RINGS WHOSE PROPER CYCLIC MODULES ARE QUASI-INJECTIVE

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A ring \( R \) with identity is a right \( \text{PCQI} \)-ring (\( \text{PCI} \)-ring) if every cyclic right \( R \)-module \( C \neq R \) is quasi-injective (injective). Left \( \text{PCQI} \)-rings (\( \text{PCI} \)-rings) are similarly defined. Among others the following results are proved: (1) A right \( \text{PCQI} \)-ring is either prime or semi-perfect. (2) A nonprime nonlocal ring is a right \( \text{PCQI} \)-ring if every cyclic right \( R \)-module is quasi-injective or \( R \cong \begin{pmatrix} D & D \\ 0 & D \end{pmatrix} \), where \( D \) is a division ring. In particular, a nonprime nonlocal right \( \text{PCQI} \)-ring is also a left \( \text{PCQI} \)-ring. (3) A local right \( \text{PCQI} \)-ring with maximal ideal \( M \) is a right valuation ring or \( M^2 = (0) \). (4) A prime local right \( \text{PCQI} \)-ring is a right valuation domain. (5) A right \( \text{PCQI} \)-domain is a right \( \text{Ore} \)-domain. Faith proved (5) for right \( \text{PCI} \)-domains. If \( R \) is commutative then some of the main results of Klatt and Levy on pre-self-injective rings follow as a special case of these results.

Since, in a commutative Dedekind domain \( D \), for each nonzero ideal \( A \), \( D/A \) is a self-injective ring, or equivalently \( D/A \) is a quasi-injective \( D \)-module, every commutative Dedekind domain is a \( \text{PCQI} \)-ring. An example of a \( \text{PCQI} \)-ring which is not a Dedekind domain is given in Levy [14]. Commutative \( \text{PCQI} \)-rings are precisely the pre-self-injective rings characterized by Klatt and Levy [11]. \( \text{PCI} \)-rings have recently been investigated by Faith [4]. Right self-injective right \( \text{PCQI} \)-rings are \( \text{qc} \)-rings which have been studied by Ahsan [1] and Koehler [13].

1. Definitions and preliminaries. Throughout all modules are unitary and right unless specified. An \( R \)-module \( X \) is called injective relative to an \( R \)-module \( M \) if for each short exact sequence \( 0 \to N \to M \to M/N \to 0 \) the sequence \( 0 \to \text{Hom}_R (M/N, X) \to \text{Hom}_R (M, X) \to \text{Hom}_R (N, X) \to 0 \) is exact. \( X \) is called quasi-injective if \( X \) is injective relative to itself. Any \( R \)-module injective relative to all \( R \)-modules is called injective. Relative projectivity is defined dually.

A ring \( R \) is called a right \( q \)-ring if each of its right ideals is quasi-injective (see Jain, Mohamed, and Singh [9]). For more results, see [7], [8], [13], [15]. Dually, a ring \( R \) is called a right \( q^* \)-ring if each cyclic right \( R \)-module is quasi-projective (see Koehler [12]).

A ring \( R \) is right \( \text{qc} \)-ring if each cyclic right \( R \)-module is quasi-injective (see Ahsan [1]). A well-known result of Osofsky [16] states...
that \( R \) is semisimple artinian iff each cyclic \( R \)-module is injective.
Koehler [13] showed that \( R \) is a right \( q\text{-}c \)-ring iff \( R \) is a finite direct sum of rings each of which is semisimple artinian or a \( r \)-\( o \) duo maximal valuation ring. As a consequence, every \( q\text{-}c \)-ring is both a \( q \)-ring and \( q^* \)-ring.

In this paper the classes of rings initially called \( q \)-rings, \( q^* \)-rings, and \( q\text{-}c \)-rings have been called \( Q \)-rings, \( Q^* \)-rings, and \( QC \)-rings respectively.

Let \( J(R) \) denote the radical of a ring \( R \). \( R \) is called semiperfect if \( R/J(R) \) is semisimple artinian and idempotents modulo \( J(R) \) can be lifted to \( R \). If \( R \) is semiperfect, then there exists a finite maximal family of primitive orthogonal idempotents \( \{e_i\}_{i=1}^{n} \) such that \( R = \bigoplus \sum_{i=1}^{n} e_i R \).

\( R \) is called a local ring if it has a unique maximal right ideal which must be the radical \( J(R) \).

\( R \) is a right valuation ring if the set of all right ideals is linearly ordered. \( R \) is a maximal valuation ring if every family of pairwise solvable congruences of the form \( x \equiv x_a (\text{mod } A_a) \) has a simultaneous solution where \( x_a \in R \) and each \( A_a \) is an ideal in \( R \). \( R \) is called an almost maximal valuation ring if each of its proper homomorphic images is a maximal valuation ring.

A ring is right duo if every right ideal is two-sided. A ring \( R \) has rank 0 if every prime ideal is a maximal ideal. By duo rings or valuation rings, we shall mean both right and left.

3. General results.

**Sublemma 1.** Let \( I \) be a right ideal in a ring \( R \) such that \( R/I \cong R \). Then \( R = I \oplus J \), where \( J \) is a right ideal, and thus \( I = eR \), \( e = e^2 \in R \).

**Proof.** \( R/I \cong R \) implies \( R/I \) is projective, and hence \( I \) is a direct summand of \( R \).

**Proposition 2.** Let \( R \) be a right \( PCQI \)-ring. If \( I \) is a right ideal of \( R \) such that \( R/I \cong R \), then \( I \) is contained in every nonzero two-sided ideal of \( R \).

**Proof.** Let \( S \) be a nonzero two-sided ideal of \( R \). Then \( R/S \) is a \( q\text{-}c \)-ring, hence is semiperfect. Let \( f: R/I \rightarrow R \) be an isomorphism. Since \( 1 + I \) generates \( R/I \), \( R = xR \), where \( x = f(1 + I) \). Then \( I = \text{ann } x = \{ r \in R | xr = 0 \} \). So there exists \( y \in R \) such that \( xy = 1 \). Since \( R/S \) is semiperfect, \( (x + S)(y + S) = 1 + S = (y + S)(x + S) \). Then \( 1 - yx \in S \). Let \( a \in I \), i.e., \( xa = 0 \). Then \( (1 - yx)a = a - yxa = a \), hence \( a \in S \). So \( I \subseteq S \).
PROPOSITION 3. Let $R$ be a right PCQI-ring. Then either $R$ is a prime ring or $R$ is semiperfect with nil radical.

Proof. Suppose $R$ is not prime, and $P \neq 0$ is a prime ideal. Then $R/P$ is a qc-ring, and hence a q-ring. So $R/P$ is simple artinian [9]. Thus $P$ is maximal, hence primitive. So the Jacobson radical is nil.

Since $R$ is not prime, there exist nonzero ideals $A, B$ such that $AB = 0$. Since $R$ is a right PCQI-ring, $R/A$ and $R/B$ are semiperfect, hence each of them has finitely many prime ideals. Since every prime ideal of $R$ contains $A$ or $B$, it follows that $R$ has finitely many prime ideals as well. Thus $R/J(R)$ is semisimple artinian, and since $J(R)$ is nil, $R$ is semiperfect.

4. Nonlocal semiperfect PCQI-rings. By Proposition 3, all nonprime right PCQI-rings are semiperfect, so the results of this section hold for the class of nonprime nonlocal right PCQI-rings. The case of local right PCQI-rings is discussed in the next section.

LEMMA 4. Let $R$ be a semiperfect ring. Then $R/A$ is a proper cyclic right $R$-module, for all nonzero right ideals $A$.

Proof. There exists a positive integer $n$ such that $R$ is a direct sum of $n$ indecomposable right $R$-modules, and $R$ cannot be expressed as a direct sum of more than $n$ right $R$-modules. Now, if $R/A \cong R$, then, by Lemma 1, $R = A \oplus B$ and $B \cong R$. So $A = (0)$, proving the lemma.

Let $R$ be a nonlocal semiperfect ring, and let $\{e_1, \ldots, e_n\}$ be a maximal set of primitive orthogonal idempotents in $R$. Then $R = \bigoplus_{i=1}^{n} e_iR$ and $n \geq 2$. Throughout this section, $e_i$'s will denote primitive idempotents. We shall often use a well-known fact that if $A \oplus B$ is a quasi-injective module then any monomorphism $A \rightarrow B$ splits.

LEMMA 5. Let $R$ be a semiperfect nonlocal right PCQI-ring. If $\sigma \in \Hom_R(e_iR, e_jR)$ such that $\sigma \neq 0$, where $i \neq j$, then $\ker \sigma = (0)$.

Proof. Suppose $\ker \sigma \neq (0)$, where $0 \neq \sigma \in \Hom_R(e_iR, e_jR)$, $i \neq j$. Then $R/\ker \sigma \cong \bigoplus_{i=1}^{n} e_iR \times \Im \sigma$, and $R/\ker \sigma$ is quasi-injective. Since $\Im \sigma \subseteq e_jR$, the inclusion map $i: \Im \sigma \rightarrow \bigoplus_{k=1}^{n} e_kR$ is a monomorphism. Since $R/\ker \sigma$ is quasi-injective, the inclusion map splits. So $\Im \sigma$ is a direct summand of $e_jR$, hence $\Im \sigma = e_jR$. Since $e_jR$ is projective, $\sigma: e_iR \rightarrow e_jR$ splits. Thus $\ker \sigma = (0)$.
LEMMA 6. Let $R$ be a semiperfect nonlocal right PCQI-ring with decomposition $\bigoplus_{i=1}^n e_i R$, where $n > 2$. Then $\text{Hom}_R(e_i R, e_j R) \neq 0$ iff $e_i R \cong e_j R$, i.e., $e_i Re_i \neq 0$ iff $e_j R \cong e_j R$.

Proof. Let $\sigma \in \text{Hom}_R(e_i R, e_j R)$ such that $\sigma \neq 0$. By Lemma 5, $\ker \sigma = 0$. Since $n > 2$, $e_i R \oplus e_j R \cong R/\bigoplus_{k \neq i, j} e_k R$ is quasi-injective. Then $\sigma$ splits, and $0 \neq \text{Im} \sigma$ is a direct summand of $e_j R$. So $\text{Im} \sigma = e_j R$, and $\sigma$ is an isomorphism. The converse is trivial.

PROPOSITION 7. Let $R$ be a semiperfect nonlocal right PCQI-ring with decomposition $R = \bigoplus_{i=1}^n e_i R$, where $n > 2$. Then $R$ is a qc-ring.

Proof. For each $i$, $e_i R \cong R/\bigoplus_{k \neq i} e_k R$. So $e_i R$ is quasi-injective, for each $i$. Let $A_i$ be the sum of all those $e_i R$ which are isomorphic to each other. Then $R = \bigoplus A_i$. We claim that $A_i$ is a two-sided ideal of $R$, for each $i$. Clearly $A_i$ is a right ideal. Consider $e_i R$ such that $e_i R \not\subset A_i$. Define $f: e_i R \to e_j R$, where $e_i R \subset A_i$, by $f(e_i r) = e_j xe_i r$, for $x \in R$. Then $f \in \text{Hom}_R(e_i R, e_j R)$. Since $e_i R$ and $e_j R$ are not isomorphic, $f = 0$ by Lemma 6. So, for $e_i R \not\subset A_i$, $e_i R A_i = 0$. So $RA_i \subset A_i$. Since $A_i$ is a finite direct sum of isomorphic quasi-injective right ideals, $A_i$ is quasi-injective, hence a qc-ring. Thus, by Koehler [13], $R$ is a qc-ring.

PROPOSITION 8. Let $R$ be a semiperfect right PCQI-ring such that $R = e_i R \oplus e_2 R$. If $e_i R \cong e_2 R$, then $R$ is a qc-ring.

Proof. Now $e_i R \cong e_2 R$ and $R/e_2 R \cong R/e_i R$, hence $e_i R$ and $e_2 R$ are quasi-injective. Since $e_i R \cong e_2 R$, $R = e_i R \oplus e_2 R$ is quasi-injective, hence right self-injective. So $R$ is a qc-ring.

PROPOSITION 9. Let $R$ be a semiperfect right PCQI-ring such that $R = e_i R \oplus e_2 R$. If $e_i Re_2 = 0$ and $e_2 Re_i = 0$, then $R$ is a qc-ring.

Proof. If $e_i Re_2 = 0$ and $e_2 Re_i = 0$, then $e_i R$ and $e_2 R$ are two-sided ideals of $R$. Thus $e_i R \cong R/e_2 R$ and $e_2 R \cong R/e_i R$ are qc-rings. Then $R = e_i R \oplus e_2 R$ is a qc-ring.

PROPOSITION 10. Let $R$ be a semiperfect right PCQI-ring such that $R = e_i R \oplus e_2 R$. If $e_i Re_2 \neq 0$ and $e_2 Re_i \neq 0$, then $R$ is a qc-ring.

Proof. $e_i Re_2 \neq 0$ and $e_2 Re_i \neq 0$ imply that there exist nonzero homomorphisms, hence monomorphisms by Lemma 5, from $e_i R$ to $e_2 R$ and from $e_2 R$ to $e_i R$. Thus, by Bumby [2], $e_i R \cong e_2 R$, and Proposition 8 yields the result.
**Proposition 11.** Let \( R = e_1 R \oplus e_2 R \) be a semiperfect right \( PCQI \)-ring where \( e_1 R \neq e_2 R \) and exactly one of \( e_1 Re_2 \) or \( e_2 Re_1 \) is zero. Then \( R \) is nonprime with nil radical.

**Proof.** It follows from that the fact that if \( e_1 Re_2 \neq 0 \), then \( e_1 Re_2 \) is a nilpotent ideal.

**Theorem 12.** Let \( R \) be a nonlocal right \( PCQI \)-ring. Then \( R \) is semiperfect iff \( R \) is nonprime or simple artinian.

**Proof.** Necessity follows by Proposition 3, and sufficiency follows from Proposition 7-11 and Koehler's characterization of \( qc \)-rings [13] (cf. definitions and preliminaries).

**Theorem 13.** Let \( R \) be a semiperfect nonlocal ring. Then \( R \) is a right \( PCQI \)-ring iff either (i) \( R = \bigoplus_{i=1}^n R_i \), where \( R_i \) is semisimple artinian or a rank 0 duo maximal valuation ring or (ii) \( R = \left( \begin{array}{cc} D & D \\ 0 & D \end{array} \right) \), where \( D \) is a division ring.

**Proof.** Let \( R \) be a right \( PCQI \)-ring. By Propositions 7-10, \( R \) is a \( qc \)-ring unless \( R = e_1 R \oplus e_2 R \), where \( e_1 R \) and \( e_2 R \) are not isomorphic and exactly one of \( e_1 Re_2 \) or \( e_2 Re_1 \) is zero, say \( e_1 Re_2 \neq 0 \) and \( e_2 Re_1 = 0 \). If \( R \) is a \( QC \)-ring, we get (i) by Koehler [13]. Otherwise, we have \( R \cong \left( \begin{array}{cc} e_1 Re_1 & e_2 Re_2 \\ 0 & e_1 Re_2 \end{array} \right) \). We claim that \( e_1 Re_2 \) and \( e_2 Re_1 \) are isomorphic division rings and \( M = e_1 Re_2 \) is a \( (D, D) \)-bimodule such that \( \dim M = 1 = \dim M_D \), where \( D \cong e_1 Re_1 \cong e_2 Re_2 \). Clearly \( e_1 Re_2 \) is nilpotent ideal and since it is nonzero, \( R \) is not prime. So, by Proposition 3, the radical \( N \) of \( R \) is a nil ideal. Thus \( e_1 Ne_2 = 0 \). Let \( e_2xe_1 \in e_2 Ne_2 \). Define \( \sigma: e_2 R \rightarrow e_2 R \) by \( \sigma(e_2y) = e_2xe_1y \). Then \( \sigma \in \text{Hom}_R(e_2 R, e_2 R) \), and since \( e_2xe_1 \) is nilpotent, \( \sigma \) is not a monomorphism. So \( \ker \sigma = (0) \). Since \( \text{Hom}_R(e_2 R, e_2 R) \neq 0 \), there exists an embedding \( \gamma: e_2 R \rightarrow e_2 R \). Now \( \gamma \sigma: e_2 R \rightarrow e_2 R \), and since \( \ker \sigma = (0) \), \( \ker \gamma \sigma = (0) \). By Lemma 5, \( \gamma \sigma = 0 \). Since \( \gamma \) is a monomorphism, we have \( \sigma = 0 \). Thus \( e_2xe_1 = 0 \), and \( e_2Ne_2 = 0 \). So \( e_2Re_2 \) is a division ring. Further \( e_2 Re_2 = e_2 R \) since \( e_2 Re_1 = 0 \). Thus \( e_2 N = 0 \), and \( e_2 R \) is a minimal right ideal. Now \( e_1 R \) is uniform because it is quasi-injective and indecomposable. Since \( 0 \neq e_1 Re_2 R \) is the sum of the images of all \( R \)-homomorphisms of \( e_1 R \) into \( e_1 R \), the fact that \( e_1 R \) is minimal and \( e_1 R \) is uniform yields that \( e_1 Re_2 R \) itself is the unique minimal right subideal of \( e_1 R \), is isomorphic to \( e_1 R \), and is contained in every nonzero right subideal of \( e_1 R \). We claim that \( e_1 Ne_1 = 0 \). Let \( 0 \neq e_1 xe_1 \in e_1 Ne_1 \). Since \( N \) is nil, \( e_1 xe_1 \) is nilpotent. Then \( \sigma: e_1 R \rightarrow e_1 R \) defined by \( \sigma(e_1 y) = e_1 xe_1 y \) is an endo-
morphism of $e,R$ with ker $\sigma \neq (0)$. Let $A = \ker \sigma$. Then $e,R \subseteq A$, and we have $e,xe,R \neq (0)$. On the other hand, $e,R \subseteq e,xe,R$ yields that $e,xe,R \neq (0)$. This is a contradiction. Hence $e,Ne_i = (0)$, and $e,R$ is a division ring. Now using the fact that Hom$_{e,R}(e,R,e,R)$ is a division ring and that $e,R$ is quasi-injective, it follows that every member of Hom$_{e,R}(e,R)$ admits a unique extension to an endomorphism of $e,R$. Further, every endomorphism of $e,R$ maps $e,R$ into itself since $e,R$ is the unique minimal subideal of $e,R$. Thus Hom$_{e,R}(e,R) \cong$ Hom$_{e,R}(e,R)$. Since $e,R \cong e,R$, we obtain $e,R \cong e,R$.

Now $e,N = e,N e_i$ because $e,N e_i = (0)$. Since $e,R \subseteq e,N$, we get $\sigma$ is an R-endomorphism, so it can be extended to an endomorphism $e,N \subseteq e,N e_i$. Thus $M = e,R e_i$ is a one-dimensional right vector space over $D = e,R e_i$. We show that $M$ is also a one-dimensional left $e,R e_i$-space. Let $X = \left( e,R e_i, M \right) \cong R/A$, where $A = \left( 0, 0 \right)$. Then $X$ is quasi-injective. Let $0 \neq x \in M$, and let $y \in M$. Consider $\sigma'(0 M) \to (0 M)$ defined by $\sigma'(0 xc) = (0 yc)$, for $c \in D$. Then $\sigma$ is an $R$-endomorphism, so it can be extended to an endomorphism $\gamma$ of $X$. Let $\gamma(0 c) = (a, b)$. Then we have $\gamma(0 y) = \sigma'(0 x) = \gamma(0 x) = (0 ax, 0)$. Thus $y = ax$, so $M = e,R e_i$. So $M$ is a one-dimensional left vector space over $e,R e_i$. Thus, for each $d \in e,R e_i$, there exists a unique $d' \in e,R e_i$ such that $dx = xd'$. Define $\theta: e,R e_i \to e,R e_i$ by $\theta(d) = d'$. Then $\theta$ is an isomorphism, and we may identify $d$ and $d'$. Then $\gamma(0 D) \to (0 D)$ defined by $\gamma(0 D) = (a, b)$. Then $\gamma(D,D) \to (D,M)$ is an isomorphism.

Conversely, if $R$ satisfies (i), then, by Koehler [13], $R$ is a QC-ring, hence a PCQI-ring. If $R$ satisfies (ii), then straightforward computation shows that $R$ is a right PCQI-ring.

Since every right QC-ring is a left QC-ring and $\left( D, D \right)$ is also a left PCQI-ring, we get the following corollary.

**Corollary.** A nonlocal semiperfect right PCQI-ring is also a left PCQI-ring.

5. **Local PCQI-rings.** Theorem 13 and Theorems 14, 15, and 16 which follow generalize Klatt and Levy's [11] theorems for commutative pre-self-injective rings which are not domains. Throughout this section $M$ will denote the unique maximal right ideal of a local ring $R$. $M$ is then the Jacobson radical of $R$, and $R/M$ is a division ring.

**Theorem 14.** Let $R$ be a local right PCQI-ring with maximal ideal $M$. Then either $R$ is a right valuation ring or $M^2 = (0)$ and $M/R$ has composition length 2.
Proof. First note that for all nonzero right ideals \( A \), \( R/A \) is indecomposable quasi-injective and hence uniform. Now we show that all nonzero right ideals are either minimal or essential. Let \( A, B \) be nonzero right ideals such that \( A \cap B = (0) \). We claim that \( A \) is minimal. Let \( C \) be a nonzero right ideal properly contained in \( A \). Then \( R/C \) is quasi-injective and not uniform since \( A/C \cap (B + C)/C = 0 \). This is a contradiction, so \( A \) is minimal. Similarly, \( B \) is minimal. In particular, it follows that any maximal independent family of minimal right ideals can contain at most two members.

If \( \text{Soc} R = (0) \), then all nonzero right ideals are essential. Let \( A, B \) be two nonzero right ideals. If neither \( A \leq B \) nor \( B \leq A \), then \( R/A \cap B \) is quasi-injective but not uniform since \( A/(A \cap B) \cap (B/(A \cap B)) = (0) \). As before, this is a contradiction. So either \( A \subseteq B \) or \( B \subseteq A \).

If \( \text{Soc} R \) consists of a unique minimal right ideal then it is clear that \( R \) is a right valuation ring.

Finally, suppose \( \text{Soc} R = A \oplus B \), where \( A, B \) are minimal right ideals. Then \( R \) cannot be prime. Let \( x \in M \), and consider \( xR \). If \( xR \) is not minimal, then \( xR \) is quasi-injective and decomposable. Then \( xR = A \oplus B \). In any case, for all \( x \in M \), \( x \in \text{Soc} R \). This implies that \( M^2 = (0) \), and the composition length of \( M \) is 2, completing the proof.

The next two theorems give the structure of non-prime local right \( PCQI \)-rings. Prime local \( PCQI \)-rings are discussed in the next section.

**Theorem 15.** For a nonprime right valuation ring \( R \), the following are equivalent:

(i) \( R \) is a right \( PCQI \)-ring.

(ii) \( R \) is a right duo almost maximal valuation ring of rank 0 such that any left ideal containing a nonzero right ideal is two-sided.

**Proof.** (i) \( \Rightarrow \) (ii). Since \( R \) is not prime, \( M \) is nil by Proposition 3. So, if \( xR \) is a nontrivial principal right ideal of \( R \), \( xR \) is quasi-injective. Since \( xR \) is essential in \( R \), the injective hull of \( xR \) is the same as that of \( R \). Hence, by Johnson and Wong [10], \( RxR \subseteq xR \). So \( xR \) is a two-sided ideal of \( R \). Thus \( R \) is a right duo ring. Since each proper homomorphic image of a \( PCQI \)-ring is a \( QC \)-ring, the proof of (i) \( \Rightarrow \) (ii) as well as that of (ii) \( \Rightarrow \) (i) is completed by a theorem of Koehler [13].

**Theorem 16.** For a local ring \( R \) with \( M^2 = (0) \) and the composition length of \( M_R \) equal to 2, the following are equivalent:
R is a right PCQI-ring.

(ii) For each nonzero right ideal $A$ in $R$ and for each $m \in A$, the congruence $x m_i \equiv m (\text{mod } A)$ has a solution, $x = \alpha$, such that $\alpha A \subseteq A$.

Proof. Under the hypothesis the only nonzero right ideals $A$ of $R$ different from $M$ and $R$ are minimal right ideals, and $M/A$ is a simple right $R$-module.

(i) $\Rightarrow$ (2) Let $A$ be a nontrivial right ideal in $R$, and let $m, m_1, m_2 \in R$ such that $m, m_1, m_2 \in A$. Then $\bar{m}_1 R = M/A = \bar{m}_2 R$, and the mapping $\sigma: M/A \to M/A$ which sends $\bar{m}_1 r$ to $\bar{m}_2 r$ is a well-defined $R$-homomorphism. Since $R/A$ is quasi-injective, $\sigma$ can be lifted to $\sigma^* \in \text{Hom}_R (R/A, R/A)$. Let $\sigma(\bar{1}) = \bar{\alpha}$. Then $\bar{\alpha} m_i = \bar{m}_2$. Hence $x m_i \equiv m_2 (\text{mod } A)$ has a solution $x = \alpha$. Clearly $\alpha A \subseteq A$.

(ii) $\Rightarrow$ (i) We only need to prove that if $A$ is a nontrivial right ideal of $R$ and $\sigma: M/A \to R/A$, is a nonzero $R$-homomorphism, then $\sigma$ can be extended to an $R$-homomorphism $\sigma^*: R/A \to R/A$. Let $m \in M$, where $m \in A$. Then $M/A = \overline{m} R$. Also, $\sigma(M/A) = M/A$. Let $\sigma(\overline{m}) = \overline{m} r$. Since $M^* = (0)$, $r \not\in M$. So $r$ is invertible, and $m r \not\in A$. Let $\alpha \in R$ be chosen such that $\alpha m = m r (\text{mod } A)$, and $\alpha A \subseteq A$. Then $\sigma^*(\overline{r}) = \overline{\alpha} R$ is well-defined, and it extends $\sigma$, completing the proof.

The example which follows shows that a local right PCQI-ring is not necessarily a left PCQI-ring.

**Example.** Let $F$ be a field which has a monomorphism $\rho: F \to F$ such that $[F: \rho(F)] > 2$. Take $x$ to be an indeterminate over $F$. Make $V = x F$ into a right vector space over $F$ in a natural way. Let $R = \{(\alpha, x \beta) \mid \alpha, \beta \in F\}$. Define

$$(\alpha_1, x \beta_1) + (\alpha_2, x \beta_2) = (\alpha_1 + \alpha_2, x \beta_1 + x \beta_2)$$

and

$$(\alpha_1, x \beta_1)(\alpha_2, x \beta_2) = (\alpha_1 \alpha_2, x (\rho(\alpha_1) \beta_2 + \beta_1 \alpha_2)).$$

Then $R$ is a local ring with identity with the maximal ideal $M = \{(0, x \alpha) \mid \alpha \in F\}$.

In fact, $M$ is also a minimal right ideal and $M^2 = (0)$. Thus $R$ is a right PCQI-ring. Further, if $\{\alpha_i\}_{i \in I}$ is a basis of $F$ as a vector space over $\rho(F)$ then straightforward computations yield that $M = \bigoplus \sum R(0, x \alpha_i)$ as a direct sum of irreducible left $R$-modules $R(0, x \alpha_i)$. Since $\text{card } I > 2$, it follows by Theorem 14 that $R$ is not a left PCQI-ring.
6. Prime local PCQI-rings.

**Theorem 17.** Let $R$ be a prime local right PCQI-ring. Then $R$ is a right valuation domain, hence right semihereditary.

**Proof.** By Theorem 14, $R$ is a right valuation ring. Let $A$ denote the intersection of all nonzero two-sided ideals of $R$. The proof that $R$ is a domain falls into three cases.

(i) $A = (0)$.

Let $x, y \in R$ such that $xy = 0$. Suppose $y \neq 0$. Then $yR$ is a nonzero right ideal of $R$. Since $R$ is right valuation and $A = (0)$, $yR$ must contain a nonzero two-sided ideal of $R$. Further, each proper homomorphic image of $R$ is a local QC-ring, hence a duo ring [13]. This implies that $yR$ is two-sided. Hence $x = 0$, and $R$ is an integral domain.

(ii) $A \neq (0)$ and $A \neq M$.

Under these hypotheses, $A$ cannot be a prime ideal. So there exist $x, y \in R$ such that $xRy \subseteq A$, $x \notin A$ and $y \notin A$. Since $R$ is right valuation, $A \subseteq xR$ and $A \subseteq yR$. So both $xR$ and $yR$ are two-sided ideals. For definiteness, let $xR \subseteq yR$. Then $(xR)^2 \subseteq (xR)(yR) \subseteq AR = A$ gives that $(xR)^2 = A$ by the minimality of $A$. Also $A = A^2$, hence $(xR)^2 = (xR)^4$. It follows that $x^2R = x'R$. Then $x^2 = x'r$, for some $r \in R$, and $x^2(1 - x^2r) = 0$. So $x^2 = 0$. Thus $A = (0)$, and this case cannot occur.

(iii) $A = M$.

Let $S \subseteq R$, and let $r(S)$ denote the right annihilator of $S$ in $R$. Let $Z(R) = \{x \in R | r(x)$ is an essential right ideal}. Then $Z(R)$ is an ideal in $R$ called the right singular ideal.

Since $R$ is a right valuation ring, $R$ is immediately a domain if $Z(R) = (0)$.

So assume that $Z(R) \neq (0)$. Then $Z(R) = M$, and each element in $M$ is a right zero divisor. So $x \in M$ implies that $xR$ is proper cyclic, hence quasi-injective. Also $xR$ is an essential right ideal in $R$. By Johnson and Wong [10], $RxR \subseteq xR$. Hence $xR$ is two-sided. So $R$ is a prime right duo ring, and it follows that $R$ is a domain.

7. **PCQI-domains.** In this section we discuss right PCQI-rings which are integral domains and prove that these are right Öre-domains. This generalizes the result of Faith [4]. Our proof, in this case, though it runs on the same lines as that of Faith, does not use Faith’s result.

**Proposition 18.** Let $R$ be a right PCQI-domain, and let $I$ be a nonessential right ideal of $R$. Then $R/I$ is an injective right $R$-
module containing a copy of $R$.

Proof. Since $I$ is nonessential, there exists a nonzero right ideal $J$ in $R$ such that $I \cap J = 0$. Let $a \in J$ such that $a \neq 0$. Then $aR \cap I \subseteq J \cap I = 0$. Consider $r(a + I) = \{x \in R | ax \in I\}$. Clearly $r(a + I) = 0$. So $R/I$ contains a copy of $R$. Since $R/I$ is also quasi-injective, this implies that $R/I$ is injective by [17].

For a right $R$-module $A$, let $\hat{A}$ denote the injective hull of $A$.

PROPOSITION 19. Let $R$ be a right PCQI-domain which is not a right Ore-domain. Then $R$ is finitely presented.

Proof. Let $a \in R$ such that $a \neq 0$ and $aR$ is not essential. Then $R/aR$ is injective. Since $R/aR$ contains a copy of $R$ and is injective, $R/aR$ contains a copy of $\hat{R}$. Then $R/aR = Y/aR \oplus X/aR$, where $X/aR \cong \hat{R}$. Now $Y/aR$ is cyclic. So $Y = aR + bR$, for some $b \in R$, and the short exact sequence $0 \rightarrow Y \rightarrow R \rightarrow R/Y \cong X/aR \cong \hat{R} \rightarrow 0$ shows that $\hat{R}$ is finitely presented.

THEOREM 20. A right PCQI-domain $R$ is a right Ore-domain.

Proof. Let $R$ be a right PCQI-domain. Suppose $R$ is not a right Ore-domain. Then, as in Proposition 19, there exists $a \in R$ such that $R/aR = Y/aR \oplus X/aR$, where $X/aR \cong \hat{R} \cong R/Y$ and $Y = aR + bR$. We also get that $R = X + Y$, where $X \cap Y = aR$. This yields an exact sequence $0 \rightarrow aR \rightarrow X \times Y \rightarrow R \rightarrow 0$ which splits. So $X \times Y \cong aR \times R \cong R \times R$. This implies that $Y = aR + bR$ is a finitely generated projective right ideal. Since $\hat{R} \cong R/Y$, $0 \rightarrow Y \rightarrow R \rightarrow \hat{R} \rightarrow 0$ is exact. Then $Y \otimes_R \hat{R} \rightarrow R \otimes_R \hat{R} \rightarrow \hat{R} \otimes_R \hat{R} \rightarrow 0$ is exact. Also, a finitely generated projective $R$-module is essentially finitely related. So, by Cateforis ([3], Proposition 1.7), $(aR + bR) \otimes_R \hat{R}$ is projective as an $\hat{R}$-module. Then $Y \otimes_R \hat{R}$ is a direct summand of a free $\hat{R}$-module. Now $Z(\hat{R}) = 0$, hence $Z(Y \otimes_R \hat{R}) = 0$ because $Y \otimes_R \hat{R}$ is a direct summand of a free $\hat{R}$-module. Now consider $Y \otimes_R \hat{R} \rightarrow R \otimes_R \hat{R} \rightarrow \hat{R} \otimes_R \hat{R} \rightarrow 0$. Again, by Cateforis ([3], Lemma 1.8), $\ker i = Z(Y \otimes_R \hat{R}) = 0$. So $0 \rightarrow Y \otimes_R \hat{R} \rightarrow R \otimes_R \hat{R} \rightarrow \hat{R} \otimes_R \hat{R} \rightarrow 0$ is exact. Since $R \otimes_R \hat{R} \cong \hat{R}$, let $f : R \otimes_R \hat{R} \rightarrow \hat{R}$ be the canonical isomorphism. Then $f \circ i : Y \otimes_R \hat{R} \rightarrow \hat{R}$ is a monomorphism, and $Y \otimes_R \hat{R} \cong Y\hat{R}$. Since $Y$ is finitely generated, $Y\hat{R}$ is a finitely generated right ideal of $\hat{R}$. So $Y\hat{R} = e\hat{R}$, where $e^2 = e$. Thus we have the following exact sequence: $0 \rightarrow e\hat{R} \rightarrow \hat{R} \otimes_R \hat{R} \rightarrow 0$, and $\hat{R} \otimes_R \hat{R} \cong \hat{R}/e\hat{R} = (1 - e)\hat{R}$. Hence $\hat{R} \otimes_R \hat{R}$ is isomorphic to a direct summand of $\hat{R}$. Since $Z(\hat{R}) = 0$, $Z(\hat{R} \otimes_R \hat{R}) = 0$. Since $\hat{R} = xR$, for some $x \in \hat{R}$, the
kernel of the canonical map \( f: \hat{R} \otimes_R B \to \hat{R} \) defined by \( f(a \otimes b) = ab \) is contained in \( Z(\hat{R} \otimes_R \hat{R}) \) and hence must be zero. Since \( f \) is surjective, \( f \) is an isomorphism. By Silver ([18], Proposition 1.1), there exists an epimorphism in the category of rings from \( R \) to \( \hat{R} \).

Let \( M \) be a right \( \hat{R} \)-module which is quasi-injective as a right \( \hat{R} \)-module. We claim that \( M \) is quasi-injective as a right \( \hat{R} \)-module. Let \( 0 \to A_{\hat{R}} \to M_{\hat{R}} \to B_{\hat{R}} \to 0 \) be exact. Consider \( 0 \to \text{Hom}_{\hat{R}}(B_{\hat{R}}, M_{\hat{R}}) \to \text{Hom}_{\hat{R}}(B_{\hat{R}}, B_{\hat{R}}) \to \text{Hom}_{\hat{R}}(A_{\hat{R}}, M_{\hat{R}}) \). By Silver ([18], Corollary 1.3), \( \text{Hom}_{\hat{R}}(N, N') \cong \text{Hom}_{\hat{R}}(N, N') \), where \( N, N' \) are right \( \hat{R} \)-modules. Also \( 0 \to \text{Hom}_{\hat{R}}(B, M) \to \text{Hom}_{\hat{R}}(M, M) \to \text{Hom}_{\hat{R}}(A, M) \to 0 \) is exact since \( M_{\hat{R}} \) is quasi-injective. Thus \( 0 \to \text{Hom}_{\hat{R}}(B, M) \to \text{Hom}_{\hat{R}}(M, M) \to \text{Hom}_{\hat{R}}(A, M) \to 0 \) is exact. So \( M_{\hat{R}} \) is quasi-injective. Let \( K \) be a cyclic right \( \hat{R} \)-module. Then \( K \) is a cyclic right \( R \)-module. Since \( R \) is a right \( PCQI \)-domain, \( K_{\hat{R}} \) is quasi-injective. Thus \( K_{\hat{R}} \) is quasi-injective. Since \( \hat{R} \) is right self-injective, \( \hat{R} \) is a \( QC \)-ring. So \( \hat{R} \) is semiperfect and simple, hence simple artinian. Thus \( \hat{R} \) is a division ring. This proves that \( R \) is a right \( \hat{O}r \)-domain.

We conclude by a remark that we have not studied arbitrary prime right \( PCQI \)-rings. This case remains open. Indeed, a characterization of right \( PCQI \)-domains has not yet been obtained.

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