ON EXTREME POINTS OF THE JOINT NUMERICAL RANGE OF COMMUTING NORMAL OPERATORS

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Let $W(T) = \{\langle Tx, x \rangle : ||x|| = 1; x \in H\}$ denote the numerical range of a bounded normal operator $T$ on a complex Hilbert space $H$. S. Hildebrandt has proved that if $\lambda$ is an extreme point of $W(T)$, the closure of $W(T)$, and $\lambda \in W(T)$ then $\lambda$ is in the point spectrum of $T$. In this note, we shall prove an analogous result for an $n$-tuple of commuting bounded normal operators on $H$.

2. Notations and terminology. Let $A = (A_1, \cdots, A_n)$ be an $n$-tuple of commuting bounded operators on $H$ and $\mathcal{Z}$, the double commutant of $\{A_1, \cdots, A_n\}$. Then $\mathcal{Z}$ is a commutative Banach algebra with identity, containing the set $\{A_1, \cdots, A_n\}$. We shall need the following definitions [3] and [4].

A point $z = (z_1, \cdots, z_n)$ of $\mathbb{C}^n$ is in the joint spectrum $\sigma(A)$ of $A$ relative to $\mathcal{Z}$ if for all $B_i$, $\cdots$, $B_n$ in $\mathcal{Z}$

$$\sum_{j=1}^{n} B_j (A_j - z_j) \neq I.$$ 

The joint numerical range of $A$ is the set of all points $z = (z_1, \cdots, z_n)$ of $\mathbb{C}^n$ such that for some $x$ in $H$ with $||x|| = 1$, $z_j = \langle A_j x, x \rangle$ i.e.,

$$W(A) = \{\langle Ax, x \rangle = (\langle A_1 x, x \rangle, \cdots, \langle A_n x, x \rangle)\}.$$ 

We say that $z = (z_1, \cdots, z_n)$ is in the joint point spectrum $\sigma_p(A)$ if there exists some $0 \neq x \in H$ such that

$$A_j x = z_j x, \quad j = 1, \cdots, n,$$

and that $z$ is in the joint approximate point spectrum $\sigma_\pi(A)$ if there exists a sequence $\{x_n\}$ of unit vectors in $H$ such that $||A_j - z_j x_n|| \to 0$ as $n \to \infty$, $j = 1, \cdots, n$.

Bunce [2] has proved that $\sigma_\pi(A)$ is a nonempty compact subset of $\mathbb{C}^n$.

If $A = (A_1, \cdots, A_n)$ is an $n$-tuple of commuting normal operators, then the extreme points of $W(A)$ are in the joint approximate point spectrum $\sigma_\pi(A)$. This is immediate from the fact that for such $A_j$'s,

$$W(A) = \text{closed convex hull of } \sigma(A) = \text{closed convex hull of } \sigma_\pi(A),$$

and that every compact set contains the extreme points of its closed
convex hull [1, Cor. 36. 11, p. 144]. We show in the following theorem that something more can be said about the extreme points of $\overline{W(A)}$, see Hildebrandt [5].

**Theorem.** Let $A = (A_1, \cdots, A_n)$ be an $n$-tuple of commuting normal operators on $H$. If $\lambda = (\lambda_1, \cdots, \lambda_n)$ is an extreme point of $\overline{W(A)}$ and $\lambda \in W(A)$, then $\lambda \in \sigma_p(A)$.

**Proof.** Firstly, we shall prove the result for commuting self-adjoint operators.

It is sufficient to show that if $(0, \cdots, 0)$ is an extreme point of $\overline{W(A)}$ and $(0, \cdots, 0) \in W(A)$, then $(0, \cdots, 0) \in \sigma_p(A)$.

Since $(0, \cdots, 0)$ is an extreme point of $\overline{W(A)}$, we may assume that

$$\overline{W(A)} \subset \{ z = (\alpha_1, \cdots, \alpha_n) \in \mathbb{C}^n ; \text{Re} \alpha_n \geq 0 \}.$$  

As $A_1, \cdots, A_n$ are commuting self-adjoint operators, there exists a measure space $(X, \mu)$ and a set of bounded measurable functions $\varphi_1, \cdots, \varphi_n$ in $L^\infty(X, \mu)$ such that each $A_i$ is unitarily equivalent to multiplication by $\varphi_i$ on $L^2(X, \mu)$. Thus

$$A_j f = \varphi_j f, \quad \text{for all } f \in L^2(X, \mu)$$

and for each $j = 1, 2, \cdots, n$ [3].

Because of the assumption (1), and since $\sigma(A) \subset \overline{W(A)}$, we have

$$\sigma(A) \subset \{ z = (\alpha_1, \cdots, \alpha_n) \in \mathbb{C}^n ; \text{Re} \alpha_n \geq 0 \}.$$  

It follows that $A_n \geq 0$ and so $\varphi_n(x) \geq 0$ a.e. Let, if possible, $(0, \cdots, 0) \in \sigma(A_1, \cdots, A_n)$. Then $|\varphi_j(x)| > 0$ a.e. for at least one $j = 1, 2, \cdots, n$. Let

$$E_1 = \{ x \in X; \text{Im} \varphi_j(x) \geq 0 \}$$

and

$$E_2 = \{ x \in X; \text{Im} \varphi_j(x) < 0 \}.$$  

Since $(0, \cdots, 0) \in W(A_1, \cdots, A_n)$, for some $f \in H$ with $\|f\| = 1$, $\langle A_j f, f \rangle = 0$, $j = 1, 2, \cdots, n$ and

$$0 = \langle A_j f, f \rangle = \int_X \varphi_j(x) |f(x)|^2 d\mu = \int_{E_1} \varphi_j |f|^2 d\mu + \int_{E_2} \varphi_j |f|^2 d\mu$$

$$= \int_X \varphi_j |\chi_{E_1} f|^2 d\mu + \int_X \varphi_j |\chi_{E_2} f|^2 d\mu$$
where \( g_kx = (\chi)_{g_k}(x)f(x), k = 1, 2, \chi \) denotes the characteristic function.

As \( A_n \geq 0 \) and \( \langle A_ng_1, g_1 \rangle + \langle A_ng_2, g_2 \rangle = 0 \), it follows that \( \langle A_ng_1, g_1 \rangle = 0 \) and \( \langle A_ng_2, g_2 \rangle = 0 \).

(i) Suppose that \( |\varphi_n(x)| > 0 \) a.e. Then \( \langle A_ng_1, g_1 \rangle = 0 \) implies that \( f \) and \( \varphi_n \) have complementary support which is a contradiction to the fact that \( ||f|| = 1 \) and \( |\varphi_n(x)| > 0 \) a.e.

(ii) If \( |\varphi_j(x)| > 0 \) a.e for \( j \neq n \), then \( \langle A_jg_1, g_1 \rangle \neq 0 \) for if \( \langle A_jg, g_1 \rangle = 0 \), then \( \langle A_jg_2, g_2 \rangle = 0 \) which means that \( f \) and \( \varphi_j \) have complementary support which is again not possible as argued in (i). Thus \( \langle A_jg_1, g_1 \rangle \neq 0 \), \( \langle A_jg_2, g_2 \rangle \neq 0 \). We write \( h_k(x) = g_k(x)/||g_k||, k = 1, 2 \) and

\[ \lambda = \{\langle A_1h_1, h_1 \rangle, \cdots, \langle A_nh_1, h_1 \rangle\} \]

and

\[ \mu = \{\langle A_1h_2, h_2 \rangle, \cdots, \langle A_nh_2, h_2 \rangle\} \]

Thus \( \lambda \) and \( \mu \) are two points in the joint numerical range with \( (0, \cdots, 0) \) as an interior point of the line segment joining these two, which is a contradiction. This proves the result for commuting self-adjoint \( A_j \)'s.

Now, we consider \( A_j \)'s to be commuting normal operators on \( H \). Since each \( A_j \) has a unique decomposition

\[ A_j = A_{j1} + iA_{j2}, \quad j = 1, 2, \cdots, n, \]

where \( A_{j1} \) and \( A_{j2} \) are self-adjoint, the \( 2n \)-tuple

\[ \{A_{i1}, A_{i2}, \cdots, A_{ni}, A_{i2}, \cdots, A_{ni}\} \]

is of commuting self-adjoint operators. Similarly if

\[ \lambda_j = \lambda_{j1} + i\lambda_{j2}, \quad j = 1, 2, \cdots, n \]

then \( \lambda' = \{\lambda_{11}, \cdots, \lambda_{ni}, \lambda_{i2}, \cdots, \lambda_{ni}\} \) is an extreme point of \( W(A_{i1}, \cdots, A_{ni}) \) and \( \lambda' \in W(A_{i1}, \cdots, A_{ni}) \). Thus \( \lambda' \in \sigma_p(A_{i1}, \cdots, A_{ni}) \). Hence \( \lambda = (\lambda_{i1}, \cdots, \lambda_{ni}) \in \sigma_p(A_{i1}, \cdots, A_{ni}) \) and the result is proved.

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**References**


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<table>
<thead>
<tr>
<th>Title</th>
<th>Authors</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weyl’s inequality and quadratic forms on the Grassmannian</td>
<td>Patricia Andresen and Marvin David Marcus</td>
<td>277</td>
</tr>
<tr>
<td>Regular lattice measures: mappings and spaces</td>
<td>George Bachman and Alan Sultan</td>
<td>291</td>
</tr>
<tr>
<td>On certain algebraic integers and approximation by rational functions with integral coefficients</td>
<td>David Geoffrey Cantor</td>
<td>323</td>
</tr>
<tr>
<td>On the value distribution of functions meromorphic in the unit disk with a spiral asymptotic value</td>
<td>James Richard Choike</td>
<td>339</td>
</tr>
<tr>
<td>Divided rings and going-down</td>
<td>David Earl Dobbs</td>
<td>353</td>
</tr>
<tr>
<td>Integrals of continuous functions</td>
<td>Mark Finkelstein and Robert James Whitley</td>
<td>365</td>
</tr>
<tr>
<td>Integrals of foliations on manifolds with a generalized symplectic structure</td>
<td>Ronald Owen Fulp and Joe Alton Marlin</td>
<td>373</td>
</tr>
<tr>
<td>Principal and induced fibrations</td>
<td>Cheong Seng Hoo</td>
<td>389</td>
</tr>
<tr>
<td>Parametrized surgery and isotopy</td>
<td>Wu-Chung Hsiang and Richard W. Sharpe</td>
<td>401</td>
</tr>
<tr>
<td>Rings whose proper cyclic modules are quasi-injective</td>
<td>Surender Kumar Jain, Surjeet Singh and Robin Gregory Symonds</td>
<td>461</td>
</tr>
<tr>
<td>On extreme points of the joint numerical range of commuting normal operators</td>
<td>Pushpa Juneja</td>
<td>473</td>
</tr>
<tr>
<td>Nth order oscillations with middle terms of order N – 2</td>
<td>Athanassios G. Kartsatos</td>
<td>477</td>
</tr>
<tr>
<td>The generalized translational hull of a semigroup</td>
<td>John Keith Luedeman</td>
<td>489</td>
</tr>
<tr>
<td>The altitude formula and DVR’s</td>
<td>Louis Jackson Ratliff, Jr.</td>
<td>509</td>
</tr>
<tr>
<td>An upper bound for the period of the simple continued fraction for √D</td>
<td>Ralph Gordon Stanton, C. Sudler and Hugh C. Williams</td>
<td>525</td>
</tr>
<tr>
<td>Global analysis and periodic solutions of second order systems of nonlinear differential equations</td>
<td>David Westreich</td>
<td>537</td>
</tr>
<tr>
<td>“Compactly cogenerated LCA groups”</td>
<td>David Lee Armacost</td>
<td>555</td>
</tr>
<tr>
<td>“On groups with a single involution”</td>
<td>Jerry Malzan</td>
<td>555</td>
</tr>
<tr>
<td>“Bifurcation of operator equations with unbounded linearized part”</td>
<td>David Westreich</td>
<td>555</td>
</tr>
</tbody>
</table>