

Pacific Journal of Mathematics

**ON EXTREME POINTS OF THE JOINT NUMERICAL RANGE
OF COMMUTING NORMAL OPERATORS**

PUSHPA JUNEJA

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Let $W(T) = \{\langle Tx, x \rangle : \|x\| = 1; x \in H\}$ denote the numerical range of a bounded normal operator T on a complex Hilbert space H . S. Hildebrandt has proved that if λ is an extreme point of $\overline{W(T)}$, the closure of $W(T)$, and $\lambda \in W(T)$ then λ is in the point spectrum of T . In this note, we shall prove an analogous result for an n -tuple of commuting bounded normal operators on H .

2. Notations and terminology. Let $A = (A_1, \dots, A_n)$ be an n -tuple of commuting bounded operators on H and \mathcal{U} , the double commutant of $\{A_1, \dots, A_n\}$. Then \mathcal{U} is a commutative Banach algebra with identity, containing the set $\{A_1, \dots, A_n\}$. We shall need the following definitions [3] and [4].

A point $z = (z_1, \dots, z_n)$ of \mathcal{E}^n is in the joint spectrum $\sigma(A)$ of A relative to \mathcal{U} if for all B_1, \dots, B_n in \mathcal{U}

$$\sum_{j=1}^n B_j(A_j - z_j) \neq I.$$

The joint numerical range of A is the set of all points $z = (z_1, \dots, z_n)$ of \mathcal{E}^n such that for some x in H with $\|x\| = 1$, $z_j = \langle A_j x, x \rangle$ i.e.,

$$W(A) = \{\langle Ax, x \rangle = (\langle A_1 x, x \rangle, \dots, \langle A_n x, x \rangle)\}.$$

We say that $z = (z_1, \dots, z_n)$ is in the joint point spectrum $\sigma_p(A)$ if there exists some $0 \neq x \in H$ such that

$$A_j x = z_j x, \quad j = 1, \dots, n,$$

and that z is in the joint approximate point spectrum $\sigma_\pi(A)$ if there exists a sequence $\{x_n\}$ of unit vectors in H such that $\|(A_j - z_j)x_n\| \rightarrow 0$ as $n \rightarrow \infty$, $j = 1, \dots, n$.

Bunce [2] has proved that $\sigma_\pi(A)$ is a nonempty compact subset of \mathcal{E}^n .

If $A = (A_1, \dots, A_n)$ is an n -tuple of commuting normal operators, then the extreme points of $\overline{W(A)}$ are in the joint approximate point spectrum $\sigma_\pi(A)$. This is immediate from the fact that for such A_j 's,

$$\begin{aligned} \overline{W(A)} &= \text{closed convex hull of } \sigma(A) \\ &= \text{closed convex hull of } \sigma_\pi(A), \end{aligned}$$

and that every compact set contains the extreme points of its closed

convex hull [1, Cor. 36. 11, p. 144]. We show in the following theorem that something more can be said about the extreme points of $\overline{W(A)}$, see Hildebrandt [5].

THEOREM. *Let $A = (A_1, \dots, A_n)$ be an n -tuple of commuting normal operators on H . If $\lambda = (\lambda_1, \dots, \lambda_n)$ is an extreme point of $\overline{W(A)}$ and $\lambda \in W(A)$, then $\lambda \in \sigma_p(A)$.*

Proof. Firstly, we shall prove the result for commuting self-adjoint operators.

It is sufficient to show that if $(0, \dots, 0)$ is an extreme point of $\overline{W(A)}$ and $(0, \dots, 0) \in W(A)$, then $(0, \dots, 0) \in \sigma_p(A)$.

Since $(0, \dots, 0)$ is an extreme point of $\overline{W(A)}$, we may assume that

$$(1) \quad \overline{W(A)} \subset \{z = (\alpha_1, \dots, \alpha_n) \in \mathcal{C}^n; \operatorname{Re} \alpha_n \geq 0\}.$$

As A_1, \dots, A_n are commuting self-adjoint operators, there exists a measure space (X, μ) and a set of bounded measurable functions $\varphi_1, \dots, \varphi_n$ in $L^\infty(X, \mu)$ such that each A_j is unitarily equivalent to multiplication by φ_j on $L^2(X, \mu)$. Thus

$$A_j f = \varphi_j f, \quad \text{for all } f \in L^2(X, \mu)$$

and for each $j = 1, 2, \dots, n$ [3].

Because of the assumption (1), and since $\sigma(A) \subset \overline{W(A)}$, we have

$$\sigma(A) \subset \{z = (\alpha_1, \dots, \alpha_n) \in \mathcal{C}^n; \operatorname{Re} \alpha_n \geq 0\}.$$

It follows that $A_n \geq 0$ and so $\varphi_n(x) \geq 0$ a.e. Let, if possible, $(0, \dots, 0) \notin \sigma_p(A_1, \dots, A_n)$. Then $|\varphi_j(x)| > 0$ a.e. for at least one $j = 1, 2, \dots, n$. Let

$$E_1 = \{x \in X; \operatorname{Im} \varphi_j(x) \geq 0\}$$

and

$$E_2 = \{x \in X; \operatorname{Im} \varphi_j(x) < 0\}.$$

Since $(0, \dots, 0) \in W(A_1, \dots, A_n)$, for some $f \in H$ with $\|f\| = 1$, $\langle A_j f, f \rangle = 0$, $j = 1, 2, \dots, n$ and

$$\begin{aligned} 0 &= \langle A_j f, f \rangle = \int_X \varphi_j(x) |f(x)|^2 d\mu \\ &= \int_{E_1} \varphi_j |f|^2 d\mu + \int_{E_2} \varphi_j |f|^2 d\mu \\ &= \int_X \varphi_j |\chi_{E_1} f|^2 d\mu + \int_X \varphi_j |\chi_{E_2} f|^2 d\mu \end{aligned}$$

$$\begin{aligned}
 &= \int_X \varphi_j |g_1|^2 d\mu + \int_X \varphi_j |g_2|^2 d\mu \\
 &= \langle A_j g_1, g_1 \rangle + \langle A_j g_2, g_2 \rangle,
 \end{aligned}$$

where $g_k x = (\chi)_{E_k}(x)f(x)$, $k = 1, 2$, χ denotes the characteristic function.

As $A_n \geq 0$ and $\langle A_n g_1, g_1 \rangle + \langle A_n g_2, g_2 \rangle = 0$, it follows that $\langle A_n g_1, g_1 \rangle = 0$ and $\langle A_n g_2, g_2 \rangle = 0$.

(i) Suppose that $|\varphi_n(x)| > 0$ a.e. Then $\langle A_n g_1, g_1 \rangle = 0$ implies that f and φ_n have complementary support which is a contradiction to the fact that $\|f\| = 1$ and $|\varphi_n(x)| > 0$ a.e.

(ii) If $|\varphi_j(x)| > 0$ a.e for $j \neq n$, then $\langle A_j g_1, g_1 \rangle \neq 0$ for $\langle A_j g, g \rangle = 0$, then $\langle A_j g_2, g_2 \rangle = 0$ which means that f and φ_j have complementary support which is again not possible as argued in (i). Thus $\langle A_j g_1, g_1 \rangle \neq 0, \langle A_j g_2, g_2 \rangle \neq 0$. We write $h_k(x) = g_k(x)/\|g_k\|$, $k = 1, 2$ and

$$\lambda = \{\langle A_1 h_1, h_1 \rangle, \dots, \langle A_n h_1, h_1 \rangle\}$$

and

$$\mu = \{\langle A_1 h_2, h_2 \rangle, \dots, \langle A_n h_2, h_2 \rangle\}.$$

Thus λ and μ are two points in the joint numerical range with $(0, \dots, 0)$ as an interior point of the line segment joining these two, which is a contradiction. This proves the result for commuting self-adjoint A_j 's.

Now, we consider A_j 's to be commuting normal operators on H . Since each A_j has a unique decomposition

$$A_j = A_{j_1} + iA_{j_2}, \quad j = 1, 2, \dots, n,$$

where A_{j_1} and A_{j_2} are self-adjoint, the $2n$ -tuple

$$\{A_{11}, A_{21}, \dots, A_{n1}, A_{12}, \dots, A_{n2}\}$$

is of commuting self-adjoint operators. Similarly if

$$\lambda_j = \lambda_{j_1} + i\lambda_{j_2}, \quad j = 1, 2, \dots, n$$

then $\lambda' = \{\lambda_{11}, \dots, \lambda_{n1}, \lambda_{12}, \dots, \lambda_{n2}\}$ is an extreme point of $\overline{W(A_{11}, \dots, A_{n2})}$ and $\lambda' \in W(A_{11}, \dots, A_{n2})$. Thus $\lambda' \in \sigma_p(A_{11}, \dots, A_{n2})$. Hence $\lambda = (\lambda_1, \dots, \lambda_n) \in \sigma_p(A_1, \dots, A_n)$ and the result is proved.

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