup of $\mathcal{S}$ iff $\mathcal{F}$ is a function group such that $\mathcal{F} \subseteq \mathcal{S}$.

Observe that every function group is a subgroup of the function group $\mathcal{C}^\infty(\omega)$. Also the intersection of two subgroups is a subgroup. In particular $\mathcal{S} \cap \mathcal{S}'$ is a subgroup of both $\mathcal{S}$ and $\mathcal{S}'$.

**Proposition 2.6.** Let $\mathcal{S}$ be a function group of rank $r$ at $p$. Then its reciprocal has rank $\rho - r$ at $p$.

**Proof.** Let $p \in M$ and let $f_1, \ldots, f_r$ be generators of $\mathcal{S}$ at $p$. Choose coordinates $x_1, \ldots, x_m$ at $p$ such that $X_i = X_{f_i} = \partial / \partial x_i$ for $1 \leq i \leq r$ and such that $\{\partial / \partial x_{r+j}\} 1 \leq j \leq m - p$ generate $\Gamma \omega$ near $p$. Then any integral of the integrable system $X_1, \ldots, X_r, \partial / \partial x_{r+1}, \ldots, \partial / \partial x_{r+m-p}$ depends only on the last coordinates. Since each
\( f \in \mathcal{S}' \) is an integral of this system it follows that \( x_{m+r-p+1}, \ldots, x_m \) generates \( \mathcal{S}' \) at \( p \).

Using arguments similar to those above we obtain the following corollary.

**COROLLARY 2.7.** Let \( \mathcal{S} \) be a function group of rank \( r \), \( \mathcal{S}' \) the reciprocal of \( \mathcal{S} \), and \( E \) the foliation determined by \( \mathcal{S} \). Then

1. \( E_p = \bigcap \{ \ker dg \mid g \in \mathcal{S}' \} \), for each \( p \in M \),
2. if \( g_1, g_2, \ldots, g_{r-p} \) generate \( \mathcal{S}' \) at \( p \in M \), then there is a neighborhood \( U \) of \( p \) such that the map \( x \mapsto (g_1(x), g_2(x), \ldots, g_{r-p}(x)) \) is constant on each leaf of the foliation \( E \mid U \) of \( U \).

We say that a subbundle \( E \) of \( TM \) is locally Hamiltonian iff \( \ker (\omega) \subseteq E \) and for each \( p \in M \) there is a neighborhood \( U \) of \( p \) such that \( \Gamma(E \mid U) \) is spanned by vector fields \( X \) which satisfy \( df = i_X \omega \) for some \( f \in C^\infty_\text{loc}(\omega) \).

**PROPOSITION 2.8.** An integrable subbundle \( E \) is the foliation determined by some function group iff \( E \) is locally Hamiltonian. Moreover, the function group which determines such an \( E \) is unique.

**Proof.** Clearly if \( E \) is determined by some function group, then \( E \) is locally Hamiltonian.

Conversely, suppose that \( E \) is locally Hamiltonian and consider the set \( \mathcal{I} \) of all local integrals of \( E \). We now show that \( \mathcal{I} \) is a function group and that \( E \) is determined by the reciprocal, \( \mathcal{S}' \), of \( \mathcal{I} \). Let \( f, g \in \mathcal{I}, p \in M \), and \( X \in \Gamma(E) \). There is no loss of generality in assuming that there is an \( H \in C^\infty_\text{loc}(\omega) \) such that \( 2\omega(X, \cdot) = dH(\cdot) \) in a neighborhood of \( p \). It follows that

\[
\begin{align*}
\mathcal{d}(f, g)(X) &= L_{xH}([f, g]) = \{f, \{g, H\}\} + \{g, \{H, f\}\} \\
&= \{L_xg, f\} + \{L_xf, g\} = 0
\end{align*}

\]

by Proposition 1.5, the Jacobi identity, and the fact that \( X \in \Gamma(E) \). Thus \( \{f, g\} \in \mathcal{I} \) and it follows that \( \mathcal{I} \) is a function group with constant rank. Since \( \mathcal{I} = \mathcal{I}'' \) it follows from Corollary 2.7 that \( E = \bigcap \{ \ker df \mid f \in \mathcal{I}'' = \mathcal{I} \} \).

**Remark.** If \( \mathcal{I} \) is any function group then \( \mathcal{I} \) determines a unique integrable locally Hamiltonian subbundle \( E \) of \( TM \) and conversely. If \( E \) is determined by \( \mathcal{I} \) then the reciprocal of \( \mathcal{I} \) is precisely the set of all local integrals of \( E \). If \( E \) is an integrable locally Hamiltonian subbundle of \( TM \) then the set of all local inte-
Integrals of foliations on manifolds

Let \( S \) be a function group of rank \( r \). We say that a set \( S \subseteq C^\infty(M) \) globally generates \( S \) provided that for each \( p \in M \) there exist functions \( f_1, f_2, \ldots, f_r \in S \) and a neighborhood \( U \) of \( p \) such that \( \{ f_i|U, f_2|U, \ldots, f_r|U \} \) generates \( S \) at \( p \). We say that a set \( T \subseteq \Gamma^* \) of closed 1-forms globally generates \( S \) provided that for each \( p \in M \) there exist forms \( \beta_1, \ldots, \beta_r \in T \), a neighborhood \( U \) of \( p \), functions \( f_1, f_2, \ldots, f_r \) satisfying \( df_i = \beta_i \) on \( U \) for \( i = 1, 2, \ldots, r \) such that \( \{ f_1|U, f_2|U, \ldots, f_r|U \} \) generates \( S \) at \( p \).

**Proposition 2.9.** Suppose that there exist closed 1-forms \( \beta_1, \beta_2, \ldots, \beta_n \) in \( \Gamma^*_\) and \( r > 0 \) such that

(i) \( \beta_i(p), \beta_2(p), \ldots, \beta_n(p) \) span an \( r \)-dimensional subspace of \( M^*_p \) for each \( p \in M \),

(ii) there exist functions \( a_{ik} \in C^\infty(M) \) such that

\[
\{ \beta_i, \beta_j \} = \sum_{k=1}^{n} a_{ik} \beta_k .
\]

Then there exists a unique function group \( S \) of rank \( r \) which is globally generated by \( \{ \beta_1, \beta_2, \ldots, \beta_n \} \). Conversely, if \( S \) is a function group of rank \( r \) which is globally generated by \( \beta_1, \beta_2, \ldots, \beta_n \) then conditions (i) and (ii) are satisfied.

**Proof.** The details of this proof are much like those of Theorem 2.2 and are left to the reader.

Recall that \( \text{inv} (\Gamma^*_\) is a Lie algebra under \( \{ , \} \). Observe that if \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are elements of \( \text{inv} (\Gamma^*_\) they span a finite dimensional subalgebra of \( \text{inv} (\Gamma^*_\) iff

\[
\{ \alpha_i, \alpha_j \} = \sum_{k=1}^{n} c_{ijk} \alpha_k
\]

for constants \( c_{ijk} \in R \).

We now give an application of function groups which is a slight generalization of certain well-known theorems.

**Theorems 2.10.** Let \( M \) be a symplectic manifold \( (\rho = m = 2N) \) and \( S \) a function group of rank \( r \) on \( M \). Suppose that the closed 1-forms \( \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \) globally generate \( S \) and that they span an \( n \)-dimensional subalgebra \( \mathcal{L} \) of \( \text{inv} (\Gamma^*_\) = \( \Gamma(T^*M) \). If the vector field \( X_{\alpha_i} \) is complete for each \( i = 1, 2, \ldots, n \), then each leaf of the foliation determined by \( S \) is diffeomorphic to a homogeneous space.
G/H where G is the unique simply connected Lie group with Lie algebra $\mathfrak{g}$ and H is a closed subalgebra of G.

**Proof.** This is a consequence of a well-known theorem due to Palais [11] (see also Loos [10]). The details of the proof of Theorem 2.10 are similar to those of Theorem 1 of [2].

**REMARK.** Note that if we take $r = 2N - 1$ we obtain a part of Theorem 1 of Andrie and Simms [2]. Note that if we take $r = N$ and assume that $\mathcal{L}$ is commutative we obtain a part of a theorem of Arnold [1] in which the leaves of the foliation turn out to be cylinders or tori (see, for example, Abraham [1, page 113]).

3. Invariant metrics and transverse structures. Let $M$ be a connected $C^\infty$-manifold of dimension $m$ and let $E$ be an integrable subbundle of $TM$ of dimension $r$. The normal bundle $TM/E$ of $E$ will be denoted by $Q$ and its dual $Q^*$ will be identified with the bundle $E^0$ where, for each $x \in M$, $E^0_x$ is the annihilator of $E_x$ in $T^*_x M$, i.e.,

$$E^0_x = \{ \beta \in T^*_x M \mid \beta(E_x) = 0 \}.$$  

Define a connection $\nabla^*$ on $\Gamma(E^0)$ along the leaves of $E$ by $\nabla^*_x \beta = L_x \beta$ for $\beta \in \Gamma(E^0)$ and $X \in \Gamma(E)$. Observe that if $f$ is any local integral of $E$ then $\nabla^*_x (df) = L_x(df) = df(X) = 0$ and thus $df$ is covariant constant along leaves of $E$. Also, if $f_1, f_2, \ldots, f_{m-r}$ are independent local integrals of $E$ defined on an open set $U \subseteq M$, then $df_1, df_2, \ldots, df_{m-r}$ span $E^0$ on $U$.

**Lemma 3.1.** If $\beta \in \Gamma(E^0)$ is closed, then $\beta$ is parallel along the leaves of $E$, i.e., $\nabla^*_x \beta = 0$ for all $X \in \Gamma(E)$.

**Proof.** $\nabla^*_x \beta = L_x \beta = (i_x d) \beta + (di_x) \beta = 0$ for all $X \in \Gamma(E)$ and $\beta \in \Gamma(E^0)$.

**Corollary 3.2.** If $\beta_1, \beta_2, \ldots, \beta_{m-r}$ are global, independent, closed elements of $\Gamma(E^0)$, then $E^0$ is parallelizable, i.e., it has $m-r$ global, independent, parallel sections.

If $\sigma$ is a Riemannian metric on $M$, then $Q$ may be identified with the orthogonal complement of $E$ in $TM$. Let $\sigma_Q = \sigma \mid (Q \times Q)$ be the induced metric on $Q$. If $\beta \in \Gamma(E^0)$, then $\text{grad} \ \beta$ is that unique vector field in $\Gamma(Q)$ such that

$$\sigma(\text{grad} \ \beta, \cdot) = \beta$$
and, for $\xi \in \Gamma(Q)$, $\beta_\xi$ is that element of $\Gamma(E^\omega)$ defined by

$$\beta_\xi = \sigma(\xi, \cdot)$$

We define the dual connection $\nabla$ of $\nabla^*$ to be that connection on $\Gamma(Q)$ along leaves of $E$ such that

$$\nabla_X(\xi) = \text{grad} \left( \nabla^*_X \beta_\xi \right)$$

for $X \in \Gamma(E)$ and $\xi \in \Gamma(Q)$. Another connection $\tilde{\nabla}$ for $\Gamma(Q)$ along the leaves of $E$ is defined by

$$\tilde{\nabla}_X(\xi) = [L_X \xi]_q$$

where $X \in \Gamma(E)$, $\xi \in \Gamma(\xi)$ and where $[Y]_q$ denotes the component of $Y$ in $Q$.

**Lemma 3.3.** If $\sigma_q$ is invariant with respect to $\tilde{\nabla}$ then $\tilde{\nabla} = \nabla$.

**Proof.** For $\xi, \eta \in \Gamma(Q)$ we have: $(\nabla^*_X \beta_\xi)(\eta) = (L_X \beta_\xi)(\eta) = i_\xi (L_X \beta_\xi) = L_X (i_\xi \beta_\xi) - i_{[X, \eta]} (\beta_\xi) = L_X (\sigma_d(\xi, \eta)) - \sigma(\xi, [X, \eta]) = [\sigma_d(\tilde{\nabla}_X \xi, \eta) + \sigma_d(\xi, \tilde{\nabla}_X \eta)] - \sigma(\xi, [X, \eta]) = \sigma_d(\tilde{\nabla}_X \xi, \eta)$. Thus $\tilde{\nabla}_X \xi = \text{grad} \left( \nabla^*_X \beta_\xi \right) = \nabla_X \xi$.

We say that $\sigma$ is invariant when $\sigma_q$ is invariant with respect to the connection $\tilde{\nabla}$ in which case $\nabla = \tilde{\nabla}$. Observe that a metric $\sigma$ satisfies this property iff it is "bundle-like" in the sense of Reinhart [12]. Also the connection $\tilde{\nabla}_X$ can be defined for all $X \in \Gamma(TM)$ in such a way that $\tilde{\nabla}$ is a "basic connection" (see Conlon [5]). Moreover the last result is a reflection of the fact that restrictions of basic connections to $\Gamma(E)$ are unique.

**Lemma 3.4.** If $\sigma$ is an invariant metric, then $\beta$ is parallel with respect to $\nabla^*$ iff $\text{grad} \beta$ is parallel with respect to $\tilde{\nabla}$.

**Proof.** It is a standard result that $\beta$ is $\nabla^*$-parallel iff $\text{grad} \beta$ is parallel relative to the dual connection $\nabla$ (see [7], Vol. II, page 342). Since $\nabla = \tilde{\nabla}$ the result follows.

**Remark.** If $\sigma$ is an invariant metric the usual one-to-one correspondence between $\Gamma(Q)$ and $\Gamma(E^\omega)$ induces a one-to-one correspondence between $\tilde{\nabla}$-parallel sections of $Q$ and $\nabla^*$-parallel sections to $E^\omega$.

**Remark.** If $\xi$ and $\eta$ are $\tilde{\nabla}$-parallel along leaves of $E$ then the invariance of $\sigma$ implies that $\sigma(\xi, \eta)$ is an integral of $E$. Thus if $\beta$ is a closed element of $\Gamma^*(E)$ we conclude that $\sigma(\text{grad} \beta, \text{grad} \beta)$ is constant on leaves of $E$. If $\sigma$ is complete as well as invariant then the vector field
\[
\frac{1}{\sigma(\text{grad } \beta, \text{grad } \beta)} \cdot \text{grad } \beta
\]

is a complete vector field for nonvanishing closed \( \beta \) in \( \Gamma(E^\circ) \).

The foliation \( E \) is \textit{transversally} parallelizable iff there exist \( m-r \) independent elements of \( \Gamma Q \) each of which is \( \bar{\nabla} \)-parallel along the leaves of \( E \).

**Theorem 3.5.** Suppose there exist \( m-r \) everywhere independent closed 1-forms \( \beta_1, \beta_2, \ldots, \beta_{m-r} \) such that

\[
\beta_i(\Gamma(E)) = 0 \quad \text{for} \quad i = 1, 2, \ldots, m - r.
\]

Then \( E \) is transversally parallelizable.

**Proof.** If we show that there exists an invariant metric on \( E \), then the theorem will be a consequence of Lemmas 3.1 and 3.4. Let \( Q \) be the orthogonal complement of \( E \) in \( TM \) relative to an arbitrary Riemannian \( \tau \) on \( TM \). Define \( \sigma \) on \( TM \) by

\[
\sigma = \tau | (E \times E) \oplus \sum_{i=1}^{m-r} (\beta_i \otimes \beta_i).
\]

Clearly \( \sigma \) is a Riemannian on \( TM \). We show that \( \sigma \) is invariant. First observe that for \( \xi, \eta \in \Gamma(Q) \) and \( X \in \Gamma(E) \),

\[
L_X(\sigma_0(\xi, \eta)) = \sum_{i=1}^{m-r} L_X(\beta_i(\xi)\beta_i(\eta)) = \sum_{i=1}^{m-r} [\beta_i(\xi)L_X(\beta_i(\eta)) + \beta_i(\eta)L_X(\beta_i(\xi))].
\]

But

\[
L_X(\beta_i(\eta)) = L_X(i_X \beta_i) = i_{[X, \eta]} \beta_i + i_\eta(L_X \beta_i) = \beta_i([X, \eta]) + i_\eta([i_X d + d i_X](\beta_i)) = \beta_i([X, \eta]) = \beta_i(\bar{\nabla}_X(\eta)).
\]

Thus

\[
L_X(\sigma_0(\xi, \eta)) = \sum [\beta_i(\xi)\beta_i(\bar{\nabla}_X(\eta)) + \beta_i(\eta)\beta_i(\bar{\nabla}_X(\xi))]
\]

\[
= \sigma_0(\xi, \bar{\nabla}_X(\eta)) + \sigma_0(\bar{\nabla}_X(\xi), \eta)
\]

as required. The theorem follows.

**Remark.** In the proof of the preceding theorem we have introduced a new metric \( \sigma = \tau |_E \oplus \sum_{i=1}^{m-r} (\beta_i \otimes \beta_i) \). Observe that the orthogonal complement of \( E \) relative to \( \sigma \) is the same as for \( \tau \), namely \( Q \). The gradient vector fields of the 1-forms \( \beta_1, \beta_2, \ldots, \beta_{m-r} \) with respect to this metric are parallel along the leaves of \( E \). In the following we will use these vector fields without specific refe-
rences to the metric $\sigma$. Thus $\text{grad} \beta_i$ is the unique section of $Q$ satisfying

$$
\sum_{j=1}^{m-r} \beta_j(\text{grad} \beta_i)\beta_j(Y) = \beta_i(Y)
$$

for all $Y \in \Gamma(Q)$.

We make a few remarks regarding completeness. First note that if the metric $\tau$ is complete then the metric $\sigma$ will also be complete if there exist numbers $l$ and $L$ such that

$$
l\tau_p(X_p, X_p) \leq \sum_{i=1}^{m-r} \beta_i(X_p)^2 \leq L\tau_p(X_p, X_p)
$$

for all $p \in M$ and $X \in \Gamma(Q)$. If this is the case then the vector fields $[1/\beta_i(\text{grad} \beta_i)] \text{grad} \beta_i$ are complete vector fields. In any case (assuming $\tau$ is complete) the vector fields $\text{grad} \beta_i$ will be complete if they are bounded in the metric $\tau$. Moreover, in this case, every linear combination in the $\text{grad} \beta_i$ is complete.

**Corollary 3.7.** If in addition to the hypothesis of Theorem 3.5 we require that every linear combination of the vector fields $\text{grad} \beta_i$ (see 3.6) be complete, then

1. any two leaves of $E$ are diffeomorphic and if any leaf of $E$ is closed in $M$ they all are,

2. if $E$ admits a closed leaf then there is a fibre bundle $p: M \to N$ where $N$ is parallelizable and $E$ is the foliation of $M$ whose leaves are the fibres of $p$.

**Proof.** The corollary follows immediately from Theorem 3.5 above and Propositions 4.3 and 4.4 of Conlon [5].

We now apply the results of this section to function groups. As an example consider the case where $M$ is symplectic and suppose there is a Hamiltonian function $H \in C^\infty(M)$ such that $dH(p) \neq 0$ for each $p \in M$. Clearly $\{H\}$ globally generates a function group $\mathcal{H}$ of rank 1. This leads to a foliation $E$ which is generated by the unique Hamiltonian vector field $X_H = X_{dH}$. The reciprocal function group $\mathcal{H}'$, which consists of all local integrals of $E$, also determines a foliation $E'$. Thus by Theorem 3.5, $E'$ is transversally parallelizable. Indeed, if the vector field $\text{grad}(dH)$ is complete then each two leaves of $E'$ are diffeomorphic. Also since the leaves are the components of the level surfaces of $H$, they are closed and hence, by Corollary 3.7, they fibre $M$ over a parallelizable manifold. The following theorem generalizes this example where
Theorem 3.8. Let $\omega$ be a closed 2-form of constant rank $p$ on $M$. Let $\beta_1, \beta_2, \cdots, \beta_r$ be closed 1-forms which globally generate a function group $\mathcal{S}$ of rank $r$. Then the foliation $E'$ determined by the reciprocal function group of $\mathcal{S}$ is transversally parallelizable. Moreover, if every linear combination of the vector fields $\nabla \beta_i$, $i = 1, 2, \cdots, r$ is complete then $E'$ is a complete transversally parallelizable foliation and each two leaves of $E'$ are diffeomorphic. Furthermore, if one of the leaves of $E'$ is closed then they all are and $M$ is a fibre bundle over a parallelizable manifold in which the fibres are the leaves of $E'$.

Proof. The theorem is an immediate consequence of what it means for $\{\beta_1, \beta_2, \cdots, \beta_r\}$ to globally generate $\mathcal{S}$, Theorem 3.5 and Corollary 3.7.

Remark. Suppose that in the above theorem we have $r = p$. In this case $E' = \ker \omega$. Moreover, if some leaf $L$ of the foliation $E'$ is closed then the manifold $M$ is fibered by $\pi: M \to N$ where $\pi^{-1}(x) \equiv L$, for each $x$, and $N$, the manifold of leaves of $\ker \omega$, is a symplectic manifold. This is true since $N_p \cong Q_p$ and $\omega | (Q \times Q)$ is nondegenerate.

Remark. If in the above theorem $r = 1$, then $E'$ is a foliation of codimension 1 and thus by [5, Proposition 5.1] we conclude that either every leaf of $E'$ is closed or else every leaf of $E'$ is dense in $M$.

Remark. If in addition to the hypothesis of the above theorem we assume that the 1-forms $\beta_1, \beta_2, \cdots, \beta_r$ are exact, then there exist functions $H_1, H_2, \cdots, H_r$ such that $dH_i = \beta_i$ and the leaves of $E'$, being components of level surfaces of $H_i = h_i$, are necessarily closed. Thus we see that if the functions $\{H_1, H_2, \cdots, H_r\}$ globally generate a function group of rank $r$ and every linear combination of the grad $(H_i)$ is complete then each two components of the level surfaces $H_i = h_i$ are diffeomorphic.

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References


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<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>Patricia Andresen and Marvin David Marcus</td>
<td>Weyl's inequality and quadratic forms on the Grassmannian</td>
</tr>
<tr>
<td>George Bachman and Alan Sultan</td>
<td>Regular lattice measures: mappings and spaces</td>
</tr>
<tr>
<td>David Geoffrey Cantor</td>
<td>On certain algebraic integers and approximation by rational functions with integral coefficients</td>
</tr>
<tr>
<td>James Richard Choike</td>
<td>On the value distribution of functions meromorphic in the unit disk with a spiral asymptotic value</td>
</tr>
<tr>
<td>David Earl Dobbs</td>
<td>Divided rings and going-down</td>
</tr>
<tr>
<td>Mark Finkelstein and Robert James Whitley</td>
<td>Integrals of continuous functions</td>
</tr>
<tr>
<td>Ronald Owen Fulp and Joe Alton Marlin</td>
<td>Integrals of foliations on manifolds with a generalized symplectic structure</td>
</tr>
<tr>
<td>Cheong Seng Hoo</td>
<td>Principal and induced fibrations</td>
</tr>
<tr>
<td>Wu-Chung Hsiang and Richard W. Sharpe</td>
<td>Parametrized surgery and isotopy</td>
</tr>
<tr>
<td>Surender Kumar Jain, Surjeet Singh and Robin Gregory Symonds</td>
<td>Rings whose proper cyclic modules are quasi-injective</td>
</tr>
<tr>
<td>Pushpa Juneja</td>
<td>On extreme points of the joint numerical range of commuting normal operators</td>
</tr>
<tr>
<td>Athanassios G. Kartsatos</td>
<td>Nth order oscillations with middle terms of order N − 2</td>
</tr>
<tr>
<td>John Keith Luedeman</td>
<td>The generalized translational hull of a semigroup</td>
</tr>
<tr>
<td>Louis Jackson Ratliff, Jr.</td>
<td>The altitude formula and DVR's</td>
</tr>
<tr>
<td>Ralph Gordon Stanton, C. Sudler and Hugh C. Williams</td>
<td>An upper bound for the period of the simple continued fraction for √D</td>
</tr>
<tr>
<td>David Westreich</td>
<td>Global analysis and periodic solutions of second order systems of nonlinear differential equations</td>
</tr>
<tr>
<td>David Lee Armacost</td>
<td>Correction to: “Compactly cogenerated LCA groups”</td>
</tr>
<tr>
<td>Jerry Malzan</td>
<td>Corrections to: “On groups with a single involution”</td>
</tr>
<tr>
<td>David Westreich</td>
<td>Correction to: “Bifurcation of operator equations with unbounded linearized part”</td>
</tr>
</tbody>
</table>